Rent Extraction with Rich Type Spaces*

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We characterize Revenue Maximizing Mechanisms in auction settings with ‘rich’ type spaces - where bidders obtain information from sources other than their own valuation. The focus of the paper is on the concept of weak Bayes-Nash implementation. By considering a relaxed problem, we provide an upper bound on Revenue Extraction that explicitly builds upon the richness of the information structure. We provide a condition under which this upper bound is attained, and also characterize a mechanism that does it. Whenever the optimal revenue is characterized, we show that it can be achieved with dominant strategy implementation. The characterization of Optimal Revenue reduces to the full surplus extraction result of Cremer and McLean (1985), when reduced to a standard setting.

Keywords: Surplus Extraction; Information Rents; Mechanism Design; Private Information; Correlated Information; Partial Implementation

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1 Introduction

In most of the literature on Mechanism Design, assumptions about beliefs of agents are of crucial importance to the characterization of incentive compatibility and individual rationality, and therefore important in determining the set of feasible allocations. In the classic literature (as in Myerson [1981]), the usual assumptions regarding uncertainty are that there is a given prior joint distribution over the payoff relevant variables of all players. This automatically pins down the interim beliefs of each agent, given each possible payoff type he might draw. Under the extreme case of iid shocks, for example, every agent has interim belief that other agents types are distributed according to their given prior and that their conditional belief on all other agents’ types is also given by the prior distribution.

The most basic concept of implementation considered is that of Bayes-Nash, which explores the conditional distribution that each player has regarding his opponents’ payoff types. In a quasi-linear environment, Cremer and McLean [1988] (henceforth CM) showed that it is generically the case that incentive compatibility is not a relevant property of allocations if one wants to use Bayes-Nash implementation as the relevant concept. If a certain linear independence condition for the prior holds, then for any direct mechanism, there is an alternative equivalent mechanism that is incentive compatible. As surprising as it sounds, this results suggests the irrelevance of private information for mechanism design under the standard assumptions. This can be interpreted as an inadequacy of the Bayes-Nash implementation concept or of the underlying assumptions regarding the distribution of private information.

One important alternative to the Bayes-Nash implementation is the stronger concept of ex-post implementation, which does not depend on assumptions regarding interim beliefs of agents. More recently, Bergemann and Morris [2005] have presented an argument for the adoption of such a concept from primitives of the model. They have showed that ex-post implementation is equivalent to interim implementation (considering interim beliefs) if we consider all possible belief systems for the uncertainty of the model (including non-common prior beliefs).

Our objective in this paper is to consider a richer structure for the private information in the model, while maintaining the same implementation criterion as the standard literature. In the standard auction model, the only piece of information that agents receive is their own valuation for the good to be traded. In light of this information, a given agent forms his

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1 So on and so forth...
2 That has the same allocation rule and generates the same conditional revenue from each agent.
interim belief regarding his opponents' valuations. Under these assumptions, two essential aspects of private information are tied together, namely, the knowledge of one's own payoff relevant variable (in this case, valuation) and information about opponents' types, which is summarized in the interim beliefs. Because of this 'smallness' property we say that the set of possible types constitutes a naive type space. This characteristic of the model is known to be critical to the result of CM. In here we will consider the more general case of 'rich' type spaces, in which information regarding beliefs and payoff relevant variables are not necessarily tied together.

A simple example is in order for clarification of what is meant by a 'rich' type space. In the specific case of auctions, it means that types might be different from valuation. Consider the scenario each bidder receives some signal about his opponents' types and after that observes his own valuation (and assume that his valuation reveals no additional information regarding his opponents). In this case an agent’s type will consist of the signal he has observed (that determines his interim beliefs regarding other players) plus his realized valuation. In this situation, beliefs and valuation are separate parts of an individual type and are not necessarily connected in any way. More generally, we say that a type space is rich whenever the payoff relevant variable realization is not the only source of information when updating beliefs in the interim stage.

Our objective in this paper is to characterize the set of feasible mechanisms in the presence of 'rich' type spaces while maintaining the Common Prior Assumption (CPA). More specifically, we want to say something about what optimal mechanisms look like in such a setting. In other words, we will consider the problem of Partial Bayes-Nash implementation in this setting.

We will maintain CPA basically for three reasons. First, it is a question of interest to check the robustness of the classical mechanism design results only with respect to the strong belief assumptions implicitly assumed in the classical models, while keeping the other parts of the standard environment unchanged. Second, in the absence of CPA the problem of optimal mechanisms might be ill-posed. For example, consider an auction setting in which a given agent has (for sure) some interim beliefs regarding other agents’ types that is different from the prior distribution maintained by the auctioneer. In this case there would not be an optimal mechanism since there is a money pump available to the auctioneer (from an ex-ante perspective) by exploring arbitrarily their differences in beliefs. Finally, for many situations we believe that CPA is a reasonable assumption. We see it as basically a consistency requirement of beliefs across agents, as different beliefs can be generated solely
by the observation of different information.

We have two main results. First, we characterize explicitly one upper bound for the optimal revenue in any auction setting with a rich type space. Second, we present an assumption under which the upper bound is achieved, and therefore is the optimal revenue. The restriction is a linear independence condition on the set of possible interim beliefs. In case the upper bound is achieved, we have automatically characterized an optimal mechanism.

The main result from standard auction theory with correlated types, considering Bayes implementation with naive type spaces, is that full rent extraction is generically possible. What this means is that we are able to implement the first best product allocation and leave no rents to the bidders while respecting the incentive constraints.

This result generally breaks down in the case of rich type spaces: a given bidder has the option of misreporting to have a lower valuation for the good, while correctly reporting his interim belief regarding his opponents’. By doing that he attains a utility level strictly above the one attained if the realized outcome was indeed the one reported, which is above zero. This implies that the bidders will have positive information rents, which reduce the revenue extracted in the mechanism.

The core of the argument presented is: even if interim beliefs are always truthfully announced, residual private information in valuation generates information rents. Accordingly, we start by considering a relaxed problem, in which we assume that bidders are only able to misreport their valuations, but always truthfully announce their interim beliefs. We characterize the optimal revenue in this problem and also one mechanism that achieves this revenue. Given that this is a relaxed problem, the characterized revenue is an genuine upper bound on the optimal revenue on the original problem.

The idea behind the optimal mechanism in the relaxed problem is to break the original problem, in which there are correlated types, into a series of smaller problems with independent types. Once a specific profile of interim beliefs is realized (and that is, for now, assumed to be truthfully revealed to the mechanism designer), the remaining uncertainty in valuations is independent across bidders. The optimal allocation in this situation is the same as in the standard optimal auction from the independent-types case. The optimal revenue is the average revenue from each of the possible independent-types auctions, where we are integrating over the possible profiles of interim beliefs.

Next we ask in which situation the upper bound characterized can be achieved. The relaxed problem only deals with payoff relevant uncertainty; it treats private information

\footnote{Assuming all types discussed receive the good, in interim expected terms, with positive probability.}
with respect to interim beliefs as irrelevant and the residual private information on valuation as the relevant one. It seems natural that the situations where the relaxed and the original problem coincide are the ones in which we can argue that private information regarding beliefs is irrelevant.

The assumption that we identify is that the set of possible interim beliefs satisfies a certain linear independence condition, that generalizes the full rank condition of CM in the case of naive type spaces. This condition allows us to adapt CM’s logic to our setting: we are able to construct a set of bets, one for a given agent and interim belief, that induce each agent to tell the truth regarding his interim belief (by looking at which bet they would choose) and do not affect average payments.

For clarification consider the following example. Suppose that there are two bidders: 1 and 2. Bidder 2 might have a low or high valuation. Bidder 1 knows his own valuation and has two possible beliefs for 2’s valuation: he might think that the low valuation is more likely (pessimistic) or that the high valuation is more likely (optimistic). In this case, whenever bidder 1 claims to be optimistic he will receive a bet that pays a positive amount when 2 has a high type and pays a negative amount when 2 has a low type. This bet will pay zero on average to an "optimistic" bidder 1, but would pay a negative amount to a "pessimistic" bidder 1.

This reasoning is an adaptation of CM’s argument, but in the case of rich type spaces, it only applies to part of the private information held by agents, namely, interim beliefs. The other portion of the private information will be revealed in general with some rents left to the bidders, and it is dealt with in the relaxed problem.

We also present some examples to better understand when the (LI) assumption is satisfied. Finally, we will discuss what happens when this condition fails, providing a specific example in which the upper bound cannot be achieved.

The rest of the paper proceeds as follows. This introduction is continued by a brief discussion of closely related papers. Section 2 presents the basic framework to be analyzed; Section 3 characterizes the Optimal Mechanism and Optimal Revenue in a relaxed problem. Section 4 uses the results from the relaxed problem to characterize an upper bound on Optimal Revenue in the original problem, states a sufficient condition under which this bound can be achieved and talks about dominant implementation. Section 5 provides an example. Section 6 discusses the importance of one assumption for which we have characterized the Optimal Revenue and presents an example to illustrate what happens when it fails. Finally,

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4The bet will depend on his opponents’ types.
1.1 Related Literature

Chung and Ely (2007) in a closely related paper, consider the problem of designing a revenue maximizing auction by an auctioneer with a given prior over valuations for the good. Since all the relevant type spaces can be embedded in the universal type space, they consider all possible direct revelation mechanisms on this type space. They consider Bayes-Nash implementation on this enlarged type space and, for any given mechanism, the expected revenue it would generate for any distribution over the universal type space that has the initial prior over valuations as a marginal distribution (we shall call it admissible). The auctioneer is assumed to have the most extreme uncertainty aversion with respect to this unknown structure and considers the worst expected revenue that can be generated by a given mechanism. First notice that any given dominant strategy mechanism is always Bayes-Incentive compatible when embedded in the set of more complex mechanisms on the universal type space, and that it generates the same expected revenue for all admissible distributions (since the revenue only depends on the payoff type profile). They show that there is one specific admissible distribution such that any Bayes-Nash implementable mechanism generates lower revenue than the optimal dominant strategy mechanism. Since they consider an ambiguity aversion-like preference for the auctioneer, this is enough to say that the best Dominant Strategy auction is the preferred one by the auctioneer. In this setting he would not like to explore extra information he might get regarding beliefs of the agents.

