Liquidity, Assets and Business Cycles

Shouyong Shi∗
University of Toronto
(shouyong@chass.utoronto.ca)

October 2011

Abstract

Equity price is cyclical and often leads the business cycle. These observations have led to the hypothesis that shocks to equity market liquidity are an independent source of the business cycle. In this paper I construct a tractable model to evaluate this hypothesis. After calibrating the model to the US data, I find that a negative liquidity shock in the equity market can generate large reductions in investment, employment and output but, opposite to what is observed in recessions, the shock generates an equity price boom. I demonstrate that this counterfactual response of equity price to liquidity shocks is not unique to the particular model here; rather, it is a robust feature of many models where equity is important either as a direct means of financing investment or as collateral for borrowing in investment finance. The culprit of this counterfactual response is the effect that a negative liquidity shock tightens a firm’s financing constraint, which is intuitively what the shock is supposed to do. This robust result indicates that liquidity shocks to the equity market are unlikely to be the primary driving force of the business cycle. For equity price to fall as it typically does in a recession, a negative liquidity shock must be accompanied or caused by other shocks that relax firms’ financing constraints on investment. I illustrate that a strong negative productivity shock is a good candidate of such concurrent shocks.

JEL classifications: E32; E5; G1
Keywords: Liquidity; Asset prices; Business cycle.

∗Address: Department of Economics, University of Toronto, 150 St. George Street, Toronto, Ontario, Canada, M5S 3G7. This paper has been presented at the Canadian Economic Association meeting (Ottawa, 2011), the International Economic Association meeting (Beijing, 2011), the Canon Institute for Global Studies (Tokyo, 2011), the Asian Meeting of the Econometric Society (Seoul, 2011), and the Chicago Federal Reserve Bank conference on money, banking and payments (Chicago, 2011). I am grateful to Nobu Kiyotaki for many conversations on the topic, to Andrea Ajello for comments, and to Andrea Ferrero for providing some computing codes. Li Li provided excellent research assistance. I gratefully acknowledge financial support from the Canada Research Chair, the Bank of Canada Fellowship and the Social Sciences and Humanities Research Council of Canada. The view expressed here is my own and does not reflect the view of the Bank of Canada.
1. Introduction

Asset prices are cyclical and, in recent episodes, lead the business cycle. Figures 1.1 - 1.3 depict the time series of a broad stock price index and some macro variables in the US from 1999 to 2011. All series are percentage deviations of the quarterly data from the trend, as signified by “dev” in the labels. Percentage deviations of investment, GDP and the bond price are rescaled by the factors indicated in the figures.1 Three regularities are clear from these figures. First, investment and output move closely with stock prices. Second, stock prices lead business cycles. In the 2008-2009 recession, stock prices peaked in the third quarter of 2007 before falling to the trough in the first quarter of 2009. Investment and output did not peak until the second quarter of 2008 and did not reach the trough until the second quarter of 2009. Similarly, in the 2001-2002 recession, stock prices reached the peak and the trough about two quarters ahead of investment and output. Third, the bond price moves oppositely to stock prices.

An intuitive explanation for these regular patterns is that shocks to the asset market are an important cause of the business cycle instead of a mere response to it. A popular hypothesis along this line is as follows. Sudden drops in the asset market liquidity cause equity price to fall and the price of liquid assets to rise. In a world where firms face financing constraints on investment, this fall in equity price reduces the funds for investment that a firm can raise by issuing equity and/or using equity as collateral in borrowing. Thus, investment falls, output falls and a recession starts. Note that liquidity shocks in this hypothesis are not necessarily related to the fundamentals of the economy. Let me refer to this hypothesis as the liquidity shock hypothesis.

The liquidity shock hypothesis has received wide attention because of its immediate policy implication. If unexpected fluctuations in equity liquidity are a cause of the business cycle, then a government can attenuate the business cycle by making the supply of liquid assets countercyclical. At the onset of a recession, a government can use liquid assets to buy up some of the illiquid equity in order to prevent equity price from falling precipitously. The increase in the supply of liquid assets relaxes firms’ financing constraints, and the stabilization of equity

---

1The stock price index is the Wilshire 5000 price full cap index (Wilshire Associates Incorporated, also available at the Federal Reserve Data Center). This is an index of the market value of all stocks actively traded in the US, weighted by market capitalization. The designation “full cap” signifies a float adjusted market capitalization that includes shares of stocks not considered available to ordinary investors. The data is available on the daily basis, but the series used here is the price of the last trading day in each quarter. Investment is real private nonresidential fixed investment and GDP is the real gross domestic product, both of which are available at the US Department of Commerce: Bureau of Economic Analysis. The bond price index is the price of the three-month Treasury bills at the secondary market rate, available from the Board of Governors of the Federal Reserve System. All variables depicted here are quarterly and filtered through the Hodrick-Prescott filter with a parameter 1600. I have multiplied the deviation of investment from its trend by 2, the deviation of GDP by 10, and the deviation of the bond price by 50.
price further improves firms’ ability to use the equity market to finance investment. This policy implication seems to provide a justification for the large and repeated injections of liquidity by the US Federal Reserve System and some other central banks in the 2008-2009 recession. Such intervention is warranted if fluctuations in market liquidity are caused by non-fundamental forces.

Figure 1.1. Deviations of stock price and investment from trend (%)

Figure 1.2. Deviations of stock price and GDP from trend (%)

Figure 1.3. Deviations of stock price and bond price from trend (%)
Given the intuitive appeal and the immediate policy implication of the liquidity shock hypothesis, it is important to evaluate the hypothesis formally and clearly. For concreteness, I focus on the version of the hypothesis modeled by Kiyotaki and Moore (2008, KM, henceforth), who place two equity-market frictions at the center. One is the difficulty to issue new equity: a firm can issue new equity on at most a fraction $\theta \in (0, 1)$ of investment. Another friction is the lack of resaleability of existing equity; that is, only a fraction $\phi \in (0, 1)$ of existing equity can be resold in any given period. Modeling a negative liquidity shock as an unexpected drop in equity resaleability $\phi$, they argue that the shock has large and persistent effects on macro variables.

I first construct a model to simplify the formulation of the liquidity shock hypothesis. The KM model formulates an individual’s decision problem as a sequence problem instead of dynamic programming. This formulation makes it difficult to illustrate how the liquidity shock works in a stochastic and dynamic environment. Another issue is aggregation. To emphasize the financing constraint, the model distinguishes entrepreneurs, who undertake investment, from other individuals called workers. To aggregate the decisions of different types of individuals, the model requires a special utility function (logarithmic) and, even in that case, aggregation is quite involved. While retaining the two equity market frictions in KM, I formulate households’ decisions with dynamic programming and determine the recursive competitive equilibrium. My model provides straightforward aggregation with general specifications of preferences.

Then I evaluate the liquidity shock hypothesis quantitatively. After calibrating the model and computing the equilibrium, I find that a strong and persistent negative liquidity shock generates large and persistent reductions in aggregate investment, employment and output. However, contrary to the conjecture by KM, these negative changes in macro quantities are not associated with a drop in equity price. Instead, a negative liquidity shock generates an asset price boom, which is opposite to what is observed during recessions.

This counterfactual response of equity price to liquidity shocks is not unique to the particular model here or the KM model. Rather, it is a robust and common feature of many models where equity is important either as a direct means of financing investment or as collateral for borrowing in investment finance. The culprit of this counterfactual response is the effect that a negative liquidity shock tightens a firm’s financing constraint, which is intuitively what the shock is supposed to do. The transparent formulation of the model enables me to clearly explain the puzzling response of equity price and demonstrate its robustness to a wide range of extensions and modifications of the model. Thus, contrary to the liquidity shock hypothesis, this finding indicates that liquidity shocks in the equity market alone are unlikely to be the primary cause of the business cycle.
Finally, I discuss possible resolutions to the puzzle, all of which rely on direct or induced changes in effective productivity to accompany the liquidity shock.

Other authors have independently discovered the puzzling response of equity price to liquidity shocks. Nezafat and Slavik (2010) show that a negative shock to $\theta$ increases equity price, and Ajello (2010) shows that a negative shock to $\phi$ increases equity price. However, these authors do not focus on the puzzling response of equity price. Instead, Nezafat and Slavik (2010) focus on the importance of shocks to $\theta$ in explaining the volatility of asset prices, and Ajello (2010) on the importance of shocks to the intermediation cost in explaining the volatility of investment and output. Another closely related paper is Del Negro et al. (2011), who quantitatively evaluate the non-standard monetary policy intervention in the 2008-2009 recession. Both Ajello (2010) and Del Negro et al. (2011) incorporate a range of elements into KM, such as wage/price rigidity, adjustment costs in investment and habit persistence in consumption. These elements are intended to be realistic for addressing the issues in the two papers, but they cloud the picture of how liquidity shocks affect equity price. I simplify the KM model rather than complicate it. As said above, the simplified model is easy for aggregation and for exploring the robustness of the response of equity price to liquidity shocks. In particular, I illustrate why adding the aforementioned elements to the model is unlikely to overturn the counterfactual response of equity price to liquidity shocks. The tractable formulation in my model should also be useful broadly for studying the role of the asset market in macro.\footnote{After I completed the first draft of this paper and communicated with Del Negro et al., they revised their paper to adopt the construct of large households from my model to simplify aggregation. Also, some parts of the current paper are summarized in Shi (2011), where the focus is on the steady state.}

More broadly, financial frictions have been the focus of business cycle research for quite some time. The literature is too large to be surveyed here (see Bernanke et al., 1999, for a partial survey). One approach emphasizes the role of financial intermediaries in economizing on the cost of lending to and monitoring entrepreneurs who have private information on their projects’ outcome (see Townsend, 1979). Williamson (1987) seems the first to use this approach to study the business cycle, and Bernanke and Gertler (1989, 1990) construct popular models along this line. The main mechanism in this approach is that net worth of entrepreneurs and/or financial intermediaries is pro-cyclical, which generates the financial multiplier. A related approach emphasizes a borrower’s assets as collateral in securing debt when there is limited enforceability on debt repayment. A notable example is Kiyotaki and Moore (1997) who study how this interaction between the asset value and debt propagates the business cycle. More recent examples are Jermann and Quadrini (2009) and Liu et al. (2011). To focus on equity market frictions, I will abstract from financial intermediaries and debt finance in the main part of the analysis, as KM
did. However, I will show that the main result is robust to the introduction of debt finance and collateral in subsection 4.2 and relate to the literature further in section 5.