Our question is different from the above one basically because we will consider a given type space and a given distribution over it, which will imply a fixed distribution over the universal type space as well. We think that the relevance of non-trivial assumptions regarding how the belief system is connected to the payoff relevant variables in the model goes almost without saying, but it does not mean that it is unreasonable for the mechanism designer (in this case, the auctioneer) to entertain a fixed hypothesis regarding how the uncertain aspects in the model behave.

Neeman (2004) considers basically the same setting, applied to a public good problem. He considers a given (rich) type space and maintains CPA. The paper shows that if beliefs do not pin down the payoff relevant variable for each agent in a strong sense\(^5\), then (Bayes-Nash) incentive compatibility, individual rationality and (ex-ante) bud-

\(^5\)It is needed that for each agent, conditional any specific belief, the probability of a valuation below the cost of the public good per agent is bounded away from zero. In fact, in the paper the author assumes the
get balance imply that the public good is never produced even though it becomes socially desirable with probability approaching one. Therefore he extends the result of Mailath and Postlewaite (1990) \cite{Mailath1990} to a setting without naive type spaces.

The basic argument is that conditional on a given profile of beliefs for all agents, the decision of the public good resembles the original problem faced by Mailath and Postlewaite (1990) \cite{Mailath1990} since valuations are independently distributed (we have conditioned on everything that connects different agents’ types) and most agents must have a small effect on the final decision for the public good. As a consequence, even when conditioning only on his own beliefs, most agents must believe (with a high probability) that they have an arbitrarily small effect on the public good decision, and therefore must be paying a very small transfer. Then budget balance implies that if not enough money is raised, then the public good must not be provided.

In this paper, we want to answer the related question of what optimal mechanisms in such a setting (i.e., rich type spaces with a common prior) would look like for a finite number of agents. More specifically, we are focusing on the problem of revenue maximization in an auction setting. A key insight present in Neeman (2004) \cite{Neeman2004} that we will use is that the result of CM extends to this setting by the observation that private information about beliefs can be elicited with no cost. This means that the problem of finding optimal mechanisms can be dealt with as if the auctioneer observed directly what beliefs each agent holds in the interim stage.

Nonetheless, residual uncertainty regarding each agent’s payoff type might still remain, and we have argued that this additional private information is independent across players. This implies the bidders must hold some information rents, and the existence of these might entail inefficiencies in the optimal allocation (as presented, in an extreme example, by Neeman \cite{Neeman2004}).

In their working paper version, Bergemann and Morris \cite{Bergemann2003} characterize what productive allocations are implementable in a quasi-linear environment with rich type spaces, but ignoring the revenue consequences due to the focus on efficiency. They show that a weak version of incentive compatibility (akin to the definition of Payoff incentive compatibility presented below) is sufficient for implementation, a result similar to Lemma \cite{Bergemann2003}. Unlike our exposition, their results do not hinge on a Linear Independence assumption. Since our paper

\footnote{An important assumption is that the unit cost of the public good grows at least linearly with the number of agents.}
focusses on revenue extraction, we build our arguments on a perturbations of transfers that reinforce incentives but do not affect expected revenue. The construction of these requires linear independence.

2 Model

We focus on an auction setting. There are a finite number of agents (bidders) \( i \in I \) that have valuation \( v_i \in V_i = \{ v_1^i, ..., v_K^i \} \subseteq \mathbb{R} \). We also assume that \( v^K_i - v^{k-1}_i = \gamma > 0 \) for any \( k \geq 2 \) and \( i \in I \). There is a single unit of a given good that can be allocated to one of the bidders. We will denote as \( x_i \in [0, 1] \) the probability that the good is allocated to agent \( i \).

Given a valuation \( v_i \), agents have preferences over an allocation decision \( x_i \in [0, 1] \) and transfers \( p_i \in \mathbb{R} \) given by

\[
U(x, p_i; v_i) = x_i v_i - p_i.
\]

The physical structure of the model is standard. As we mentioned in the introduction, we are interested in situations where the possible types that can be realized for a given agent are richer than simply the different possible valuation levels. So it is necessary to introduce formally what is meant by a type space.

A type space is \( (T^i, T^i, \tilde{v}_i, \tilde{\beta}_i)_{i \in I} \), which includes the following objects for each bidder. \( (T^i, T^i) \) is a measurable space and represents the set of possible types. We denote \( T = \times_{i \in I} T^i \) and endow it with product sigma-algebra \( \otimes_{i \in I} T^i \). For any vector \( (x_i)_{i \in I} \), we will use \( x \) to refer to the whole vector and \( x_{-i} \) to refer to \( (x_j)_{j \neq i} \). Finally, for any function \( f : T \rightarrow D \) and any set \( A \subseteq D \) we use the notation \( f^{-1}(A) \equiv \{ t \in T | f(t) \in A \} \).

Each agent observes his own type \( t_i \in T^i \), which contains information regarding his valuation and his beliefs. This means that for each \( i \in I \), there are two random variables \( \tilde{v}_i : T \rightarrow V \) and \( \tilde{\beta}_i : T \rightarrow \Delta (T^i) \) and both are \( t_i \)-measurable, so that we will also write this as functions of \( t_i \) only. For any random variable \( \tilde{x}_i \), we will denote as \([\tilde{x} = x] = \{ t \in T | \tilde{x}(t) = x \} \). We will also denote as \( \tilde{t}_i : T \rightarrow T^i \) the random variable \( \tilde{t}_i(t) = t_i \). Naturally, whenever we have a list of random variables \( (\tilde{x}_i)_{i=1}^I \) we use \( \tilde{x} \) to refer to this ordered list.

We assume that there exists a common prior distribution \( \mathbb{P} \) over this space. The assumption CPA means that beliefs satisfy

\[
\mathbb{P} [\tilde{t}_i = t_i] \tilde{\beta}_i (t_{-i} | t_i) = \mathbb{P} [\tilde{t} = \tilde{t}],
\]

\footnote{This assumption is irrelevant for any of the results, but just simplifies some of the expressions.}
for all $t \in T^t$. This condition is basically the use of Bayes Rule whenever possible (events with zero probability are irrelevant).

Since we will be considering the problem of partial implementation, the revelation principle applies. We define as a direct mechanism a pair of measurable functions $M = (\xi, \pi)$, where $\xi = (\xi_i)_{i \in I}$, $\xi_i : T \rightarrow [0, 1]$ represents the productive decision rule and $\pi = (\pi_i)_{i \in I}$, where $\pi_i : T \rightarrow \mathbb{R}$ represents the conditional payments provided by each agent. It also has to be that case that $\pi_i$ is integrable for each $i \in I$, $\xi_i \geq 0$ for all $i \in I$ and that $\sum_{i \in I} \xi_i \leq 1$ a.e.-$\mathbb{P}$. The set of such mechanisms is denoted by $\mathcal{M}$, and $\mathcal{M}^q$ denotes the set of possible allocation decisions $\xi$ that are part of a mechanism.

We will consider the notion of Bayes-Nash implementation, which goes contrary to the agnostic approach of the standard Robust Mechanism design literature regarding beliefs. First it is necessary to define the notion of (Bayesian) individual rationality (IR), that will have to hold for a mechanism to be feasible.

**Definition 1.** A mechanism $M = (\xi, \pi)$ is individually rational (IR) if for all $i \in I$,

$$\int U(\xi_i(t), \pi_i(t); \nu_i(t_i)) d\beta_i(t_{-i}|t_i) \geq 0,$$

for all $t_i \in T^i$.

We define incentive compatibility in the usual way.

**Definition 2.** A mechanism $M = (\xi, \pi)$ is incentive compatible (IC) if for all $i \in I$,

$$\int U(\xi_i(t), \pi_i(t); \nu_i(t_i)) d\beta_i(t_{-i}|t_i) \geq \int U(\xi_i(t', t_{-i}), \pi_i(t', t_{-i}); \nu_i(t_i)) d\beta_i(t_{-i}|t_i),$$

for all $t_i, t'_i \in T^i$.

As discussed in Section 1, we will start the analysis by considering a relaxed version of the problem of maximizing revenue, by considering that bidders can only misreport their valuation, but are unable to lie about their interim beliefs. This means that for an player $i \in I$ and type $t_i$, it must not be profitable to announce alternative type $t'_i$ such that $\bar{\beta}_i(t'_i) = \bar{\beta}_i(t_i)$, or equivalently, $\bar{\beta}_i^{-1}(\bar{\beta}_i(t_i))$. Accordingly, we define a weaker notion of incentive compatibility that formalizes this idea.

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8This is actually the definition for the case each $T^i$ is countable and the $\sigma$-algebras contain all the singletons. More generally it means that for any measurable function $f : T \rightarrow \mathbb{R}$, $\int f(t) d\mathbb{P}(t) = f \left[ \int f(t_{-i}, t_i) \bar{\beta}_i(dt_{-i}|t_i) \right] d\mathbb{P}(t_i)$. 