2. A Macro Model with Asset Market Frictions

2.1. The model environment

Consider an infinite-horizon economy with discrete time. The economy is populated by a continuum of households, with measure one. Each household has a unit measure of members. At the beginning of each period, all members of a household are identical and share the household’s assets. During the period, the members are separated from each other and each member receives a shock that determines the role of the member in the period. With probability \( \pi \in (0,1) \), a member will be an entrepreneur and, with probability \( 1 - \pi \), the member will be a worker in the period. These shocks are iid across the members and time. An entrepreneur has an investment project and no labor endowment, while a worker has one unit of labor endowment and no investment project. The members’ preferences are aggregated and represented by the following utility function of the household:

\[
\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{ \pi \ u(c^e_t) + (1 - \pi) [U(c^w_t) - h(\ell_t)] \}, \quad \beta \in (0,1).
\]

Here, the expectation is taken over aggregate shocks \((A, \phi)\) which will be described below. The variable \( c^e_t \) is an entrepreneur’s consumption, \( c^w_t \) a worker’s consumption, and \( \ell_t \) a worker’s labor supply. The functions \( u, U \) and \( h \) are assumed to have standard properties. The household maximizes the above utility function by choosing the actions for the members, and the members implement these choices. In the presence of ex post heterogeneity among the individuals, this household structure facilitates aggregation.\(^3\)

Let me describe the technologies in the economy together with the timing of events in an arbitrary period \( t \). The time subscript \( t \) is suppressed and the variables in period \( t \pm j \) are given subscripts \( \pm j \). A period is divided into four stages: households’ decisions, production, investment, and consumption. In the stage of households’ decisions, all members of a household are together to pool their assets. Aggregate shocks \((A, \phi)\) are realized.\(^4\) The household holds (physical) capital \( k \), equity claims \( s \), and liquid assets \( b \). Capital resides in the household and will be rented later to firms that produce consumption goods in the second stage. On every unit of capital there is a claim which is either sold to the outsiders or retained by the household. Thus,

\(^3\)A similar household structure has been used in monetary theory by Shi (1997).

\(^4\)This timing of aggregate shocks simplifies the analysis. If \( A \) and \( \phi \) are realized in the second stage, instead, there may be precautionary holdings of assets.
a household holds a diversified portfolio of equity claims on other households’ capital as well as retained claims on the household’s own capital. Liquid assets are government bonds. Because all members of the household are identical in this stage, the household evenly divides the assets among the members. The household also gives each member the instructions on the choices in the period contingent on whether the member will be an entrepreneur or a worker in the second stage. For an entrepreneur, the household instructs him to consume an amount $c^e$, invest $i$, and hold a portfolio of equity and liquid assets $(s_{+1}^e, b_{+1}^e)$ at the end of the period. For a worker, the household instructs him to consume an amount $c^w$, supply labor $\ell$, and hold a portfolio $(s_{+1}^w, b_{+1}^w)$ at the end of the period. After receiving these instructions, the members go to the market and will remain separated from each other until the beginning of the next period.

At the beginning of the production stage, each member receives the shock whose realization determines whether the individual is an entrepreneur or a worker. Competitive firms rent capital from the households and hire labor from workers to produce consumption goods according to $y = AF(k^d, l^d)$, where the superscript $d$ indicates the demand. The function $F$ has diminishing marginal productivity of each factor and constant returns to scale. Total factor productivity $A$ follows a Markov process. After production, a worker receives wage income, and an individual who holds equity claims receives the rental income of capital. Then, a fraction $(1 - \sigma)$ of existing capital depreciates, where $\sigma \in (0, 1)$, and every existing equity claim is rescaled by a factor $\sigma$.

The third stage in the period is the investment stage where entrepreneurs undertake their investment projects. To simplify, I assume that all investment projects are identical and each project can transform any amount $i \geq 0$ units of consumption goods into $i$ units of new capital that will be added to next period’s capital stock. In this stage, the asset market and the goods market are open. Individuals trade assets to finance new investments and to achieve the portfolio of equity and liquid assets instructed earlier by their households.

In the final stage of the period, a worker consumes $c^w$ and an entrepreneur consumes $c^e$. Then, individuals return to their households, arriving at the beginning of the next period.

There are two frictions in the equity market, as emphasized by KM. The first is a constraint on new equity: an entrepreneur can issue equity in the market on only a fraction $\theta \in (0, 1)$ of investment. The remaining fraction is equity retained by the entrepreneur’s household. The second friction is a constraint on re-selling equity: an individual can sell only a fraction $\phi \in (0, 1)$ of existing equity in his portfolio. One may be able to explicitly specify the impediments in the

---

5 As in KM, I assume that the claims on the household’s own capital and other households’ capital have the same liquidity, and so they have the same price. This assumption simplifies the analysis.

6 I will discuss this assumption on timing at the end of subsection 3.3.
asset market that generate these bounds, $\theta$ and $\phi$. As a first pass, however, I take $\theta$ and $\phi$ as exogenous as KM did. Let me refer to $\theta$ as the equity-issuing bound and $\phi$ as equity resaleability. Also following KM, I focus on equity resaleability by assuming that $\theta$ is constant and $\phi$ follows a Markov process. Shocks to $\phi$ are interpreted as shocks to equity liquidity.

The asset market frictions amount to putting a lower bound on the amount of equity that an entrepreneur must hold at the end of the period. Because of the equity-issuing bound, an entrepreneur must retain $(1 - \theta)i$ claims on the new capital formed by his investment $i$. After capital depreciates, the entrepreneur also has $\sigma s$ claims on existing capital, of which the entrepreneur must hold onto at least the amount $(1 - \phi)\sigma s$. These new and old equities will both become existing equity at the beginning of the next period. Thus, the amount of equity that the entrepreneur holds at the end of the period, $s_{t+1}^e$, must satisfy the following liquidity constraint:

$$s_{t+1}^e \geq (1 - \theta)i + (1 - \phi)\sigma s. \quad (2.1)$$

For (2.1) to be binding, an entrepreneur must face a tight borrowing limit. I set this limit to zero for now and will introduce debt finance and collateral in subsection 4.2. Note that the borrowing constraint is enforced by temporary separation of the members from each other in a period. This separation ensures that a household cannot shift funds from workers to entrepreneurs in the investment stage to circumvent entrepreneurs’ liquidity constraint. This role of temporary separation is similar to that in the literature of limited participation, e.g., Lucas (1990).

Government policies are kept simple. In each period, the government spends $g$ per household, redeems all matured bonds, and issues an amount $B$ of new real bonds per household, where $g$ and $B$ are positive constants. The government collects lump-sum taxes $\tau$ per household to balance the budget in each period. (If $\tau < 0$, they are transfers to the households.) Let $p_b$ be the price of bonds. Then, the government budget constraint is

$$g = \tau + (p_b - 1)B. \quad (2.2)$$

### 2.2. A household’s decisions

In a period, a household chooses $(i, c^e, s_{t+1}^e, b_{t+1}^e)$ for each entrepreneur and $(\ell, c^w, s_{t+1}^w, b_{t+1}^w)$ for each worker. In addition to the liquidity constraint, the household faces a resource constraint on

---

7For example, if new investment differs in quality which is the entrepreneur’s private information, then the entrepreneur may not be able to finance the investment entirely with equity. Also, if investment requires an entrepreneur’s (non-contractible) labor input as well as the input of goods, then moral hazard on labor input may put an upper bound on $\theta$ (see Hart and Moore, 1994). The difficulty in re-selling equity, as captured by $\phi < 1$, may be caused by the lemons problem in the asset market that induces asset prices to fall sharply as the quantity sold increases. Instead of modeling this difficulty with a smoothly decreasing function, I use the two-step function in KM to simplify the analysis.
each member. On an entrepreneur, the resource constraint is:

$$rs + q(i + \sigma s - s^e_{t+1}) + (b - p_0 b^e_{t+1}) - \tau \geq i + e^o,$$  \hspace{1cm} (2.3)

where $r$ is the rental rate of capital and $q$ the price of an equity claim, measured in consumption goods. This constraint is explained as follows. An entrepreneur has three items of expenditure: consumption $c^e$, investment $i$, and the tax liability $\tau$. The entrepreneur must finance these expenditures by three sources of cash flows available to him. The first is the rental income of capital, $rs$. The second is the net receipts from trading equity. After capital depreciates in the period, the entrepreneur holds $s^e_{t+1}$ claims on existing equity. The entrepreneur’s investment creates $i$ units of new capital. There is one claim on each unit of new capital, which is either sold to other households or retained by the entrepreneur for the household. Thus, the entrepreneur’s total holdings of equity claims are $(i + \sigma s)$. Because the entrepreneur holds onto only $s^e_{t+1}$ claims at the end of the period, the remainder is sold to the market. Thus, the entrepreneur’s net receipt from trading equity claims is $q(i + \sigma s - s^e_{t+1})$. Note that the liquidity constraint (2.1) ensures that the receipt from trading equity is strictly positive, which prevents the entrepreneur from going short on equity. The third cash flow available to the entrepreneur is the net receipt from trading liquid assets, $(b - p_0 b^e_{t+1})$, which is the amount obtained from redeeming matured bonds minus the amount spent on new bonds.

It is useful to consolidate the liquidity constraint, (2.1), with an entrepreneur’s resource constraint, (2.3), to eliminate $s^e_{t+1}$. Doing so yields

$$(r + \phi \sigma q)s + (b - p_0 b^e_{t+1}) - \tau \geq e^c + (1 - \theta q)i.$$ \hspace{1cm} (2.4)

This financing constraint reveals two features. First, the resaleability of equity increases an entrepreneur’s ability to finance investment. Second, an entrepreneur’s “downpayment” on each unit of investment is $1 - \theta q$, because the entrepreneur can raise an amount $\theta q$ by issuing equity in the market. The downpayment must be financed by other cash flows such as the rental income of capital and the receipts from selling existing assets. Note that an entrepreneur’s resource constraint (2.3) holds with equality, provided that the entrepreneur’s marginal utility of consumption is strictly positive. Thus, an entrepreneur’s liquidity constraint (2.3) is binding if and only if the consolidated constraint (2.4) is binding. For this reason, I refer to (2.4) as the equity liquidity constraint.

A worker faces a resource constraint similar to (2.3), except that a worker has labor income but no investment project. Let $w$ be the real wage rate. This constraint is:

$$rs + w\ell + q(\sigma s - s^w_{t+1}) + (b - p_0 b^w_{t+1}) - \tau \geq c^w.$$ \hspace{1cm} (2.5)
A worker’s equity holdings at the end of the period should also satisfy the constraint: \( s^w_{+1} \geq (1 - \phi)s \). However, this constraint is not binding because workers are the buyers of the new and existing equity sold by entrepreneurs in the equilibrium.

Denote average consumption per member as \( c \) and the average holdings of the portfolio per member at the end of the period as \((s_{+1}, b_{+1})\). Then,

\[
c = \pi c^e + (1 - \pi)c^w, \tag{2.6}
\]
\[
s_{+1} = \pi s^e_{+1} + (1 - \pi)s^w_{+1}, \quad b_{+1} = \pi b^e_{+1} + (1 - \pi)b^w_{+1}. \tag{2.7}
\]

Multiply (2.3) by \( \pi \) and (2.5) by \( 1 - \pi \). Adding up yields the household’s resource constraint:

\[
(r + \sigma q)s - qs_{+1} + (1 - \pi)w\ell + (q - 1)\pi i + (b - p_b b_{+1}) - \tau \geq c. \tag{2.8}
\]

Now I can formulate a household’s decisions with dynamic programming. The aggregate state of the economy at the beginning of a period is \((K, Z)\), where \( K \) is the stock of capital per household and \( Z = (A, \phi) \) is the realizations of the exogenous shocks to total factor productivity and equity resaleability. I omit the amount of equity per household and the supply of liquid assets from the list of aggregate state variables because the former is equal to \( K \) and the latter is a constant \( B \geq 0 \). Let equity price be \( q(K, Z) \), the price of liquid assets \( p_b(K, Z) \), the rental rate of capital \( r(K, Z) \), and the wage rate \( w(K, Z) \). All prices are expressed in terms of the consumption good, which is the numeraire.\(^8\)

A household’s state variables consist of equity claims, \( s \), and liquid assets, \( b \), in addition to the aggregate state. Denote the household’s value function as \( v(s, b; K, Z) \). The household’s choices in a period are \((i, c^e, s^e_{+1}, b^e_{+1})\) for each entrepreneur, \( \ell \) for each worker, and \((c, s_{+1}, b_{+1})\) for the average quantities per member. Note that I have replaced the choices for each worker, \((c^w, s^w_{+1}, b^w_{+1})\), with the corresponding choices of the quantities per member. Similarly, I can use the household’s resource constraint (2.8) in lieu of a worker’s resource constraint (2.5). The household’s choices \((i, c^e, s^e_{+1}, b^e_{+1}, \ell, c, s_{+1}, b_{+1})\) solve:

\[
v(s, b; K, Z) = \max \{ \pi \ u(c^e) + (1 - \pi) \left[ U(c^w) - h(\ell) \right] + \beta \ E v(s_{+1}, b_{+1}; K_{+1}, Z_{+1}) \} \tag{2.9}
\]

subject to (2.1), (2.4), (2.8), and the following constraints:

\[
i \geq 0, \; c^e \geq 0, \; s^e_{+1} \geq 0, \; b^e_{+1} \geq 0, \tag{2.10}
\]
\[
c^w \geq 0, \; s^w_{+1} \geq 0, \; b^w_{+1} \geq 0, \tag{2.11}
\]

\(^8\)As is standard, the price of equity is the so-called post-dividend price; i.e., it is measured after the rental income of capital is distributed to shareholders.
where \((c^w, s^w_{+1}, b^w_{+1})\) are functions of \((c, s_{+1}, b_{+1})\) and \((c^e, s^e_{+1}, b^e_{+1})\) defined through (2.6) and (2.7). The expectation in the objective function is taken over next period’s aggregate state \((K_{+1}, Z_{+1})\), and I have suppressed the arguments of price functions \(r, w, q\) and \(p_b\) in the constraints. The non-negativity constraints in (2.10) ensure that an entrepreneur cannot borrow, and the constraints in (2.11) are similar constraints on a worker.

Let \(\lambda^e \pi U'(c^w)\) be the Lagrangian multiplier of the equity liquidity constraint, (2.4), where the rescaling by \(\pi U'(c^w)\) simplifies various expressions below. The multiplier \(\lambda^e\) is a measure of liquidity services provided by cash flows, and the equity liquidity constraint binds if and only if \(\lambda^e > 0\). If \(\lambda^e > 0\), then it is optimal for an entrepreneur to hold only the minimum amount of each asset at the end of the period, which is \(s^e_{+1} = (1 - \theta)i + (1 - \phi)\sigma s\) for equity and \(b^e_{+1} = 0\) for liquid assets. Moreover, the optimal choices of \((\ell, c^e, i)\) yield:

\[
\frac{h'(\ell)}{U'(c^w)} = w, \tag{2.12}
\]

\[
u'(c^e) = U'(c^w)(1 + \lambda^e), \tag{2.13}
\]

\[
q - 1 \leq (1 - \theta q)\lambda^e \text{ and } i \geq 0, \tag{2.14}
\]

where the two inequalities in (2.14) hold with complementary slackness.\(^9\) Condition (2.12) is the standard condition for optimal labor supply. Condition (2.13) captures the fact that a marginal unit of the resource is more valuable to an entrepreneur than to a worker if an entrepreneur’s liquidity constraint is binding, in which case the additional value to an entrepreneur is captured by \(\lambda^e U'(c^w)\). The conditions in (2.14) characterize the optimal choice of investment. As explained above, the downpayment on each unit of investment in terms of goods is \(1 - \theta q\), the cost of which in terms of utility is \((1 - \theta q)\lambda^e U'(c^w)\). For the household, a unit of investment increases the resource by \((q - 1)\), the benefit of which in terms of utility is \((q - 1)U'(c^w)\). Investment is zero if the cost exceeds the benefit, and positive if the cost is equal to the benefit. The liquidity constraint is binding, i.e., \(\lambda^e > 0\), if and only if \(1 < q < 1/\theta\). Put differently, equity price exceeds the replacement cost of capital (unity) if and only if the equity liquidity constraint binds.

Finally, the optimality conditions on asset holdings at the end of the period and the envelope conditions on asset holdings together give rise to the asset-pricing equations below:

\[
q = \beta \mathbb{E} \left\{ \frac{U'(c^w_{+1})}{U'(c^w)} \left[ r_{+1} + \sigma q_{+1} + \pi \lambda^e_{+1} (r_{+1} + \phi_{+1} \sigma q_{+1}) \right] \right\}, \tag{2.15}
\]

\[
p_b = \beta \mathbb{E} \left[ \frac{U'(c^w_{+1})}{U'(c^w)} (1 + \pi \lambda^e_{+1}) \right]. \tag{2.16}
\]

\(^9\)The constraints \(c^e \geq 0, c^w \geq 0, b_{+1} \geq 0\) and \(s_{+1} \geq 0\) do not bind. The constraint \(s^e_{+1} \geq 0\) is not binding under (2.1), because the latter constraint imposes a strictly positive lower bound on \(s^e_{+1}\).
These asset-pricing equations incorporate liquidity services provided by the assets as implicit returns. The shadow price \( \lambda_{e+1} \) enters the right-hand sides of both pricing equations because existing equity and liquid assets can both be sold to raise funds for new investment, thereby relaxing the liquidity constraint on an entrepreneur. However, only a fraction \( \phi_{e+1} \) of existing equity can be sold next period while all liquid assets can be sold. Thus, \( \phi_{e+1} \) appears in the pricing equation for equity but not in that for liquid assets.

### 2.3. Definition of a recursive equilibrium

The formulation thus far suggests a straightforward definition of an equilibrium. Let \( K \subset \mathbb{R}_+ \) be a compact set which contains all possible values of \( K \) and \( Z \subset \mathbb{R}_+ \times [0,1] \) a compact set which contains all possible values of \( Z \). Let \( C_1 \) be the set containing all continuous functions that map \( K \times Z \) into \( \mathbb{R}_+ \), \( C_2 \) the set containing all continuous functions that map \( K \times [0,B] \times K \times Z \) into \( \mathbb{R}_+ \) and \( C_3 \) the set containing all continuous functions that map \( K \times [0,B] \times K \times Z \) into \( \mathbb{R} \). A **recursive competitive equilibrium** consists of asset and factor price functions \((\theta, \pi, \beta, \omega)\) belonging in \( C_1 \), a household’s policy functions \((\tau, \xi, \epsilon, \sigma, \phi_{e+1}, \phi_{e+1})\) belonging in \( C_2 \), the value function \( \vartheta \in C_3 \), the demand for factors by final-goods producers, \((\kappa, \ell)\), and the law of motion of the aggregate capital stock that meet the following requirements:

1. **Requirement (i)** Given price functions and the aggregate state, a household’s value and policy functions solve a household’s optimization problem in (2.9);
2. **Requirement (ii)** Given price functions and the aggregate state, factor demands satisfy
   \[ r = AF'(k^d, \ell^d), \]
   and \( w = AF''(k^d, \ell^d) \), where the subscripts of \( F \) indicate partial derivatives;
3. **Requirement (iii)** Given the law of motion of the aggregate state, prices clear the markets:
   - **goods**:
     \[ c(s, d; K, Z) + \pi i(s, d; K, Z) + g = AF(k^d, \ell^d), \]
   - **labor**:
     \[ \ell^d = (1 - \pi)\ell(s, d; K, Z), \]
   - **capital**:
     \[ k^d = K = s, \]
   - **liquid assets**:
     \[ b_{e+1}(s, b; K, Z) = b \equiv B, \]
   - **equity**:
     \[ s_{e+1}(s, b; K, Z) = \sigma s + \pi i(s, b; K, Z); \]
4. **Requirement (iv)** The law of motion of the aggregate capital stock is consistent with the aggregation of individual households’ choices:
   \[ K_{e+1} = \sigma K + \pi i(K, B; K, Z). \]

Requirements (i) and (ii) are self-explanatory. So are the market clearing conditions for goods, labor, capital and liquid assets in requirement (iii). In the capital market clearing condition, the
equality $K = s$ states the fact that there are claims on all capital. In the equity market clearing condition, new equity claims are equal to new investment, $\pi i$, because I define $s$ to include not only equity claims sold in the market but also claims retained by the household. Condition (iv) is also self-explanatory. I impose this requirement explicitly here because it is used by the households to compute the expectations in (2.9). However, because $K = s$, the law of motion of the capital stock duplicates the equity market clearing condition – a reflection of the Walras’ law.