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**Definition 3.** A mechanism $M = (\xi, \pi)$ is payoff incentive compatible (PIC) if for all $i \in I$,

$$\int U(\xi_i(t), \pi_i(t), \tilde{\nu}_i(t)) \ d\beta_i(t \mid t_i) \geq \int U(\xi_i(t', t_{-i}), \pi_i(t', t_{-i}), \tilde{\nu}_i(t_i)) \ d\beta_i(t \mid t_i),$$

for all $t_i \in T_i, t'_i \in \bar{\beta}_i^{-1}(\tilde{\beta}_i(t_i))$.

A useful simplification is in place due to the linearity assumptions imposed on the preferences. The relevant aspects of a type $t$ are: the belief induced by it, $\tilde{\beta}_i(t) \in \Delta(T_{-i})$, and the valuation induced by it, $\tilde{\nu}_i(t_i) \in V_i$. Therefore we can restrict attention to mechanisms that depend on $t$ solely through the vector $\left(\tilde{\beta}(t), \tilde{\nu}(t)\right) = \left(\tilde{\beta}_i(t), \tilde{\nu}_i(t_i)\right)_{i \in I}$, so we will also treat a given vector $(v, \beta)$ as a type (when in fact it means the set of types $[\tilde{\nu}_i = v_i$ and $\tilde{\beta}_i = \beta_i]$ among which there is no problem of misreporting) and also write the mechanism directly as a function of $(v, \beta) = (v_i, \beta_i)_{i \in I}$.

### 3 Relaxed Problem

In this section we characterize the Optimal Revenue obtained under any direct mechanism that is Individually Rational (IR) and is Payoff Incentive Compatible (PIC). We will refer to this as the relaxed problem throughout the analysis.

The thorough analysis of this simplified problem is illuminating for two reasons. The first one is that it proposes an upper bound on Rent extraction in any model with rich type spaces. Neeman (2004) argued that the full rent extraction result from CM relies on the implicit assumption that beliefs determine preferences. The intuition for this is that in general we might be able to adapt the construction of bets that make it very costly for a given agent to lie about their interim beliefs, but if it is possible for an agent to have a given belief but accompanied by different valuations, we cannot distinguish those without leaving rents. The notion of Payoff incentive compatibility builds on this intuition to formalize the limit on rent extraction suggested above. We are only using the private character of the residual information, once beliefs have been truthfully revealed (in this case, by assumption).

Secondly, in the next section, we will introduce an assumption on the type space for which we can conclude that the subset of incentive compatibility constraints considered here is the

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9If this were not the case one could simply consider the mechanism with allocation $\tilde{\xi}(t) = \int \xi_i(t) \ d\bar{P}(t \mid \tilde{\beta} = \beta(t) \wedge \tilde{\nu} = \nu(t))$ and similarly for $\tilde{\pi}$. All the incentive constraints, individual rationality constraints and total resource constraints are linear in quantities and transfers, and therefore are satisfied for the modified mechanism.
relevant one. We do this by using a notion of linear independence (LI) to adapt the famous CM construction. We find a set of bets, each connected with one of the possible interim beliefs that can arise in the model. Whenever an agent is allocated a bet that is connected with his own belief, the bet pays zero in expected terms. On the other hand, whenever an agent is endowed with a bet that is not connected with his own belief, it is expected value is negative (and can be made arbitrarily large). By including these bets into the transfers that are part of the mechanism, we can make it arbitrarily costly for a given agent to lie about his beliefs. Since this bets pay zero on average, this has no cost for the auctioneer. Under this condition the relaxed and the original problem are equivalent (from an ex-ante perspective) and we have characterized the Optimal Revenue.

To a lesser extent, the relaxed problem can also be useful when taken at face value. We can imagine that there might be situations where, for reasons outside the model, it is natural to assume that all the information held by the bidders is also observed by the auctioneer, except for their valuations. In this case the analysis is obviously useful.

The main result in this section is the characterization of the Optimal Mechanism for the relaxed problem. The main idea is that the original problem is divided into a set of smaller problems, each of which corresponding to one possible realization of the profile of interim beliefs. The interim beliefs can be treated as information available to the auctioneer since the implementation concept used in the relaxed problem precludes the bidders from lying about it. For each profile of interim beliefs, the mechanism implements the allocation decision and transfers that is optimal given the conditional distribution over valuations. Since this distribution is independent across bidders (Lemma 1) this allocation is well known from the standard auction literature (Myerson 1981).

The statement that the proposed mechanism satisfies the constraints for the relaxed problem is simple and is formally stated in the first result of this section. The remainder of it deals with the statement that it is indeed optimal. The difficulty with this is that in the mechanism proposed, each bidder would be willing to announce truthfully his valuation even if he observed the complete profile of interim beliefs. Nonetheless, they only observe their own interim belief. We show that this coarsening of the incentive constraints does not generate extra rents to the auctioneer.

In order to state our results, first we need to define some auxiliary notation. We use \( \mathbb{P}_\beta \) as shorthand for the distribution over the type space \((T, T)\) conditional on profile of interim beliefs \( \beta \), i.e., \( \mathbb{P}(\cdot | \bar{\beta} = \beta) \). A distribution over the type space implies a distribution for valuations, so for a given distribution \( \mu \in \Delta(T) \) denote as \( M(\mu) \in \Delta(V) \) the distribution.
over valuations implied by $\mu$ and $\bar{v}$. It means that $M_{\mu}(v) = \mu(\{t \in T : \bar{v} = v\})$ for any $v \in V$. Finally let $Q$ denote the distribution over beliefs (over $\times_i \Delta(T^{-i})$) induced by the prior $\mathbb{P}$ and $\tilde{\beta}$ (notice that given assumption (LI) the support of this distribution is finite).

Now we define an auxiliary problem. Consider $\mu = \otimes_{i \in I} \mu_i$ where $\mu_i \in \Delta(V_i)$ is a distribution over valuations. Now define as $R(\mu)$ the maximal revenue that can be achieved in a setting where agents have independent valuations (jointly) distributed according to $\mu$, i.e.,

$$R(\mu) \equiv \arg \max_{p, q} \int \left[ \sum_{i=1}^{I} p_i(v) \right] (\otimes_i \mu_i)(dv),$$

subject to for all $i \in I$, $v_i, v'_i \in V_i$

$$\int [v_i q_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] (\otimes_{j \neq i} \mu_j)(dv_{-i}) \geq$$

$$\int [v'_i q_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i})] (\otimes_{j \neq i} \mu_j)(dv_{-i}),$$

and

$$\int [v_i q_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] (\otimes_{j \neq i} \mu_j)(dv_{-i}) \geq 0.$$

The domain of $p, q$ are the sets of functions from $V$ to $\mathbb{R}^I$, with the extra restriction that quantities are nonnegative and that $\sum_{i \in I} q_i(v) \leq 1$, for all $v \in V$. This is a linear programming problem, therefore for any $\mu$ with finite support we can say that it has a solution, which we will denote as $(q^\mu, p^\mu)$.

The connection between this auxiliary problem and the relaxed problem is as follows. We are assuming that every agent’s belief is revealed truthfully since bidders do not consider the possibility of misreporting it in the definition of (PIC). Therefore the auctioneer basically always has knowledge of this information, but nonetheless needs to extract the residual private information regarding beliefs. The crucial point is that, by conditioning on the interim beliefs, we have already controlled for all the correlated information across different bidders, so the remaining uncertainty is independent across bidders. The following Lemma states this conclusion formally.

**Lemma 1.** Valuations are independent, conditional on beliefs, i.e., for any $\beta \in \times_{i \in I} \tilde{\beta}_i(T)$ and sets $(A_i)_{i \in I} \in \times_{i \in I} T^i$,

$$\mathbb{P}_\beta(\bar{t} \in \times_i A_i) = \prod_{i \in I} \mathbb{P}_\beta(\bar{t}_i \in A_i).$$
Finally we are in position to state the first result. The result uses one specific mechanism to provide a revenue level that can be achieved in the relaxed problem, and this revenue is reached through the breakdown of the initial problem into a series of conditional auction problems with independent types. Each of this conditional auxiliary problems is connected to the extraction of revenue, conditional on a given profile of interim beliefs.

Assume that the profile of interim beliefs is public information (observed by all bidders and the auctioneer). This means that maximizing ex-ante revenue is equivalent to maximizing revenue conditional on each realization of beliefs $\beta$. Notice that given any specific profile of beliefs $\beta$, payoff incentive compatibility would be equivalent to the incentive compatibility constraint for the problem of independent valuations distributed according to $M_\nu \mathbb{P}_\beta$. This means that the optimal revenue achieved, once the profile $\beta$ is realized, is given by $R (M_\nu \mathbb{P}_\beta)$ by definition of $R (.)$. If this revenue is achieved for each possible interim belief profile, the achieved ex-ante revenue is given through integration across $\beta$, which reduces to

$$\int R (M_\nu \mathbb{P}_\beta) Q (d\beta).$$

Nonetheless, in the relaxed problem the realized beliefs profile $\beta$ is not public information. By dealing with constraint (PIC), we are assuming that the auctioneer knows each player’s belief $\beta_i$, since no player can lie about their beliefs. However, each player knows only $t_i$, which means that he knows his valuation $\tilde{v}_i (t_i)$ and his interim belief $\tilde{\beta}_i (t_i)$, but not his opponents’ interim beliefs. The fact that the bidders have less information means that the auctioneer can potentially extract more revenue than proposed before, as confirmed by the next Lemma.