Determining an equilibrium amounts to solving for asset price functions $q(K, Z)$ and $p_b(K, Z)$. Once these functions are determined, other equilibrium functions can be recovered from a household’s first-order conditions, the Bellman equation in (2.9), the market clearing conditions and factor demand conditions. To solve for asset price functions, I can use the right-hand sides of the asset pricing equations, (2.15) and (2.16), to construct a mapping $T$ that maps a pair of functions in $C_1$ back into $C_1$ (see Appendix A). The pair of functions $(q, p_b)$ in an equilibrium is a fixed point of $T$. I will implement this procedure numerically in subsection 3.1.

I relegate the discussion on the value of liquidity and the equity premium to Appendix B and the steady state to Appendix C. For comparative statics of the model, see Shi (2011).

3. Equilibrium Response to Shocks

I calibrate the model and examine how the equilibrium responds to equity liquidity shocks.

3.1. Calibration and computation

For the utility and production functions, I choose the following standard forms:

$$U(c^u) = \frac{(c^u)^{1-\rho} - 1}{1 - \rho}, \quad u(c^c) = u_0 U(c^c),$$

$$h(\ell) = h_0 \ell^\theta, \quad F(K, (1 - \pi)\ell) = K^\alpha \left[(1 - \pi)\ell\right]^{1-\alpha}.$$

For the exogenous state of the economy $(A, \phi)$, I assume that $\log A$ and $-\log(1/\phi - 1)$ obey:

$$\log A_{t+1} = (1 - \delta_A) \log A^* + \delta_A \log A_t + \varepsilon_{A,t+1}, \quad (3.1)$$

$$- \log(1/\phi_{t+1} - 1) = -(1 - \delta_\phi) \log \left( \frac{1}{\phi^*} - 1 \right) + \delta_\phi \log \left( \frac{1}{\phi_t} - 1 \right) + \varepsilon_{\phi,t+1}. \quad (3.2)$$

The superscript * indicates the non-stochastic steady state. These processes ensure $A \geq 0$ and $\phi \in [0, 1]$. The quantitative analysis below will take $\varepsilon_A$ and $\varepsilon_\phi$ as one-time shocks.

I choose the length of a period to be one quarter and calibrate the non-stochastic steady state to the US data. The steady state and the calibration are described in Appendix C. The value
of the discount factor $\beta$ and the relative risk aversion are standard; so are the following targets and parameter values. The elasticity of labor supply is equal to two and aggregate hours of work in the steady state are 0.25. The share of labor income in output is $1 - \alpha = 0.64$, the ratio of annual investment to capital in the steady state is $4(1 - \sigma) = 0.076$, and the ratio of capital to annual output is 3.32. The steady state value of productivity is normalized to $A^* = 1$ and the persistence of productivity is $\delta_A = 0.95$. Government spending $g$ is set to be 18% of the steady state level of output. Note that the parameter $u_0$ is identified by the ratio of capital to output because $u_0$ affects entrepreneur’s consumption which in turn affects the rental rate of capital and the capital stock.

Table 1. Parameters and calibration targets

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
<th>calibration target</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$: discount factor</td>
<td>0.992</td>
<td>exogenously chosen</td>
</tr>
<tr>
<td>$\rho$: relative risk aversion</td>
<td>2</td>
<td>exogenously chosen</td>
</tr>
<tr>
<td>$\pi$: fraction of entrepreneurs</td>
<td>0.06</td>
<td>annual fraction of investing firms = 0.24</td>
</tr>
<tr>
<td>$u_0$: constant in entrep. utility</td>
<td>44.801</td>
<td>capital stock/annual output = 3.32</td>
</tr>
<tr>
<td>$h_0$: constant in labor disutility</td>
<td>17.005</td>
<td>hours of work = 0.25</td>
</tr>
<tr>
<td>$\eta$: curvature in labor disutility</td>
<td>1.5</td>
<td>labor supply elasticity $1/(\eta - 1) = 2$</td>
</tr>
<tr>
<td>$\alpha$: capital share</td>
<td>0.36</td>
<td>labor income share $(1 - \alpha) = 0.64$</td>
</tr>
<tr>
<td>$\sigma$: survival rate of capital</td>
<td>0.981</td>
<td>annual investment/capital = 0.076</td>
</tr>
<tr>
<td>$A^*$: steady-state TFP</td>
<td>1</td>
<td>normalization</td>
</tr>
<tr>
<td>$\delta_A$: persistence in TFP</td>
<td>0.95</td>
<td>persistence in TFP = 0.95</td>
</tr>
<tr>
<td>$B$: stock of liquid assets</td>
<td>2.0204</td>
<td>fraction of liquid assets in portfolio = 0.12</td>
</tr>
<tr>
<td>$\phi^*$: steady-state resaleability</td>
<td>0.276</td>
<td>annual return to liquid assets = 0.02</td>
</tr>
<tr>
<td>$\delta_\phi$: persistence in resaleability</td>
<td>0.9</td>
<td>exogenously chosen</td>
</tr>
<tr>
<td>$\theta$: fraction of new equity</td>
<td>0.276</td>
<td>set to equal to $\phi^*$</td>
</tr>
<tr>
<td>$g$: government spending</td>
<td>0.1928</td>
<td>government spending/GDP = 0.18</td>
</tr>
</tbody>
</table>

Let me discuss the remaining identification restrictions. First, the parameter $\pi$ can be interpreted as the fraction of firms that adjust their capital in a period. The estimate of this fraction on the annual basis ranges from 0.20 (Doms and Dunne, 1998) to 0.40 (Cooper et al., 1999). I choose a value 0.24 in this range, which leads to $\pi = 0.06$. Second, I set $\theta$ to be equal to $\phi^*$ as a benchmark. Third, liquidity shocks must be persistent in order to generate persistent effects. Thus, I set $\delta_\phi = 0.9$ in the baseline calibration. Fourth, the rate of return to liquid assets and the fraction of liquid assets in the total value of assets come from the evidence in Del Negro et al. (2011). These authors report that the annualized net rate of return to the US government liabilities is 1.72% for one-year maturities and 2.57% for ten-year maturities. I choose the value $\delta_\phi = 0.9$.10

10Nezafat and Slavik (2010) use the US Flow of Funds data for non-financial firms to estimate the stochastic process of $\theta$. Interpreting $\theta$ as the ratio of funds raised in the market to fixed investment, they find that the mean of $\theta$ is 0.284. This is close to the value $\theta = 0.276$ that I use here.

13
Finally, Del Negro et al. (2011) use the US Flow of Funds between 1952 and 2008 to compute the share of liquid assets in asset holdings. Their measure of liquid assets consists of all liabilities of the Federal Government, that is, Treasury securities net of holdings by the monetary authority and the budget agency plus reserves, vault cash and currency net of remittances to the Federal Government. The sample average of the share of liquidity assets is close to 0.12, which I target in the calibration.

Let me briefly mention some of the identified values. First, equity resaleability in the steady state is $\phi^* = 0.276$. Because this is significantly less than one, the resale market for equity is far from being liquid. Notice that $\phi^*$ is identified by the target that the annual yield on liquid assets is 0.02. If assets were liquid, then the yield on liquid assets would be $\beta^{-1/4} - 1 = 0.0327$. So, another way to interpret the identified value of $\phi^*$ is that equity resaleability needs to fall from one to 0.276 in order for the steady state to generate a differential of 127 basis points between the annual discount rate and the yield on liquid assets. Second, in the steady state, the rental rate of capital is $r^* = 0.0271$ and the price of equity is $q^* = 1.0367$. It is difficult to map this price of equity to a specific price index of stocks because equity in this model represents a broad collection of assets other than the particular group of liquid assets mentioned above. Third, the annualized equity premium in the steady state is $4(r^*/q^* + \sigma - 1/p_b^*) = 0.0087$. This premium is significant, considering that it is associated with the steady state where no risk is present.

Suppose that the error terms in the processes of $A$ and $\phi$ are zero for all $t > 1$. That is, the paths of $A$ and $\phi$ are all realized at the beginning of $t = 1$, which is the case with one-time shocks. I follow the procedure in Appendix A to compute equilibrium asset price functions $(q, p_b)(K, Z)$, where $Z = (A, \phi)$. Then, I recover an individual household’s policy functions $x(s, b; K, Z)$, where $x$ is any element in the list $(c, i, c^e, s_{+1}^e, \ell, c^{we}, s_{+1}^w, b_{+1})$. Since the equilibrium has $s = K$ and $b = B$, where $B$ is a constant, I shorten the notation $x(K, B; K, Z)$ as $x(K, Z)$.

Most of the policy functions have predictable properties. For example, consumption, investment and output are increasing functions of the capital stock, $K$. An exception might be the dependence of asset prices on the capital stock. For most values of the capital stock, equity price and the price of liquid assets are decreasing functions of the capital stock. On equity price, a plausible explanation is that as the capital stock increases, the rental rate of capital falls which reduces equity price. On the price of liquid assets, a plausible explanation is that as the capital stock increases, the need for further investment falls, which reduces the demand for liquid assets.

---

Note that the pricing equation for liquid assets in the steady state imposes the constraint $\beta^0 \geq \beta$. Thus, given the value of $\beta$, the upper bound on the annual rate of return to liquid assets is $\beta^{-4} - 1 = 0.0327$. Thus, the value chosen for the rate of return to liquid assets is in this feasible region.
3.2. Equilibrium response to an asset liquidity shock

Suppose the economy is in the non-stochastic steady state at time $t = 0$. At the beginning of $t = 1$, there is an unanticipated drop in equity resaleability from $\phi^*$ to $\phi_1 < \phi^*$. After this shock, $\phi$ follows the process in (3.2), with $\varepsilon_{\phi,t} = 0$ for all $t \geq 2$. To focus on this shock, let me assume for the moment that the fraction of new investment that can be financed by issuing equity, $\theta$, does not change. Similarly, total factor productivity, $A$, is assumed to remain at the steady state level. I compute the dynamics of the equilibrium using the procedure described in Appendix C. Note that I use the policy functions and asset pricing functions to compute the dynamics instead of linearizing the equilibrium system. This approach has the advantage of being able to deal with large shocks. I assume that the shock reduces equity resaleability by 46% from 0.276 to 0.146.

Figure 2.1 graphs aggregate investment ($I = \pi \ell$) and equity resaleability, where the vertical axis is percentage deviations of the variables from their steady-state levels and the horizontal axis is the number of quarters after the shock. In period 1, the negative shock to equity resaleability reduces investment by 40%. Although the negative shock to the resaleability is large by construction, the size of the reduction in investment may still be surprising in the following sense. Because $\theta$ is not reduced with $\phi$ in this experiment, entrepreneurs can still issue new equity to finance new investment. The large fall in investment indicates that a majority of new investment is financed by selling existing equity and other cash flows rather than issuing new equity. Figure 2.1 also shows that investment closely follows the dynamics of equity resaleability. Because the liquidity shock is assumed to be persistent, the shock has persistent effects on investment. Three years after the shock, investment is still about 7.9% below the steady state.

Figure 2.2 exhibits the percentage deviations of aggregate employment ($L = (1 - \pi)\ell$) and output ($Y$) from the steady state. Both variables fall by large amounts when the negative shock to equity resaleability hits. Employment falls by 12.4% and output by 8.1% in period 1. Such magnitudes are comparable to the ones in the 2008-2009 recession. Note that the reduction in output in period 1 comes entirely from the reduction in employment, because the capital stock in period 1 is predetermined and total factor productivity is fixed. After period 1, however, the capital stock also falls below the steady state due to lower investment, which keeps output low. The responses of these aggregate variables are persistent. Three years after the shock, employment and output are still 2% below the steady state. These responses of aggregate quantities seem to suggest that liquidity shocks to the equity market can be a potent cause of aggregate fluctuations.
One can question whether shocks to equity resaleability are as persistent as I assumed. Instead of getting into this debate, let me check how asset prices respond to the liquidity shock. As Figure 2.3 shows, a negative liquidity shock to the equity market generates an asset price boom! Immediately after the shock, equity price increases by 13% and the price of liquid assets increases by 5.7%. Asset prices stay above the steady state for quite a long time. Three years after the shock, equity price is still 2.5% above the steady state and the price of liquid assets is 1.2% above the steady state. Thus, the negative liquidity shock to the asset market can generate large and persistent fluctuations in equity price, but the problem is that the response of equity price is opposite to what is conjectured by KM and opposite to the pattern observed in the business cycle (e.g., Figure 1.1).
3.3. What is the cause of this problem?

One suspect is the fixed $\theta$, the fraction of investment that can be financed by issuing new equity. It is likely that $\theta$ falls when $\phi$ falls. I will investigate this scenario later. Another suspect is that the model is too simplistic. By abstracting from many realistic elements, this model might have forced some variables to respond to the liquidity shock in the wrong magnitude or direction, in which case equity price might respond to the liquidity shock in the wrong direction in order to make up for the unrealistic responses in other variables. The following is a partial list of the usual elements that I have omitted:

(i) wage rigidity: the absence of it in my model may imply that output does not fall enough after the negative liquidity shock;

(ii) habit persistence in consumption: the absence of this ingredient may imply that consumption responds to the liquidity shock by too much or in the wrong direction;

(iii) adjustment costs of investment: the absence of these costs may imply that investment may fall by too much immediately after the negative liquidity shock.

These items are related. If the fall in output is sufficiently deficient and the fall in investment is sufficiently large, then consumption must increase after the negative liquidity shock in order to clear the goods market. In fact, after the negative liquidity shock described above, an entrepreneur’s consumption falls, a worker’s consumption increases, and aggregate consumption increases (not depicted here). Given this apparent defect of the model, one may be tempted to put items (i)-(iii) above into the model.\footnote{Subsection 4.1 will show that large adjustment costs in investment can make aggregate consumption fall together with investment, employment and output in response to a negative liquidity shock. In the absence of}
Such an effort will be futile in overturning the response of equity price to the liquidity shock. So will be the effort of allowing $\theta$ to fall together with the shock. To explain, let me examine the condition for optimal investment, (2.14). For the sake of argument, let me focus on the case where investment is positive. In this case, (2.14) becomes:

$$q - 1 = (1 - \theta q)\lambda^e.$$  \hfill (3.3)

Because this equation is central to the argument, let me repeat the meanings of the terms in it. For each unit of capital formed by investment, the price is $q$ and the direct marginal cost is one. So, the benefit of investment is $(q - 1)$. If there were no frictions in the equity market, this term would be the net benefit of investment, in which case investment could be positive and finite in the equilibrium if and only if $q = 1$. When there are frictions in issuing new equity, as modeled by $\theta < 1$, the funds raised by issuing new equity are $\theta q$. The entrepreneur must use other cash flows to finance the remainder of investment, $1 - \theta q$. The cost of this downpayment depends on the tightness of the equity liquidity constraint (2.4), which is measured by the shadow price $\lambda^e$. Thus, $(1 - \theta q)\lambda^e$ is the implicit marginal cost of a unit of investment. Condition (3.3) requires the marginal benefit of investment to be equal to the marginal cost.

The condition above provides a simple explanation for why asset prices increase after a negative liquidity shock. The condition contains only two variables, $q$ and $\lambda^e$. Moreover, the marginal benefit of investment is a strictly increasing function of $q$ and the downpayment on investment a strictly decreasing function of $q$. Thus, for any given $\lambda^e$, the net marginal benefit of investment is strictly increasing in $q$. Intuitively, when there is a negative shock to liquidity, the implicit cost of raising funds to finance the downpayment of investment should increase. That is, the equity liquidity constraint (2.4) should become tighter and its shadow price $\lambda^e$ should increase. The higher $\lambda^e$ raises the implicit marginal cost of investment for any given equity price. To restore the balance between the marginal benefit and cost of investment, equity price must increase. As liquid resources become more scarce, the price of liquid assets, $p_b$, also increases.

This argument is quite general, because it only requires the negative liquidity shock to tighten the liquidity constraint, which is what the shock is supposed to do. At the risk of over-simplifying the problem, let me phrase it in terms of the demand for and the supply of equity. A reduction in equity resaleability reduces the supply of equity. Because the demand for equity is not affected so much by the reduction, the price of equity must increase to clear the equity market. With this generality, the argument can survive a wide range of extensions/modifications of the model and adjustment costs, combined shocks to liquidity and productivity can also generate this positive comovement (see section 5). Ajello (2010) emphasizes the importance of nominal wage rigidity in producing this positive comovement in response to financial shocks when nominal prices are rigid.
the liquidity shock. I discuss some of these variations in the remainder of this section and the
extensions in the next section.

Consider first the possibility that \( \theta \) falls with \( \phi \). This case is quite plausible. Using the micro
data in the US, Covas and den Haan (2011) find that equity finance relative to firm output is
significantly procyclical for almost all but the top 1% of the firms. However, the concurrent fall in
\( \theta \) with \( \phi \) exacerbates the problem in the response of equity price to \( \phi \). To see this, consider (3.3)
again. For any given equity price, a fall in \( \theta \) increases the downpayment needed for each unit
of investment, which increases the implicit marginal cost of investment. To restore the balance
between the marginal benefit and cost of investment, equity price must rise even further after a
negative liquidity shock. It is clear from this explanation that a negative shock to \( \theta \) increases
equity price even if \( \phi \) is fixed.\(^{13}\)

Next, consider the construct of large households used in this model. With this construct,
entrepreneurs and workers in a household pool their assets at the beginning of each period, and
so heterogeneity in asset holdings among individuals created by trade during a period lasts only
for one period. In contrast, the KM model allows this heterogeneity to persist and so their
model admits aggregation only when entrepreneurs’ utility function is logarithmic. However, the
construct of large households cannot be the reason why equity price increases after a fall in equity
liquidity. Specifically, the pooling of assets allows entrepreneurs in the next period to use some
of workers’ assets, which reduces the persistence of the negative liquidity shock on the liquidity
constraint. Without such pooling, the liquidity constraint can be tighter, which requires equity
price to increase by even more in response to a negative liquidity shock.

Another assumption in this model and in KM is that an entrepreneur has an immediate
access to capital income in the period. That is, the income \( r_s \) is available for financing current
investment, as can be seen from (2.3). One may consider the alternative timing according to
which the income \( r_s \) is available only for financing consumption at the end of the period but
not for financing investment in the current period. In this case, a fall in equity resaleability will
tighten the liquidity constraint to a greater extent than it does in the current model and, hence,
will increase equity price by even more.

4. Robustness to Debt Finance and Other Elements

In this section, I explain why the response of equity price to the liquidity shock is robust to the
introduction of debt/collateral and the following elements of a macro model: (i) wage rigidity;

\(^{13}\)This explains a result in Nezafat and Slavik (2010). Setting \( \phi = 1 \) and focusing on the volatility of asset prices,
they find that a negative shock to \( \theta \) increases equity price.
(ii) habit persistence in consumption; and (iii) adjustment costs of investment.

4.1. Introducing additional elements of a macro model

Wage rigidity and habit persistence in consumption do not directly affect the marginal benefit and cost of investment, as it is clear from (3.3). Their indirect effects may tighten the equity liquidity constraint, (2.4), even further and exacerbate the problem in the response of equity price to a liquidity shock. To see this, consider wage rigidity first. When there is a fall in equity liquidity, output is likely to fall by more with rigid wages than with flexible wages. As a result, the rental income of capital will fall by more when wages are rigid. Because capital income is part of the resource that an entrepreneur uses to finance investment, the equity liquidity constraint (2.4) is likely to become tighter, and so equity price is likely to rise by more with rigid wages than with flexible wages. Next consider habit persistence in consumption. When an entrepreneur cannot adjust consumption quickly because of habit persistence, the entrepreneur needs resource not only to finance investment but also to support persistently high consumption. Again, the liquidity constraint (2.4) is likely to be even tighter after a negative liquidity shock, which requires equity price to increase by more than if there is no habit persistence.