**Proposition 1.** The optimal revenue in the relaxed problem is weakly above

$$\int R (M_\nu \mathbb{P}_\beta) Q (d\beta).$$

*Proof.* Consider the direct mechanism $(\bar{\xi}, \bar{\pi})$ given by

$$\bar{\xi} (t) = q^{M_\nu \mathbb{P}_\beta (t)} (\tilde{v}_i (t));$$

$$\bar{\pi} (t) = p^{M_\nu \mathbb{P}_\beta (t)} (\tilde{v}_i (t)).$$

Remember that $q^\mu, p^\mu$ is the solution to the independent values problem with distribution $\mu$.
Fix an arbitrary agent $i$ and types $t_i, t'_i \in [\tilde{\beta}_i = \beta_i]$. Define $U_i(t_{-i}) \equiv \tilde{v}_i(t) \tilde{\xi}(t) - \tilde{\pi}(t)$ and $U'_i(t_{-i}) \equiv \tilde{v}_i(t) \tilde{\xi}(t'_i, t_{-i}) - \tilde{\pi}(t'_i, t_{-i})$, which are the realized truth telling and lying payoffs. Now notice that by construction of $(\tilde{\xi}, \tilde{\pi})$ we know that for all $\beta_{-i}$ (such that $\mathbb{P}[\tilde{\beta} = \beta_i, \beta_{-i}] > 0$)

$$\int U_i(t) \, d\mathbb{P}[\beta_i, \beta_{-i}] (t_{-i} | t_i) \geq \int U'_i(t) \, d\mathbb{P}[\beta_i, \beta_{-i}] (t_{-i} | t_i),$$

(notice that this equation is not necessarily true if $\tilde{\beta}_i(t'_i) \neq \tilde{\beta}_i(t_i)$) and

$$\int U_i(t) \, d\mathbb{P}[\beta_i, \beta_{-i}] (t_{-i} | t_i) \geq 0.$$

These are just the incentive and individual rationality constraints for the problem $R(M \mathbb{P}[\beta_i, \beta_{-i}]).$

Now the relevant incentive constraints follow by integrating the expressions above. For incentive constraints we have

$$\int U_i(t) \, d\mathbb{P} (t_{-i} | t_i) = \int U_i(t) \, d\mathbb{P}[\beta_i, \beta_{-i}(\hat{t}_{-i})] (t_{-i} | t_i) \, d\mathbb{P}(\hat{t}_{-i} | t_i) \geq \int U'_i(t) \, d\mathbb{P}[\beta_i, \beta_{-i}(\hat{t}_{-i})] (t_{-i} | t_i) \, d\mathbb{P}(\hat{t}_{-i} | t_i) = \int U'_i(t) \, d\mathbb{P} (t_{-i} | t_i),$$

and for individual rationality we have

$$\int U_i(t) \, d\mathbb{P} (t_{-i} | t_i) = \int U_i(t) \, d\mathbb{P}[\beta_i, \beta_{-i}(\hat{t}_{-i})] (t_{-i} | t_i) \, d\mathbb{P}(\hat{t}_{-i} | t_i) \geq 0,$$

since both inequalities hold point-wise in $\hat{t}_{-i}$, i.e., agent $i$ has no gain from lying about his payoff type conditional on all vector $\beta$, therefore he is not willing to lie if he only conditions on $\beta_i$.

It then follows that the mechanism $(\tilde{\xi}, \tilde{\pi})$ satisfies (1) and (3). \qed

We will prove in the remainder of this section that this mechanism is indeed optimal. We show that the result follows and this is in fact the Optimal Revenue. In order to do this, we need to further characterize the optimal allocation in the relaxed problem. We start by doing this in the following Lemma, which recasts standard rent extraction results in our setting. It states that the interim expected allocation has to be weakly increasing in valuation, and
that all downward incentive constraints will bind, as well as the lowest valuation incentive constraint.

**Lemma 2.** Assume the type space is finite. Consider a mechanism \((\xi^*, \pi^*)\) that maximizes Expected Revenue in the relaxed problem, then for each \(i = 1, \ldots, I\), \(\beta_i \in \widehat{\beta} (T_i)\) and \(k \geq 2\),

\[
\int \xi^*_i \left( v^k_i, \beta_i, t_{-i} \right) d\beta_i (t_{-i}) \geq \int \xi^*_i \left( v^{k-1}_i, \beta_i, t_{-i} \right) d\beta_i (t_{-i}),
\]

and

\[
\int [\xi^*_i \left( v^k_i, \beta_i, t_{-i} \right) v^k_i - \pi^*_i \left( v^k_i, \beta_i, t_{-i} \right)] d\beta_i (t_{-i}) = 
\int [\xi^*_i \left( v^{k-1}_i, \beta_i, t_{-i} \right) v^k_i - \pi^*_i \left( v^{k-1}_i, \beta_i, t_{-i} \right)] d\beta_i (t_{-i});
\]

also

\[
\int [\xi^*_i \left( v^1_i, \beta_i, t_{-i} \right) v^1_i - \pi^*_i \left( v^1_i, \beta_i, t_{-i} \right)] d\beta_i (t_{-i}) = 0.
\]

Furthermore, any mechanism that satisfies this conditions is Payoff Incentive-compatible (PIC).

**Proof.** In the Appendix. \(\square\)

The important consequence of the Lemma above is that, by using the binding incentive constraints, we can see that expected transfers are pinned down by the allocation. Define \(U_i (v_i, \beta_i)\) as the expected interim utility of type \(t_i\) with valuation and belief pair \((v_i, \beta_i)\) when telling the truth. For any transfer \(\pi_i\) satisfying the referred constraints we have that:

\[
\mathbb{E} [\pi_i (v_i, \beta_i, t_{-i}) | \beta_i] = \mathbb{E} [v_i \xi_i (v_i, \beta_i, t_{-i}) | \beta_i] - \sum_k p^{i, \beta_i}_k U_i (v^k_i, \beta_i)
\]

\[
= \mathbb{E} [v^k_i \xi_i (v^k_i, \beta_i, t_{-i}) | \beta_i] - \sum_{k \geq 2} p^{i, \beta_i}_k \left\{ \sum_{k' = 2}^k \gamma \mathbb{E} \left[ \xi^*_i \left( v^{k-1}_{i}, \beta_i, t_{-i} \right) | \beta_i, v^k_i \right] \right\}
\]

\[
= \sum_k p^{i, \beta_i}_k \left[ \left( v^k_i - \gamma \frac{1 - F^{i, \beta_i}_k}{p^{i, \beta_i}_k} \right) \xi_i (v^k_i, \beta_i, t_{-i}) | \beta_i, v^k_i \right]
\]

\[
= \mathbb{E} \left[ V^{i, \beta_i}_k \xi_i (v^k_i, \beta_i, t_{-i}) | \beta_i \right],
\]
where we have defined the virtual valuation by\textsuperscript{10}

\[ V_{i,\beta}^k \equiv v_i^k - \gamma \frac{1 - F_k^{i,\beta}}{p_k^{i,\beta}}. \]

We also denote the virtual valuation indexed by valuation level when necessary, by using

\[ V_{\nu_i}^k = V_{i,\beta}^k. \]

Now notice that the only restriction over the allocation \( \xi \) is monotonicity. We summarize this into a Corollary.

**Corollary 1.** The Optimal Revenue in the relaxed problem is characterized by

\[
\max_{\xi \in \mathcal{M}^i} \mathbb{E} \left[ \sum_{t \in I} \xi_i (t) V_{\nu_i(t)}^i \right],
\]

subject to

\[
\int \xi_i^* (v_i^k, \beta_i, t_{-i}) d\beta_i (t_{-i}) \geq \int \xi_i^* (v_i^{k-1}, \beta_i, t_{-i}) d\beta_i (t_{-i}),
\]

for all \( i \in I, \beta_i \in \tilde{\beta}_i (T) \) and \( 1 \leq k \leq K_i \).

The virtual valuation captures the direct rents to be extracted from a given type and the indirect losses related to higher rents left to the types with higher valuation levels. The relevant probabilities in the definition are the ones used at the level of incentive constraints, which means conditioning on \( \beta_i \). Nonetheless, knowledge about other bidders’ type is noninformative about \( t_i \), once we have conditioned on \( \beta_i \) already (since it contains all the information that links \( \hat{r} \)’s types with opponents’). This implies that for any \( \beta \in \tilde{\beta} (T) \) and \( k \) so that \( p_k^{i,\beta} > 0 \),

\[ V_{i,\beta}^k = v_i^k - \gamma \frac{1 - F_k^{i,\beta}}{p_k^{i,\beta}} = v_i^k - \gamma \frac{1 - \mathbb{P} [\tilde{v}_i \leq v_i^k | \beta_i]}{\mathbb{P} [\tilde{v}_i = v_i^k | \beta_i]} = v_i^k - \gamma \frac{1 - \mathbb{P} [\tilde{v}_i \leq v_i^k | \beta]}{\mathbb{P} [\tilde{v}_i = v_i^k | \beta]}. \]

The last term is relevant because this is the virtual valuation in the reduced problem with type distributed (independently) according to \( \mathbb{P}_\beta \). So the relevant definition of virtual valuation in the relaxed problem coincides with the relevant notion of virtual valuation in each auxiliary problem with independent types. Therefore the only difference in the characterization of both problems is the level of conditioning used in the monotonicity constraints. It

\textsuperscript{10}In the argument presented here we assume that \( p_k^{i,\beta} > 0 \), but full support is not needed. In the formal proofs the arguments are adapted accordingly.
is useful to restate the problem of revenue maximization in the case of independent types through its representation using virtual valuations.

**Lemma 3.** For each \( \beta \in \bar{\beta} (T) \), \( R (M_v \mathbb{P}_\beta) \) and \((\xi^{M_v \mathbb{P}_\beta}, \pi^{M_v \mathbb{P}_\beta})\) are characterized by

\[
\max_q \mathbb{E}^{M_v \mathbb{P}_\beta} \left[ \sum_{i \in I} q_i (v) V_{v_i}^{i, \beta_i} \right],
\]

subject to
\[
\int q_i (v_i^k, \ldots, v_i) \, dM_v \mathbb{P}_\beta (v) \geq \int q_i (v_i^{k-1}, \ldots, v_i) \, dM_v \mathbb{P}_\beta (v).
\]

The most important conclusion from this lemma is that the relevant virtual valuation that is relevant for revenue maximization is the same in the relaxed problem as in the respective auxiliary problem. The main difference between the two is that the monotonicity constraint on quantities for the relaxed problem is coarser than the one that is required in each of the auxiliary problems.

Therefore, in order to prove that we cannot improve upon the proposed mechanism that breaks down the original problem into a series of conditional auxiliary problems, it suffices to prove that the solution to the relaxed problem actually will satisfy each of the monotonicity constraints of the auxiliary problems. This is shown in the next Lemma.

**Lemma 4.** Assume the type space is finite. Suppose the mechanism \((\xi^*, \pi^*)\) maximizes expected revenue and that for any \( \beta \in \bar{\beta} (T) \), player \( i \in I \), \( \beta_i \in \bar{\beta_i} (T) \) and \( k \geq 2 \)

\[
\int \xi_i^* (v_i^k, \beta_i, t_{-i}) \, d\mathbb{P}_{[\beta]} (t_{-i}) \geq \int \xi_i^* (v_i^{k-1}, \beta_i, t_{-i}) \, d\mathbb{P}_{[\beta]} (t_{-i}).
\]

Then the revenue achieved is
\[
\int R (M_v \mathbb{P}_\beta) \, Q (d\beta).
\]

**Proof.** Since \((\xi^*, \pi^*)\) maximizes expected revenue in the relaxed problem, we know that its revenue is given by

\[
\mathbb{E} \left[ \sum_{i \in I} \xi_i (t) V_{\xi_i (t)}^{i, \beta_i (t)} \right] = \int \left\{ \int \left[ \sum_{i \in I} \xi_i^* (v, \beta) V_{v_i}^{i, \beta_i} \right] \, dM_v \mathbb{P}_\beta (v) \right\} \, dQ (\beta).
\]

Now notice that, by assumption of the proposition, we know that for each \( \beta \in \bar{\beta} (T) \), \( q^\beta (\cdot) = \xi_i (\cdot, \beta) \) satisfies all the monotonicity constraints in the problem of optimizing revenue.
given (independent) distribution of valuations \( M_v \mathbb{P}_\beta \in \Delta (V) \). So it follows that

\[
\int \left[ \sum_{i \in I} \xi^*_i (v, \beta) V^{\xi^*_i, \beta_i}_{v_i} \right] dM_v \mathbb{P}_\beta (v) \leq R (M_v \mathbb{P}_\beta ) .
\]

By integrating both sides over \( \beta \), we find the relevant level is an upper bound on Optimal Revenue. The reverse one follows from Lemma \( \square \).

The remaining issue now is to check whether the allocation that is part of the optimal mechanism, \( \xi^* \), satisfies the condition from the previous Lemma. For now suppose that all virtual valuation are increasing in one’s own valuation level. This assumption is not important and the general case is treated in Lemma \( \square \). From what Corollary \( \square \) says about the relaxed problem, the optimal allocation is given by

\[
\xi^*_i (v, \beta) = \begin{cases} 
\frac{1}{\# \arg \max_j V^{j, \beta_j}_{v_j}} , & \text{if } i \in \arg \max_j V^{j, \beta_j}_{v_j} \text{ and } \max_j V^{j, \beta_j}_{v_j} > 0 ; \\
0 , & \text{otherwise} .
\end{cases}
\]

Then, the optimal allocation for each problem is indeed increasing in agents’ own valuation, even ex-post, i.e., after we have knowledge of \((v_{-i}, \beta_{-i})\). The main trade-off in extracting rents from a given agent \( i \) is summarized in his virtual valuation, which only depend on his own belief \( \beta_i \) and his valuation \( v_i \). This argument also holds whenever virtual valuations are not monotone, and the result is presented as follows.

**Lemma 5.** Consider any finite type space. The optimal revenue can be achieved by a mechanism with allocation rule \( \xi^* \) such that for all \( i \in I \), \( \beta_i \in \tilde{\beta}_i (T) \) and \( t_{-i} \),

\[
\xi^*_i (v^k_i, \beta_i, t_{-i}) \geq \xi^*_i (v^{k-1}_i, \beta_i, t_{-i}) , \text{ for every } k \geq 2 .
\]

**Proof.** In the Appendix. \( \square \)

Now, since we know that the optimal allocation of a given bidder \( i \) is increasing in his own valuation at an ex-post level, it is certainly going to be weakly increasing when we use expectations at any coarser level of information. In particular, it is going to be true for the level of expectation conditional on the profile of beliefs \( \beta \), which is the necessary condition in Lemma \( \square \).

**Proposition 2** (Relaxed Problem Solution). Consider any finite type space. The optimal
revenue in the relaxed problem is

$$\int R (M_r \mathbb{P}_\beta) Q (d\beta).$$

Proof. Fix a given agent $i \in I$ and $\beta_i \in \tilde{\beta}_i (T)$. Consider the direct mechanism that achieves optimal revenue and satisfies the condition from Lemma (5). We then know that for any $k \geq 2$

$$\int [\xi^*_i (v^k_i, \beta_i, t_{-i}) - \xi^*_i (v^{k-1}_i, \beta_i, t_{-i})] d\mathbb{P}_{|\beta} (t_{-i}) \geq 0,$$

since the integrand is always nonnegative. It follows from Lemma (4).

4 Optimal Revenue

We are interested in characterizing the Optimal Revenue attained by any direct mechanism that is Individually Rational (IR) and Incentive Compatible (IR). The problem considered so far has been a relaxed problem, that considers a weaker version of the incentive compatibility constraints, namely, Payoff Incentive Compatibility (PIC).

In this section, we use the characterization of the Optimal Revenue in the relaxed problem to better understand how much rents can be extracted in the original problem. The first important conclusion is that the Revenue characterized in the relaxed problem is an upper bound on revenue in the original problem. This is a trivial consequence of the weaker constraints that are considered in the former.

The second result is a characterization of a condition for which the Optimal Revenue in the relaxed problem is feasible in the original problem, and therefore is equal to the optimal revenue. This is achieved by proving that the additional incentive constraints present in the original problem do not interfere with rent extraction, from an ex-ante point of view. The key assumption that is made to obtain the optimality result is that the set of possible interim beliefs satisfies a Linear Independence assumption (LI), which is akin to the one originally required in CM.

The first result formalizes the argument that, in the original problem, since we consider a larger set of constraints, the Optimal Revenue can only be lower than the one achieved in the Relaxed Problem.
Corollary 2. The Optimal Revenue is weakly lower than

\[ \int R(M_v P_\beta) \, dQ(\beta). \]

Proof. Trivial. \qed

Notice that, since we have characterized this revenue level by considering only Payoff Incentive Compatibility, this upper bound on revenue is driven by the rents that agents hold for sure solely due to private information about valuations. So this is the formalization of the argument presented in the introduction for why full rent extraction is not possible even with correlated types once we focus on rich type spaces. The argument is that agents’ possibility of misreporting regarding only their valuation guarantees them rents, as in the model with independent types. It also reduces the original problem to a set of smaller problems with independent types, which clarifies why rich type spaces are connected with rents held by the bidders in general.

In general, it will be the case that this bound already precludes total rent extraction. For any (independent) distribution \( \mu \in \Delta(V) \), if the support of \( \mu \) is not a singleton and we know that an agent \( i \in I \) receives the good with positive interim probability given all possible valuation levels, then we know that

\[ R(\mu) < \int \max_i v_i d\mu(v). \]

A relevant situation where \( R \) reduces to full information extraction is when the support \( \mu \) is a singleton. This will be discussed in further detail in part 4.2.

4.1 Linear Independence

The main assumption maintained in this section is Linear Independence (LI), which is defined as follows.

Definition 4. The type space \( (T^i, T^i, \tilde{v}_i, \tilde{\beta}_i)_{i \in I} \) satisfies Linear Independence (LI) if, for each \( i \in I \),

\[ \tilde{\beta}_i(T) = \{b_1^i, ..., b_L^i\} \subseteq \mathbb{R}^{T^{-i}} \]

and this set is linearly independent.

The assumption has two parts. The first one requires that the set of possible beliefs is finite. This effectively reduces the type space to a finite set, since we have already assumed
that the set of possible valuations $V$ be finite. The second and more important part requires that the set of possible beliefs is linearly independent.