In contrast to wage rigidity and habit persistence, adjustment costs in investment directly affect the condition of optimal investment. For concreteness, let me adopt the conventional assumption that entrepreneurs purchase newly installed capital goods from capital-goods producers who are perfectly competitive. Producing and installing $I$ units of new capital costs $[I + I^*\Psi(I/I^*)]$ units of consumption goods, where $I^*$ is steady-state investment and $\Psi$ satisfies $\Psi(1) = 0, \Psi'(1) = 0, \text{and } \Psi'' > 0$.\(^{14}\) Let $p_I$ denote the price of newly installed capital. Then, a capital-goods producer maximizes profit, $p_I I - I - I^*\Psi(I/I^*)$, and the optimal choice of $I$ satisfies $1 + \Psi'(I/I^*) = p_I$. Profit of such a firm is zero in the steady state. Outside the steady state, profit can be non-zero, which is rebated to the household in a lump sum and hence added to the resource side of (2.4), (2.5) and (2.8). Because a unit of investment costs an entrepreneur $p_I$ units of consumption goods, the term $i$ on the right-hand side of an entrepreneur’s resource constraint, (2.3), is replaced with $p_I i$; the term $\pi(q - 1)i$ in the household’s resource constraint (2.8) is replaced with $\pi(q - p_I)i$; and the term $(1 - \theta q)i$ in the consolidated liquidity constraint (2.4) is replaced with $(p_I - \theta q)i$. A household’s optimal choice of investment satisfies:

$$q - (1 + \Psi') = (1 + \Psi' - \theta q)\lambda^e.$$  \(^{4.1}\)

where I have substituted the result $p_I = 1 + \Psi'(I/I^*)$.

\(^{14}\)Del Negro et al. (2010) use a similar specification.
Adjustment costs add two effects on the condition for optimal investment. One is the effect on $\lambda^e$ through the total downpayment, $(p_f - \theta q)i$, and this effect is ambiguous analytically. On the one hand, adjustment costs prevent investment from falling by as much as in the baseline model. This has a positive effect on the total downpayment on investment. One the other hand, the presence of adjustment costs allows the replacement cost of capital ($p_f$) to fall, which reduces the total downpayment. After a negative liquidity shock, the shadow price $\lambda^e$ increases by more than in the baseline model if the total downpayment payment falls by less than in the baseline model. The second effect of adjustment costs is that the marginal cost of adjustment directly enters (4.1). For any given $(q, \lambda^e)$, the marginal benefit of investment is a decreasing function of $i$ and the marginal cost an increasing function of $i$. When investment falls after a negative liquidity shock, the marginal adjustment cost falls, which reduces the replacement cost. Such savings at the margin increase the net marginal benefit of investment for any given $(q, \lambda^e)$ and mitigates the upward pressure on equity price caused by the increase in $\lambda^e$.

Although the overall effect of adjustment costs on (4.1) is analytically ambiguous, the effect is unlikely to overturn the positive response of equity price to a negative liquidity shock. For the marginal savings from reduced investment to be significant, the marginal cost of adjustment, $\Psi'$, must be sufficiently steep, which implies that the reduction in investment must be sufficiently small. This implication may be inconsistent with the observed large reduction in investment at the beginning of a recession (see Figure 1.1). Moreover, for equity price to fall with a negative liquidity shock, the savings from adjustment costs have to be so large that they wipe out the direct tightening effect of the shock on the liquidity constraint. This does not seem plausible.

For a concrete illustration, I set $\Psi(\frac{f^*}{f}) = \frac{1}{\psi} \left| \frac{f^*}{f} - 1 \right|^\psi$ and $\psi = 2$. Figures 3.1 - 3.3 depict the responses of some variables to the negative liquidity shock examined in subsection 3.2. The adjustment cost is sizable in this example. The replacement cost of capital falls by 16% on impact of the shock, and investment falls by less than a half of that in Figure 2.1.\textsuperscript{15} Despite such large savings from a lower cost of capital, the negative liquidity shock tightens the liquidity constraint substantially, as depicted by the large increase in $\lambda^e$ in Figure 3.2, where the percentage change in $\lambda^e$ is rescaled by a factor 1/100. As in the baseline model, equity price and the bond price increase after the shock.

\textsuperscript{15}With the particular function $\Psi$ and the value $\psi = 2$, the percentage deviation of $p_f$ from the steady state is equal to that of $f$. This is why the two lines in Figure 3.1 coincide.
Figure 3.1. Investment and the cost of capital after a negative $\phi$ shock

Figure 3.2. Asset prices and $\lambda_e$ after a negative liquidity shock

Figure 3.3 shows that employment and output fall by smaller amounts than in the baseline model in response to the negative liquidity shock, because investment falls by a much smaller amount when adjustment costs are large. However, the reductions in employment and output are still sizable. Notice that aggregate consumption falls after the negative liquidity shock, in contrast to the rise in the baseline model. The reason is that total expenditure on investment, which is equal to $p_t I$, falls by 32% instead of 40% as in the baseline model. With this smaller reduction in investment expenditure, consumption also needs to fall to help accounting for the reduction in GDP. Thus, large adjustment costs are useful for producing the positive comovement between aggregate consumption and other major macro quantities.
4.2. Debt finance and collateral

I have abstracted from debt finance so far. Debt finance can be important for the business cycle, as shown in the literature discussed at the end of the introduction. In particular, Kiyotaki and Moore (1997) have shown that cyclical fluctuations in the asset value can amplify and propagate the business cycle by affecting the value of collateral in borrowing. The question is whether this interaction between the asset value and debt finance can resolve the problem above in the response of equity price to liquidity shocks.

To address this question, suppose that individuals can borrow and lend through a perfectly competitive intermediary. In a period, let \( \delta_e^{t+1} \) be the amount borrowed by an entrepreneur in terms of consumption goods and \( \delta_w^{t+1} \) the amount borrowed by a worker. Define \( d_{t+1} = \pi d_{t+1}^{e} + (1 - \pi) d_{t+1}^{w} \) as the amount of borrowing per member in the household. Such borrowing among the households should be distinguished from the borrowing between the government and the households which is still denoted \( b \). Assume that borrowing is in the form of one-period debt. At the beginning of each period, the household pools all members’ outstanding debts and divide them evenly among the members before they go to the market. During the period, each member repays the outstanding debt allocated to him. The amount of outstanding debt per member in the household at the beginning of the period is \( d \). Let \( R \) be the gross interest rate on debt.\(^{16}\) The net receipt from new borrowing minus the repayment on the outstanding debt is \( (d_{t+1}^{e} - Rd) \) for an entrepreneur, which is added to the resource side of the entrepreneur’s resource constraint, (2.3).

---

\(^{16}\)In the equilibrium, workers are the lenders. Because a worker is indifferent between lending to the intermediary and lending to the government, the gross interest rate of lending to the intermediary must be equal to \( 1/p_0 \). Moreover, because there is perfect competition in intermediation, the borrowing rate must be equal to the lending rate. Thus, \( R = 1/p_0 \) in the equilibrium.
Similarly, the term \((d^w_{t+1} - Rd)\) is added to the resource side of a worker’s resource constraint, (2.5), and the term \((d_{t+1} - Rd)\) is added to the resource side of a household’s resource constraint, (2.8). The liquidity constraint on an entrepreneur, (2.1), still applies.

I specify a borrowing limit that depends on the asset value, as emphasized by Kiyotaki and Moore (1997). In each period, an entrepreneur faces the following constraint:

\[
\xi(\phi)qs^e_{t+1} \geq d^e_{t+1}, \quad \text{where } \xi(\phi) \in [\phi, 1).
\]  

(4.2)

This borrowing constraint can be justified by limited enforceability on debt repayment. That is, because a borrower can renege on the repayment, a lender asks the borrower to put up collateral and is only willing to lend only up to the liquidation value of the collateral. In the above specification, the collateral given by an entrepreneur is equity claims that the entrepreneur retains at the end of the period. The fraction \(1 - \xi\) is the discount in value of the collateral due to liquidation. The lower bound on \(\xi\) is \(\phi\), which is the fraction of equity that can be sold immediately. In general, \(\xi\) can be a function of \(\phi\), and \(\phi \leq \xi(\phi) < 1\).\(^{17}\)

Focusing on the interesting case where an entrepreneur’s liquidity constraint, (2.1), and the borrowing constraint, (4.2), are both binding. Using these binding constraints to substitute \((s^e_{t+1}, b^e_{t+1})\) into an entrepreneur’s resource constraint, I obtain the following consolidated liquidity constraint on an entrepreneur:

\[
[r + \phi \sigma q + (1 - \phi)\xi(\phi)\sigma q] s + (b - p_0 b^e_{t+1}) - Rd - \tau \geq c^e + [1 - \theta q - (1 - \theta)\xi(\phi)q] i.
\]  

(4.3)

This constraint modifies (2.4) in obvious ways. First, there is repayment on the outstanding debt, \(Rd\). Second, the effective downpayment on each unit of investment is reduced from \((1 - \theta q)\) to \([1 - \theta q - (1 - \theta)\xi(\phi)q]\). The additional reduction, \((1 - \theta)\xi(\phi)q\), comes from the role of assets as collateral in borrowing. That is, although a fraction \((1 - \theta)\) of investment cannot be financed by issuing new equity, the capital created by this fraction of investment can be used as collateral to secure the amount of borrowing \((1 - \theta)\xi(\phi)q\), which reduces the cash flow needed for investment. Third, the cash-flow from each claim on existing equity is increased from \(r + \phi \sigma q\) to \([r + \phi \sigma q + (1 - \phi)\xi(\phi)\sigma q]\). Again, the additional amount, \((1 - \phi)\xi(\phi)\sigma q\), comes from the role of assets as collateral in borrowing. That is, the fraction \((1 - \phi)\) of each existing equity claim that cannot be immediately sold to the market can be used as collateral to secure the amount of borrowing, \((1 - \phi)\xi(\phi)\sigma q\).

The outstanding debt in each period is a state variable for a household, and so the household’s value function is modified as \(v(s, b, d; K, Z)\). I can reformulate the household’s decision problem

\(^{17}\)Note that a worker faces a similar constraint, \(\xi(\phi)qs^e_{t+1} \geq d^e_{t+1}\), but this constraint is not binding, because a worker is a lender in the equilibrium.
by adding $d_{+1}$ to the list of choices, where the constraints are (2.1), (2.10), (2.11), (4.3), and the modified resource constraint on the household. Let $\lambda^e \pi U'(e^w)$ be the Lagrangian multiplier of the consolidated liquidity constraint, (4.3). It is straightforward to verify that if optimal investment is positive, then its following first-order condition is:

$$q - 1 = [1 - \theta q - (1 - \theta)\xi(\phi) q] \lambda^e. \tag{4.4}$$

If $\xi$ is independent of $\phi$, then the same argument as before applies. That is, when a negative liquidity shock increases the shadow price of the liquidity constraint, $\lambda^e$, equity price must increase in order to equate the marginal benefit and cost of investment.