This condition is similar to the one assumed in the classical paper of CM. It is important to clarify what is the difference between the current assumption and the classic linear independence assumption in the naive type space case. Consider the example presented in the introduction in which every single agent observes an informative signal about his opponent’s type and also observes his own valuation (which contains no additional information). This assumption requires that the set of all second stage distributions, interim distributions generated by all possible signals, be linear independent. The fact that several different types have same interim beliefs, and yet different valuation levels, is not a problem.

The classical linear independence assumption of CM requires that the matrix formed by the interim beliefs of all types have full row rank. But remember that a type in this case is a pair of valuation and signal observed. So in general it will be impossible for this condition to be satisfied, because if we consider that there are two types with same observed signal, but different valuations, the full rank condition fails.

Its role lies in constructing a system of bets that extract information regarding beliefs without cost. Under this assumption, we can find a set of bets (depending on all agents’ types), each connected with an agent and an interim belief, with two properties. The first one is that whenever an agent receives the bet that is connected with his real interim belief, his expected gain from it is zero. The second one is that whenever an agent is endowed with a bet connected with an interim belief different than his realized one, the expected gains from the bet is negative. Given any mechanism that is feasible in the relaxed problem, i.e., satisfies (IR) and (PIC), we can find a new mechanism that is feasible in the original problem and achieves same ex-ante revenue, by adding the the candidate mechanism the aforementioned bets multiplied by a large number.

**Lemma 6.** Suppose assumption (LI) holds. Consider a mechanism $M = (\xi, \pi)$ that satisfies (PIC), then there exists another payment rule $\pi'$ that satisfies:

1. $\mathbb{E}^P [\pi_i'(\tilde{t}) | \tilde{t}_i = t_i] = \mathbb{E}^P [\pi_i(\tilde{t}) | \tilde{t}_i = t_i]$, for all $i \in I$ and $t_i \in T^i$;

2. $M' = (\xi, \pi')$ satisfies (IC).

**Proof.** Consider a mechanism $(\xi, \pi)$ that satisfies (PIC). Notice that a simple argument from linear algebra can be used to argue that we can find, for each agent $i$ and belief $\tilde{b}_i \in \tilde{\beta}_i(T)$,
a function \( \tau^{b_i} : T_{-i} \to \mathbb{R} \) so that
\[
\mathbb{E} \left[ \tau^{b_i} \mid \tilde{\beta}_i = \bar{b}_i \right] = 0
\]
and
\[
\mathbb{E} \left[ \tau^{b_i} \mid \tilde{\beta}_i = \bar{b}_i \right] < 0,
\]
for any \( l' \neq l \). Now define a new mechanism by using \( \xi = \xi \) and
\[
\pi^K_i (v_i, b_i^l, t_{-i}) = \pi_i (t_i, t_{-i}) + K \tau^{b_i} (t_{-i})
\]
By construction we know that \( \mathbb{E} [\pi_i (t_i, t_{-i}) \mid t_i] = \mathbb{E} [\pi_i' (t_i, t_{-i}) \mid t_i] \), since the added term has average zero for a truth-telling agent. Now consider a belief \( b'_i \in \tilde{\beta}_i (T) \) and a type \( (v'_i, b'_i) \) with \( l' \neq l \). In this case we have that for any \( t'_i \in [\tilde{\beta}_i = \bar{b}_k] \)
\[
\mathbb{E} \left[ \pi^K_i (v'_i, b'_i, t_{-i}) \mid \tilde{\beta}_i = \bar{b}_i \right] = \mathbb{E} [\pi_i (v'_i, b'_i, t_{-i}) \mid t_i] + K \mathbb{E} \left[ \tau^{b_i} (t_{-i}) \mid \tilde{\beta}_i = \bar{b}_i \right] < 0.
\]
As long as valuations and interim expected transfers are bounded, we can find \( K \) sufficiently large so that no agent wants to pretend to be a type in \( [\tilde{\beta}_i \in \bar{b}_i] \).

The Lemma above shows that, under the condition (LI), the relevant concept of incentive compatibility for ex-ante revenue maximization is (PIC), which is the one considered in the relaxed problem. In this case, the extra incentive constraints for revelation of beliefs can be dealt with without affecting overall revenue.

The fact that the relevant incentive constraints in the relaxed and the original problem coincide under this assumption means that the Optimal Revenue for the relaxed problem can actually be achieved under the more restrictive feasible set of the original problem.

**Proposition 3.** In case assumption (LI) holds, the Optimal Revenue is equal to
\[
\int R (M_\beta \mathbb{P}_\beta) Q (d\beta).
\]

**Proof.** Consider a mechanism that is Optimal in the relaxed problem characterized in Section 3. We can modify it by including the bets used in the proof of Lemma 6 in the transfers so that this allocation satisfies (PIC) and (IR).
4.2 Naive type spaces as a special case

For its central role in the literature, we relate our result to the one for the case of naive type spaces. As expected, naive type spaces are specific cases of the more general structure of rich types. We show that under the conditions maintained in CM, the result reduces to full rent extraction.

Now consider the trivial case where $T_i = V_i = \{v_i^1, ..., v_i^n\}$, for each $i \in I$, with prior distribution $\mathbb{P}$ over $T = \times_{i \in I} V_i$ and with the standard assumption that, for each $i \in I$, the matrix whose columns are the interim beliefs $\mathbb{P}(\cdot | \tilde{v}_i = v_i)$ for each $v_i \in V_i$, has full column rank. Notice that this implies that types are correlated and that condition (LI) holds since the set $\tilde{\beta}_i (T) = \{\mathbb{P}(\cdot | \tilde{v}_i = v_i) : v_i \in V_i\}$ is going to be formed by all the columns of the matrix described before.

Now for any vector $\beta \in \tilde{\beta} (T)$, the assumption of full rank implies that $[\tilde{\beta} = \beta]$ is always a singleton. Let us call the single element connected with $\beta$ as $v(\beta)$. In this case we have that $M_i \mathbb{P}_\beta = \mathbb{P}_\beta = \delta_{\{v(\beta)\}}$, where $\delta_{\{v(\beta)\}}$ is the Dirac measure of the singleton $\{v(\beta)\}$ and the optimal revenue reduces to $R(\delta_{\{v\}}) = \max_i v_i$ for all $v \in \mathbb{R}_+^I$. The ex-ante optimal revenue becomes

$$\int R(M_v \mathbb{P}_\beta) Q (d\beta) = \int R(\delta_{\{v(\beta)\}}) Q (d\beta) = \int \max_i [v_i (\beta)] Q (d\beta) = \int \max_i [v_i] d\mathbb{P} (v),$$

since there is a one-to-one relation between belief profiles $\beta$ and types $v$. In this out result states that full rent extraction is possible.

The proposed mechanism extends the intuition of full rent extraction in the naive type space model to the more general class of type spaces, with the relevant adaptation that in cases where payoff relevant variables are not pinned down solely by beliefs, there will be some information rents foregone due to this private information.

4.3 Public Beliefs

Another relevant situation that satisfies the assumption above is the case of public beliefs with finite support. This would hold in case information updating is derived from publicly observed signal, and each bidder also observes his own valuation (which contains no additional information). The assumption that the auctioneer does not observe any information regarding the bidders’ types is maintained.
The requirement on the model is as follows. For any given bidder \( i, j \in I \), bidder \( i \) has knowledge of \( j \)'s belief, i.e., \( \tilde{\beta}_j \) is \( t_i \)-measurable. In order to check the validity of (LI), consider agent \( i \)'s possible beliefs. If \( \beta_i \) arises with positive probability, then necessarily \( \text{supp} \beta_i \subseteq \{ t_{-i} \in T^{-i} : \tilde{\beta}_j \left[ \beta_i = \beta_i \right] | t_j \} = 1 \text{ for all } j \neq i \} \), which means that player \( i \) knows that players \( j \neq i \) know his belief. This implies that all the possible interim belief \( \beta_i \) have pairwise disjoint supports, and therefore have to be linear independent. Therefore, as a consequence of Proposition 3, we have characterized the Optimal Revenue obtained in the case of public beliefs.

Even though the finiteness assumption on the set of possible beliefs is part of the definition of (LI), we know that in the case of public beliefs it could be relaxed without any loss. The reason is that there is an alternative argument for the irrelevance of the incentive constraints involving lies about interim beliefs: eliciting public information is already known to be an easy task in weak implementation theory with more than a single agent. A direct way to provide incentives for revelation of the public information is to punish all the agents in case of disagreement. We present the case of public beliefs as a special case of assumption (LI) for compactness of exposition and further clarification of what the assumption (LI) means.

4.4 Dominant Implementation

The construction of the optimal mechanism has depended strongly in the structure of interim beliefs each agent might have. The set of interim beliefs is important by two reasons: it is embedded in the prior distribution used by the auctioneer to calculate ex-ante revenue of any given mechanism, and it is also present in the interim concepts of individual rationality (IR) and incentive compatibility (IC). In this section we ask how much does the implementation of the optimal mechanism depend on the interim implementation criteria. Under linear independence, we are able to allow for dominant strategy incentive compatibility (DIC), which is equivalent to ex-post incentive compatibility in private values settings. However, we maintain the interim individual rationality constraints considered so far.

**Definition 5.** A mechanism \( M = (\xi, \pi) \) is dominant-strategy incentive compatible (DIC) if for all \( i \in I \),

\[
U (\xi_i (t) , \pi_i (t) ; \tilde{\nu}_i (t_i)) \geq U (\xi_i (t'_i , t_{-i}) , \pi_i (t'_i , t_{-i}) ; \tilde{\nu}_i (t_i)),
\]

for all \( t \in T \) and \( t'_i \in T^i \).