Allowing $\xi$ to depend on $\phi$ does not help resolving the problem and may be likely to exacerbate the problem. In fact, Covas and den Haan (2011) document the evidence in the micro data that the ratio of debt finance to firm output is procyclical for an overwhelming majority of firms. This suggests $\xi'(\phi) > 0$ if the liquidity shock is an important cause of the business cycle. Given this dependence of debt finance on the equity market liquidity, a negative liquidity shock reduces the amount that an entrepreneur can borrow under any given equity price, increases the effective downpayment needed for each unit of investment, and tightens the liquidity constraint further. To maintain optimality of investment in this case, equity price must increase further after the negative liquidity shock. Therefore, a negative liquidity shock creates an even larger equity price boom when debt finance is present than when debt finance is absent.

A notable special case of the above model is $\theta = 0$. In this case, an entrepreneur must finance investment entirely with debt and the receipts from selling existing assets, rather than issuing new equity to the market. All new equity is retained by the entrepreneur for the household for the period and it serves only as collateral to secure borrowing. Even in this case, a negative shock to liquidity generates an equity price boom, provided $\xi'(\phi) \geq 0$.

5. Some Solutions to the Problem

For equity price to fall after a negative liquidity shock, as it often does during recessions, the liquidity constraint must become less tight. To generate this paradoxical outcome, there must be other changes concurrent with the liquidity shock that sufficiently reduce the need for investment. In this section I discuss some candidates of these concurrent changes and relate the analysis to other attempts in the literature. These concurrent changes may even be the cause, rather than the consequence, of the change in equity liquidity.

One obvious candidate is a negative shock to $\pi$ – the fraction of individuals who have invest-
ment projects in a period. But a reasonable cause of a reduction in $\pi$ is a negative shock to productivity. Another candidate is a fall in the quality of capital. If market participants perceive the quality of capital to deteriorate quickly at the onset of a recession, they will move resources from equity to relative safe and liquid assets. This will depress equity price and drive up the price of liquid assets. To model this process, let me assume that the effective capital stock, instead of the raw capital stock, appears in the production function. Let the effective capital stock be $\kappa K$, where $\kappa$ is the quality of capital. In this case, total factor productivity is $A\kappa^\alpha$, and so a negative shock to the quality of capital is similar to a negative shock to productivity $A$.

Consider a negative shock to total factor productivity, $A$. This shock reduces investment by reducing the marginal productivity of capital. If the shock is sufficiently persistent, then a household will also scale down consumption. These reductions in investment and consumption reduce an entrepreneur’s expenditure, given by the right-hand side of the liquidity constraint (2.4). However, the negative shock to productivity may also reduce the rental income of capital, $rs$, which appears on the left-hand side of (2.4). If the reductions in investment and consumption dominate the reduction in the rental income, then the equity liquidity constraint becomes less tight and, by (3.3), equity price falls.

As an illustration of this possibility, I return to the baseline model without adjustment costs or debt finance and compute the response of the equilibrium to simultaneous shocks to $\phi$ and $A$. Suppose that the economy is in the steady state before $t = 1$ and, at the beginning of $t = 1$, there are unanticipated reductions in $\phi$ about 17% and in $A$ about 5%. After these shocks, $A$ and $\phi$ follow the deterministic dynamics of the processes in (3.1) and (3.2), as depicted in Figure 4.1. Figures 4.2 and 4.3 depict the responses of aggregate quantities and asset prices to the two shocks. As shown in Figure 4.2, the two shocks reduce investment, consumption and output. As mentioned earlier, the negative shock to liquidity alone increases consumption when there are no adjustment costs in investment. In contrast, consumption falls when the liquidity shock is accompanied by a strong negative shock to productivity. Moreover, as shown in Figure 4.3, equity price falls and the bond price increases after the two shocks. Supporting the above intuitive explanation, equity price falls because the negative productivity shock relaxes the liquidity constraint, as captured by the fall in $\lambda^e$. (In Figure 4.1, the percentage change in $\lambda^e$ is rescaled by a factor $1/20$).

These responses are broadly consistent with the patterns in the two recessions depicted in Figures 1.1 - 1.3, although some of the responses do not match in the magnitude. The broad consistency suggests that productivity shocks are important for explaining the cyclical behavior.

---

18 Note that a shock to $\pi$ also affects preferences because $\pi$ and $(1 - \pi)$ are the weights used in a household’s utility function.
of asset prices with other macro variables. On the other hand, if productivity shocks were the only shocks, then prices of both equity and bonds would fall in response to a negative productivity shock (not depicted here). Thus, liquidity shocks seem necessary to generate the opposite cyclical behavior of the two assets’ prices depicted in Figure 1.3.

Figure 4.1. The dynamics of \((A, \phi)\) and the cost of liquidity \(\lambda^c\)

Figure 4.2. Dynamics of \((c, I, Y)\) after negative shocks to \(A\) and \(\phi\)

Figure 4.3. Dynamics of asset prices after negative shocks to \(A\) and \(\phi\)
Let me relate this analysis to some papers that generate a negative response of equity price to negative liquidity shocks. One example is Del Negro (2011), who introduce into KM an interest-rate policy rule and non-standard policy interventions as well as the elements discussed in subsection 4.1. They illustrate that a persistent negative shock to $\phi$ can reduce equity price, aggregate investment and output. Their analysis points the cause to the interaction between nominal rigidities and the expectation of the zero lower bound on the nominal interest rate, but the exact mechanism is not clear. As shown by Ajello (2010) (discussed below), nominal rigidities alone do not generate a negative response of equity price to a negative liquidity shock.

Ajello (2010) incorporates into KM nominal price/wage rigidities and the elements discussed in subsection 4.1. In addition, he makes investment projects heterogeneous in quality among entrepreneurs and the intermediation process costly to channel funds to investment. Despite all these elements, he still finds that a negative shock to $\phi$ increases equity price, which is consistent with the analysis in 4.1. He then illustrates that a shock that increases the cost of intermediation can reduce equity price and other aggregate quantities. By increasing the spread between the rates of return to an entrepreneur and the intermediary, the shock to intermediation affects the amount of funds that can be effectively channeled to investment and the quality distribution of investments that are undertaken in the equilibrium. As such, the intermediation shock acts as a combination of negative shocks to effective productivity and liquidity. It is then conceivable that such a negative shock can reduce equity price. Ajello’s result suggests the need for future research to explicitly model why the cost of intermediation increases during recessions.

Jermann and Quadrini (2009) use a model that differs from KM in at least the following dimensions: (i) investment is undertaken by the same firms that produce consumption goods, rather than by a separate group of individuals called entrepreneurs; (ii) a firm needs working capital for all expenditures in a period, including wage payments, instead of just investment and consumption; and (iii) a firm faces a borrowing limit similar to the one in (4.2) where $\xi$ is related to the equity market liquidity. Despite all these differences, Jermann and Quadrini (2009) find that a negative shock to $\xi$ increases equity price under the baseline calibration. They then illustrate that introducing sizable adjustment costs to investment can make equity price fall after a negative shock to $\xi$. This result is surprising at first glance, given the analysis in the previous section. However, a close inspection reveals that Jermann and Quadrini (2009) have estimated a vector autoregression (VAR) between $\xi$ and total factor productivity. Because the off-diagonal elements in this VAR are non-zero, a negative shock to $\xi$ acts like a combination of shocks to current liquidity and future productivity. It is remarkable that this interaction between $\xi$ and productivity by itself is not strong enough to make equity price fall after a negative shock to $\xi$;
rather, sizable adjustment costs in investment are also needed. Jermann and Quadrini’s result is instructive in revealing the importance of the inter-dependence between asset liquidity and productivity as in the VAR. However, one needs to explain why this inter-dependence arises in order to understand the role of liquidity shocks in the business cycle.

6. Conclusion

In this paper, I have constructed a tractable model to evaluate the hypothesis that shocks to equity market liquidity are an independent cause of the business cycle. After calibrating the model to the US data and computing the dynamic equilibrium, I have found that a large and persistent negative liquidity shock can generate large drops in investment, employment and output. Contrary to the popular hypothesis, however, these large reductions in aggregate quantities are not associated with a fall in equity price; rather, a negative liquidity shock generates an equity price boom. I have explained why this counterfactual response of equity price occurs as long as a negative liquidity shock tightens firms’ financing constraints on investment. Also, I have demonstrated that this response of equity price is a robust feature of many models where equity is important for investment either as a direct means of finance or as collateral for borrowing in investment finance. For equity price to fall as it typically does in a recession, the negative liquidity shock must be accompanied or caused by other shocks that reduce the need for investment sufficiently and relax firms’ financing constraints on investment. I have discussed some candidates of these concurrent shocks, all of which can be traced to changes in factor productivity.

The main message of this analysis is not that shocks to equity market liquidity are not important in general but, rather, that such shocks are unlikely to be the primary driving force of the business cycle. In this paper, equity resaleability is taken exogenously as in many models in this literature. It is plausible that equity resaleability is affected by other components of the economy. For example, negative shocks to productivity or the quality of capital may increase the difficulty to re-sell equity. If this is the case, then fluctuations in equity resaleability can amplify or propagate the business cycle. Another message of this paper is that pro-cyclical fluctuations in liquidity seem useful for generating the observed negative comovement between equity price and the price of liquid assets. Productivity shocks alone would move the two assets’ prices in the same direction.

The strong doubt cast on the role of asset market liquidity in the business cycle provides a caution on policy intervention. Policymakers should first find the causes of the shortfall in liquidity during a recession and then see whether there is need to use policy to correct these causes. If a
shortfall in liquidity is driven by purely non-fundamental events, then supplying liquidity seems a good policy. If a shortfall in liquidity is generated instead by a deterioration in productivity or the quality of investment, simply pumping liquidity into the market seems a bad policy because it merely subsidizes low-quality investment. If a shortfall in liquidity is generated by an increase in the intermediation cost, then the corrective policy should be to subsidize intermediation rather than inject liquidity directly into the market. On the theoretical side, it is important to endogenize asset resaleability and explicitly study how such liquidity interacts with factor productivity. The tractable model in this paper seems useful for these endeavors because it is easy for aggregation.
Appendix

A. The Mapping $T$ on Asset Price Functions

To construct the mapping $T$ on asset price functions, start with arbitrary functions $q, p_b \in C_1$. The following procedure constructs the updated asset price functions $Tq$ and $Tp_b$:

(i) Substitute the factor market clearing conditions (2.18) and (2.19) into requirement (ii) of the equilibrium definition. This step generates $r$ and $w$ as functions of $(\ell, K, Z)$.