Now we present the main result of this section. It is proved by construction of a mechanism that achieves the the optimal revenue while satisfying all the relevant ex-post incentive
Proposition 4. Suppose linear independence is satisfied. There exists a revenue maximizing mechanism that satisfies individual rationality (IR) and dominant-strategy incentive compatibility (DIC).

Proof. Consider a revenue maximizing mechanism $M^* = (\xi^*, \pi^*)$ satisfying (IC), (IR) and with an ex-post monotone allocation rule $\xi^*$ (possible by Lemma 5). Let us define transfer function $t = (t_i)_{i \in I}$, with $t_i : T \to \mathbb{R}$, the following way: Fix an agent $i \in I$, interim belief $\beta_i \in \bar{\beta}_i (T)$ and types $t_{-i} \in T^{-i}$. The transfer function $t_i (\cdot; t_{-i})$ is such that

$$\xi^*_i (v^k_i, \beta_i; t_{-i}) v^k_i - t_i (v^k_i, \beta_i; t_{-i}) = \xi^*_i (v^{k-1}_i, \beta_i; t_{-i}) v^{k-1}_i - t_i (v^{k-1}_i, \beta_i; t_{-i}),$$

for all $k = 2, \ldots, K_i$. Also let $t_i (v^1_i, \beta_i; t_{-i}) = 0$.

Notice that mechanism $M' = (\xi^*, t)$ satisfies (DIC) since all the downward ex-post incentive constraints are binding and the allocation is increasing in one’s own valuation and preferences satisfy the standard single-crossing condition. They also satisfy (IR) since the any type $(v^1_i, \beta_i)$ (which has lowest possible valuation) has ex-post utility

$$\xi^*_i (v^1_i, \beta_i; t_{-i}) v^1_i - t_i (v^1_i, \beta_i; t_{-i}) = \xi^*_i (v^1_i, \beta_i; t_{-i}) v^1_i \geq 0,$$

and all the other individual rationality constraints follow from (DIC).

As a consequence, we know that mechanism $M'$ satisfies (IC) and (IR) as well. By construction, all the downward interim incentive constraints are binding, which implies that expected transfer paid by agent $i$ is given by

$$\mathbb{E} [t_i (t)] = \mathbb{E} [\pi^*_i (t)] + \mathbb{E} \left[ U_i \left( v^1_i, \bar{\beta}_i (\cdot) \right) \right],$$

since transfers are pinned down by the allocation $\xi^*_i$ (which is fixed throughout) up to a constant depending on $\beta_i$.

Finally, construct transfer functions $\tau = (\tau_i)_{i \in I}$ with $\tau_i : T^{-i} \to \mathbb{R}$ such that for all $i \in I$ and $\beta_i \in \bar{\beta}_i (T)$

$$\int \left[ U \left( \xi^*_i (v^1_i, \beta_i; t_{-i}) , t_i (v^1_i, \beta_i; t_{-i}) ; v^1_i \right) - \tau_i (t_{-i}) \right] d\mathbb{P}_{\beta_i} (t_{-i}) = 0.$$

Now mechanism $M'' = (\xi^*, t + \tau)$ satisfies (DIC) and (IR). All interim downward incentive constraints are binding and it delivers zero interim utility to all bidders with realized constraints, which depends on the linear independence assumption.
type corresponding to the lowest possible valuation \((v_i^1)\), which means that its revenue is equal to that of \(M = (\xi^*, \pi^*)\).

Our result generalizes the dominant strategy implementation of Cremer and McLean \cite{CremerMcLean1988} to ‘rich’ type spaces, under a suitable generalization of their assumptions to this setting. This result is also related to the equivalence between bayesian and dominant implementation presented by Gershkov et al. \cite{Gershkov2011} for naive and independent types.

5 An example of rich type space with Linear Independence

We adapt the example presented in Neeman \cite{Neeman2004} to the auction setting considered in the current paper. We have two players that can have valuation \(v_L = 1\) or \(v_H = 2\). First consider the situation where all the agents observe is their own valuation, which is distributed according to

<table>
<thead>
<tr>
<th></th>
<th>(v_L)</th>
<th>(v_H)</th>
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<tbody>
<tr>
<td>(v_L)</td>
<td>(1/3)</td>
<td>(1/6)</td>
</tr>
<tr>
<td>(v_H)</td>
<td>(1/6)</td>
<td>(1/3)</td>
</tr>
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</table>

This is one example of naive type space, in which CM’s result holds. In this model, since the beliefs pin down the exact valuation of each bidder, the optimal revenue proposed reduces to full surplus extraction, which gives rents

\[
R^{\text{naive}} = \frac{1}{3} + \frac{2}{3} \cdot 2 = \frac{5}{3}.
\]

Now consider the following modification to the standard model. Agents can have types \(\{L, LH, H\}\), where we have \(\tilde{v}_i(L) = \tilde{v}_i(LH) = 1\) and \(\tilde{v}_i(H) = 2\). We have introduced type \(LH\), that has same valuation as the low type, but shares the same belief as the high type. Beliefs over \(t_{-i}\) are given by \(\tilde{\beta}_i(L) = \left(\frac{2-2a}{3(1-a)}, \frac{a(1-2a)}{3(1-a)}, \frac{1-2a}{3(1-a)}\right)\) and \(\tilde{\beta}_i(LH) = \tilde{\beta}_i(H) = \left(\frac{1-2a}{3}, \frac{2a}{3}, \frac{2}{3}\right)\), where \(a \in [0, \frac{1}{2}]\). Notice that \(a = 0\) represents that we are in the naive type space above, and \(a\) increases the probability of "mixed" type \(LH\), both ex-ante and conditionally. We have constructed the modified prior distribution so that the marginal distribution over profile of valuations is equal to the one presented in the naive type space. The modified model has a rich type space. The prior distribution that generates these types is...
\[
\begin{array}{cccc}
& L & LH & H \\
L & \frac{1-a(1-a)}{3} & \frac{a(1-2a)}{3} & \frac{1-2a}{6} \\
LH & \frac{a(1-2a)}{6} & \frac{a^2}{3} & \frac{a}{3} \\
H & \frac{1-2a}{6} & \frac{a}{3} & \frac{1}{3}
\end{array}
\]

The optimal revenue in the modified model is characterized by Proposition (3). Notice that the assumption (LI) is satisfied in this setting: for each agent \(i = 1, 2\), there are two possible interim beliefs that are different, therefore they are linearly independent.

Let us verify what the lower bound is in this case. If the belief profile is (for player 1 and 2 respectively) \((\bar{\beta}_i(L), \bar{\beta}_i(L))\), we know that both agents have valuation \(v_L\) for sure, than revenue is \(v_L = 1\). In case beliefs are \((\bar{\beta}_i(L), \bar{\beta}_i(H) = \bar{\beta}_i(LH))\), we know the first player has valuation \(v_L\) and player two has valuation \(v_L\) or \(v_H\), so we can full rent (in case of tie, we give the good to player 1 so that player 2 has no information rent) and the revenue is \(\frac{1-a}{1+a} + 2\frac{1}{1+a} = \frac{2+a}{1+a}\) (the same is true in the symmetric case). Now in the case of belief profile \((\bar{\beta}_i(H) = \bar{\beta}_i(LH), \bar{\beta}_i(H) = \bar{\beta}_i(LH))\), we know that valuations for both players are distributed independently with probabilities \(\left(\frac{a}{1+a}, \frac{1}{1+a}\right)\) for types \(LH\) and \(H\) respectively. In this case optimal revenue is characterized by maximal virtual valuations, in case one has type \(LH\), the virtual valuation is given by \(v_L - \frac{1}{a} = 1 - \frac{1}{a} < 0\) (since \(a \leq \frac{1}{2}\)), and in case of type \(H\), the virtual valuation is just \(v_H = 2\). The good then will only be allocated to someone with valuation \(v_H = 2\). So expected revenue is given by \(2 \left[1 - (\frac{a}{1+a})^2\right] + (\frac{a}{1+a})^2 0\). So the lower bound on revenue is given by (summing over the four possible belief profiles)

\[
R^{rich} = \frac{1-a(1-a)}{3} + 2 \left(\frac{2+a}{1+a}\right) \left[\frac{(1+a)(1-2a)}{6}\right].
\]

Notice that the situation \(a = 0\) reduces to the case of full rent extraction, as \(a\) increases type \(LH\) becomes more relevant in the third conditional case analyzed, and the rents generated by type \(H\) pretending to be \(LH\) become more problematic. The following figure illustrates how the optimal revenue behaves as \(a\) increases.

Notice that, even though the distribution over valuations does not change with \(a\), the optimal revenue is strictly decreasing in \(a\). The difference is due to the richness of the type space. The separation of beliefs and types, which in this model is represented by type \(LH\), introduces back into the model informational rents held by bidders since they have
private information beyond their interim beliefs. As the fraction of types $LH$ increases, their allocation becomes more important, and they are always connected to higher rents to type-$H$ bidders.

6 Beyond Linear Independence

The Linear Independence assumption is crucial to the achievement of the lower bounds proposed above for rent extraction. It allows us to ignore incentives for lying that involve a misstatement of one’s private information about the other players’ types. The consequence of this is that pure differences in private information about other agents’ types will not generate rents, and so are irrelevant from the point of view of revenue maximization. In this case, the only rents held by the agents come from the residual uncertainty over their private valuation once one fixes a specific belief profile. The fact that valuations are conditionally independent means that in general agents will have some rents due to their unobservable valuations. This separation allows us to trace back how the case of correlated types differs from the independent case, in specifically characterizing what dimension of the correlation between types matters.