(ii) Substitute the functions for $(r, w)$ in step (i) and the conditions (2.18) - (2.21) into the optimality conditions (2.4) - (2.14). Then, solve $(i, c^e, c^w, c, \lambda^e)$ as functions of $(\ell, K, Z)$.

(iii) Substitute the resulting functions in steps (i) and (ii) into (2.17) to solve $\ell$ as a function of $(K, Z)$. With this function $\ell(K, Z)$, the functions solved in steps (i) and (ii) express $(r, w)$ and $(i, c^e, c^w, c, \lambda^e)$ as functions of $(K, Z)$. Similarly, $(r_{+1}, w_{+1})$ and $(\ell, i, c^e, c^w, c, \lambda^e)_{+1}$ can be expressed as functions of $(K_{+1}, Z_{+1})$.

(iv) Substitute the functions obtained in step (iii) and the condition (2.22) into the right-hand sides of the asset pricing equations (2.15) and (2.16). The result is a pair of functions of $(K, Z)$, provided that $Z$ follows a Markov process. These are the updated asset price functions, denoted as $T(q, p_b)(K, Z)$.

It can be verified that for any $q, p_b \in C_1$, $Tq$ and $Tp_b$ are continuous functions of $(K, Z)$ and their values lie in $\mathbb{R}_+$. That is, $T$ maps a pair of elements in $C_1$ back into $C_1$, where $C_1$ is the set containing all continuous functions from $K \times Z$ into $\mathbb{R}_+$. A fixed point of $T$ is a pair of asset price functions in the equilibrium. After obtaining these asset price functions, I can retrieve the policy functions of individuals’ decisions and other variables.

B. Value of Liquidity and the Equity Premium

Liquidity has a positive value in this model only if the liquidity constraint, (2.4), is binding. Precisely, the value of liquidity in the aggregate state $(K, Z)$ can be measured as $\pi \lambda^e(K, Z)$ units of consumption, where $\lambda^e \pi U'(c^w)$ is the shadow price of (2.4). Note that the price of liquid assets in a standard model without the liquidity constraint is $\beta E_U U'(c^e) / U(c^w)$. According to (2.16), liquid assets have a higher price in the current model than the standard one if and only if liquidity is expected to have a positive value in the next period.

While a liquid asset provides full liquidity, equity provides only partial liquidity. By assumption, the return to existing equity can be used to finance new investment. So can the fraction $\phi$ of the equity itself. Thus, an intuitive measure of the value of liquidity provided by a unit of
existing equity is \((\pi \lambda^e)^{r+\phi \sigma q/r+\sigma q}\). The value of the additional liquidity generated by liquid assets relative to existing equity is:

\[
\Delta \equiv \pi \lambda^e \left[ 1 - \frac{r + \phi \sigma q}{r + \sigma q} \right] = \pi \lambda^e \frac{(1 - \phi) \sigma q}{r + \sigma q}.
\]

Intuitively, this additional value is strictly positive if and only if liquidity has a positive value (i.e., if \(\lambda^e > 0\)) and if existing equity is not fully liquid (i.e., if \(\phi < 1\)).

The measure \(\Delta\) is in terms of consumption units. Another measure is the equity premium, i.e., the additional rate of return to equity that is needed to compensate for the lower liquidity of equity. The gross rate of return to equity is \((\lambda^e + 1 + \phi \sigma)\), and to liquid assets is \(1/p_b\). The realized equity premium in the next period will be \(\left(\frac{r+\sigma q}{q} - \frac{1}{p_b}\right)\). The equity premium is closely related to the measure \(\Delta\). The asset-pricing equations (2.15) and (2.16) imply that the equity premium is strictly positive on average if and only if the value of the additional liquidity provided by liquid assets relative to existing equity is positive on average. Precisely, the following two inequalities are equivalent:

\[
E \left[ \frac{U'(c_{+1}^w)}{U'(c^w)} \left( 1 + \pi \lambda^e \right) \left( \frac{r+\sigma q}{q} - \frac{1}{p_b} \right) \right] > 0
\]

\[
E \left[ \frac{U'(c_{+1}^w)}{U'(c^w)} \left( \frac{r+\sigma q}{q} \right) \Delta_{+1} \right] > 0.
\]

C. Steady State, Calibration, and Computing Dynamics

I characterize the non-stochastic steady state first, which is indicated with the superscript \(*\). In the non-stochastic steady state, the exogenous state is constant at \(Z = Z^* = (A^*, \phi^*)\), and all endogenous variables are constant over time. In the steady state, investment is strictly positive because it is equal to \(i^* = (1 - \sigma)K^*/\pi > 0\). By (2.14), the shadow price of the equity liquidity constraint in the steady state is

\[
\lambda^{e*} = \frac{q^* - 1}{1 - \theta q^*}.
\]

The asset-pricing equations, (2.15) and (2.16), yield the following steady-state relations:

\[
\lambda^{e*} = \frac{(\beta^{-1} - \sigma) q^* - r^*}{\pi (r^* + \phi^* \sigma q^*)},
\]

\[
p_b^* = \beta (1 + \pi \lambda^{e*}).
\]

The equations, (C.1) and (C.2), determine \((\lambda^{e*}, q^*)\). Substituting these solutions into (C.3) yields \(p_b^*\). Other steady-state conditions are:

\[
u'(c^{e*}) = U'(c^{e*}) \frac{(1 - \theta) q^*}{1 - \theta q^*},
\]

32
\[ c^* = AF(K^*, (1 - \pi)\ell^*) - (1 - \sigma)K^* - g, \quad (C.5) \]

\[ p_b^*B = g + c^* - \left[ r^* + \phi^*\sigma q^* - \frac{1 - \sigma}{\pi}(1 - \theta q^*) \right] K^*. \quad (C.6) \]

Equation (C.4) is the steady-state version of the first-order condition of \( c^* \), (2.13), with \( \lambda^* \) being substituted. Equation (C.5) is the steady-state version of the goods-market clearing condition, and (C.6) is the steady-state version of the liquidity constraint, (2.4). Together with \( r^* = AF_1 \) and \( h'(\ell^*) = U'(c^*)AF_2 \), (C.2) - (C.6) solve for \((q^*, p_b^*, c^*, K^*, r^*, \ell^*)\).

Next, I identify the parameters. The values of \( \beta, \rho, A^*, \delta_A, \delta_\phi \) and \( \pi \) are exogenously set and the explanations for these values are given in the text. The parameter \( \eta \) is calculated from the elasticity of labor supply, \( \frac{1}{\eta - 1} = 1 \), and the capital share in output is \( \alpha = 0.36 \). Since aggregate investment in the steady state is \( \pi i^* = (1 - \sigma)K^* \), the ratio of annual investment to capital is \( 4\pi i^*/K^* = 4(1 - \sigma) \). Equating this to the target, 0.076, yields \( \sigma \).

Setting total hours of work to the target yields \( (1 - \pi)\ell^* = 0.25 \). Given \( \pi \), this solves \( \ell^* \). Setting the ratio of capital to annual output in the steady state to the target yields \( K^*/(A^*F^*) = 3.32 \). With the value of \( \alpha \) identified above, this condition solves \( r^* \). Since \( r^* = AF_1 \), I can solve \( K^* \) and recover \( i^* = (1 - \sigma)K^*/\pi \). Also, \( w^* = AF_2 \). These values of \((\ell^*, K^*, i^*, r^*, w^*)\) will be used to identify some parameters below. Since the ratio of government spending to output is \( g/(A^*F^*) = 0.18 \), I can solve \( g \). Setting the annualized net rate of return to liquid assets to the target, I have \( 1/(p_b^*)^4 - 1 = 0.02 \). This solves for \( p_b^* \). Substituting \( p_b^* \) into the asset pricing equations in the steady state, (C.2) and (C.3), using the value of \( r^* \) identified above, and using \( \theta = \phi^* \), I can solve \( \phi^* \), \( \theta \) and \( q^* \). Using the target on the share of liquid assets, I have \( p_b^*B/(p_b^*B + q^*K^*) = 0.12 \). Because \( p_b^*, q^* \) and \( K^* \) are all solved by now, this condition solves \( B \).

Two parameters are still to be solved, \((u_0, h_0)\). To solve them, I substitute the values of \((r^*, q^*, \phi^*, K^*, i^*, B, g)\) into the steady-state version of (2.4) to solve \( c^* \). The goods-market clearing condition yields \( c^* = AF^* - \pi i^* - g \). Then, the definition of \( c \) yields \( c^{aw} = (c^* - \pi c^*)/(1 - \pi) \). Substituting the values of \( (c^{aw}, \ell^*, w^*) \) into the steady-state version of (2.12), I obtain the value of \( h_0 \). Note that \( i^* > 0 \). Combining the steady-state versions of (2.13) and (2.14) to eliminate \( \lambda^* \), I obtain an equation in which \( u_0 \) is the only item still to be solved. Solve this equation for \( u_0 \).

Finally, I describe how to compute the dynamics of the equilibrium after some shocks. Suppose that there are shocks to \( A \) or \( \phi \) or both. Let me focus on various cases where the new paths of \( A \) and \( \phi \) are completely known after the shocks are realized at the beginning of time \( t = 1 \), because only such cases are examined in the text. An example is a one-time shock to \( A \) or \( \phi \) or both that occurs at \( t = 1 \), after which \( A \) and \( \phi \) follow (3.1) and (3.2) with the error terms being zero for
all $t \geq 2$. Another example is a change to the path of $A$ or $\phi$ or both that becomes known at $t = 1$. With such shocks, the entire path of $Z = (A, \phi)$ becomes known immediately after the beginning of $t = 1$. Given this path of $Z$, I can use the equilibrium asset price functions and the policy functions to compute the paths of equilibrium variables. Specifically, at $t = 1$, asset prices are $q_1 = q(K_1, Z_1)$ and $p_{b,1} = p_b(K_1, Z_1)$, where $K_1$ is predetermined and $Z_1$ is known. An individual household’s optimal decisions in period 1 are given by $x_1 = x(K_1, B; K_1, Z_1)$, where $x$ is the policy function for any variable in the list $(c, i, c^e, s^e_{+1}, \ell, c^w, s_{+1}, b_{+1})$. Similarly, I can compute factor prices $(w_1, r_1)$ and aggregate variables in period 1. Using (2.22), I can obtain $K_2$. With $(K_2, Z_2)$, I repeat the process to obtain all equilibrium variables in period 2. Continuing this process yields the dynamic path of the equilibrium after the shocks.
References