The same argument does not hold once we drop the assumption (LI). The objective of this section is to show what part of the arguments still hold and present an example where the result does not hold, i.e., the optimal revenue is strictly below $\int R(M_v)\, Q(d\beta)$.

First let us define the weaker case of finite beliefs.

**Definition 6.** We say that the type space $\left(\mathcal{T}_i, \mathcal{F}_i\right)_{i=1}^I$ satisfies finite beliefs
(FB) if, for each \( i \in I \), \( \tilde{\beta}_i (T) = \{ \bar{b}_i^1, ..., \bar{b}_i^{k_i} \} \subseteq \mathbb{R}^{T_i} \).

In this case, given that the construction of the bets that insure that agents do not lie about their beliefs depends upon the linear independence of different interim beliefs, so it is natural to expect that this property will be relevant in the more general case. So let us define \( B_i = \{ \bar{b}_i^1, ..., \bar{b}_i^{k_i} \} \subseteq \tilde{\beta}_i (T) \) be a base for the set \( \tilde{\beta}_i (T) \), with \( k_i \leq k_i \) (if \( k_i = k_i \), we are in the (LI) case). Also let \( T_i^B = \tilde{\beta}_i^{-1} (B_i) \) denote the set of types that have beliefs within the base \( B_i \).

As before, the possibility of using large bets that exploit the differences in interim beliefs allows us to weaken the concept of incentive compatibility to be used.

### 6.1 Weak Incentive Compatibility

**Definition 7.** We say that a mechanism \( M = (\xi, \pi) \) is weakly incentive compatible (WIC) if for all \( i \in I \),

\[
\int U (\xi (t), \pi (t); \bar{\nu}_i (t_i)) d\beta (t_{-i} | t_i) \geq \int U (\xi (t'_i, t_{-i}), \pi (t'_i, t_{-i}); \bar{\nu}_i (t_i)) d\beta (t_{-i} | t_i), \tag{8}
\]

for all \( t \in T, t'_i \in \tilde{\beta}_i^{-1} (\tilde{\beta}_i (t_i)) \cup T_i \setminus T_i^B \).

Weak incentive compatibility is weaker than (IC) but stronger than (PIC). Under this concept, under we consider a fixed type, we have to insure that lying solely about their valuation, plus also the possibility of claiming to have an alternate belief out of the specified base (remember that the base is not unique). The difference between (WIC) and (IC) is the fact that a given type \( t_i \), lying about being any type in \( B_i \setminus \tilde{\beta}_i = \tilde{\beta}_i (t_i) \) is not considered relevant. The next lemma shows that the rationale of Lemma (6) applies in this case.

**Proposition 5.** Suppose Finite Beliefs assumption holds. Consider a mechanism \( M = (\xi, \pi) \) that satisfies (WIC), then there exists another payment rule \( \pi' \) that satisfies:

1. \( \mathbb{E}^P [\pi'_i (\bar{t}) | \bar{\nu}_i = t_i] = \mathbb{E}^P [\pi_i (\bar{t}) | \bar{\nu}_i = t_i] \), for all \( i \in I \) and \( t_i \in T_i^i \);
2. \( M' = (\xi, \pi') \) satisfies Incentive Compatibility (IC).

**Proof.** Consider a mechanism \( (\bar{\xi}, \bar{\pi}) \) that satisfies (WIC). Notice that a simple argument from linear algebra can be used to argue that we can find, for each agent \( i \) and belief \( \bar{b}_i^k \in B_i \), a function \( \tau^k : T_{-i} \to \mathbb{R} \) so that

\[
\mathbb{E} [\tau^k | \tilde{\beta}_i = \bar{b}_i^k] = 0
\]
and
\[ \mathbb{E} \left[ \tau_{i}^{b_k} \mid \tilde{\beta}_i = \tilde{b}_i \right] < 0, \]
for any \( l \neq k \). Now define a new mechanism as \( \xi = \tilde{\xi} \) and
\[
\pi_i^K (t_i, t_{-i}) = \begin{cases} 
\pi_i (t_i, t_{-i}) + K \tau_{i}^{b_k} (t_{-i}) & \text{if } \tilde{\beta}_i (t_i) = \tilde{b}_i \in B_i \\
\pi_i (t_i, t_{-i}) & \text{otherwise.}
\end{cases}
\]
By construction we know that \( \mathbb{E} [\pi_i (t_i, t_{-i}) \mid t_i] = \mathbb{E} [\pi'_i (t_i, t_{-i}) \mid t_i] \), since the added term has average zero for a truth-telling agent. Now consider a belief \( \tilde{\beta}_i \in B_i \) and a type \( t_i \) such that \( \tilde{\beta}_i (t_i) = \sum_{k' = 1}^{k_i} \alpha_k \tilde{b}_i^{k'} \), where \( \alpha_k > 0 \) for some \( k \neq k' \). In this case we have that for any \( t'_i \in [\tilde{\beta}_i = \tilde{b}_i] \)
\[
\mathbb{E} [\pi_i (t'_i, t_{-i}) \mid t_i] = \mathbb{E} [\pi'_i (t'_i, t_{-i}) \mid t_i] + \sum_{k' = 1}^{k_i} \alpha_k \mathbb{E} \left[ K \tau_{i}^{b_k} (t_{-i}) \mid \tilde{\beta} = \tilde{b}_i^{k'} \right] \\
\leq \mathbb{E} [\pi'_i (t'_i, t_{-i}) \mid t_i] + K \alpha_k \mathbb{E} \left[ \tau_{i}^{b_k} (t_{-i}) \mid \tilde{\beta} = \tilde{b}_i^{k'} \right] < 0.
\]
As long as valuations and transfers are bounded, we can find \( K \) sufficiently large so that no agent wants to pretend to be a type in \([\tilde{\beta}_i \in B_i] \).

Differently from the analysis in the (LI) case, we cannot rule out that agents enjoy rents due to differences in beliefs. If this is the case, the stark separation of the private information between beliefs (which do not generate rents) and residual valuation uncertainty (which, being independent across agents, generates rents) does not apply in this case. We present an example to illustrate that the results do not extend to the general setting.

### 6.2 Example

Assume that there are 2 agents, and their possible types are \( T_1 = \{ t_1^{1}, t_1^{2}, t_1^{3} \} \) and \( T_2 = \{ t_2^{1}, t_2^{2} \} \). Consider the following prior over the types for agents 1 and 2:

<table>
<thead>
<tr>
<th>( t_1 )</th>
<th>( t_1^{1} )</th>
<th>( t_1^{2} )</th>
<th>( t_1^{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_2 )</td>
<td>( \frac{2}{5} )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{1}{5} )</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>( \frac{3}{5} )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{2}{5} )</td>
</tr>
</tbody>
</table>
The values for the types are $\tilde{v}_1(t^1_1) = \tilde{v}_1(t^2_1) = 2$, $\tilde{v}_1(t^3_1) = 4$, $\tilde{v}_2(t^1_2) = 1$ and $\tilde{v}_2(t^2_2) = 3$. One can see immediately that, for both agents, the belief function $\tilde{\beta}_i(\cdot)$ is injective, so that beliefs identify valuation.

It follows that in this case, the lower bound established in the paper is equal to total surplus extraction, i.e.,

$$\int R(M_v \mathbb{P}_\beta) \, dQ(\beta) = 2 \frac{7}{18} + 3 \frac{5}{18} + 4 \frac{1}{3} = \frac{53}{18}.$$

From (IR) one can see that in order to obtain full surplus extraction, the product allocation has necessarily to be

<table>
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<tr>
<th>$\xi_1(\cdot)$</th>
<th>$t^1_1$</th>
<th>$t^2_1$</th>
<th>$t^3_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^1_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$t^2_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

and that all the individual rationality constraints must bind. But notice that combining the incentive constraints of $t^1_1$ and $t^3_1$ to pretend to be $t^2_1$ (with weights $\frac{1}{2}$) we get that

$$0 \geq \frac{\mathbb{E} [2\xi_1(t^2_1, t_2) \, | \, t^1_1]}{2} + \mathbb{E} [4\xi_1(t^2_1, t_2) \, | \, t^3_1] - \mathbb{E} [\pi_1(t^2_1, t_2) \, | \, t^2_1]$$

$$\geq \frac{\mathbb{E} [2\xi_1(t^2_1, t_2) \, | \, t^1_1]}{2} + \mathbb{E} [2\xi_1(t^2_1, t_2) \, | \, t^3_1] - \mathbb{E} [\pi_1(t^2_1, t_2) \, | \, t^2_1]$$

$$= \mathbb{E} [2\xi_1(t^2_1, t_2) \, | \, t^3_1] - \mathbb{E} [\pi_1(t^2_1, t_2) \, | \, t^3_1] = 0,$$

which is a contradiction. We have used in the first line the binding individual rationality constraints of $t^1_1$ and $t^3_1$, while in the last equality we have used the individual rationality of $t^2_1$. So it is impossible to obtain full rent extraction in this case.

In fact the allocation in the optimal auction is given by

<table>
<thead>
<tr>
<th>$\xi^*_1(\cdot)$</th>
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<th>$t^2_1$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$t^1_2$</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$t^2_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
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</table>

while $\xi^*_2 = 1 - \xi^*_1$. So in this case the designer decides to distort the allocation of $t^3_1$, so that it does not interfere with the extraction of rents from $t^3_1$ and $t^1_1$. 
7 Conclusion

In this paper, we have considered the problem of Revenue extraction in auction setting in the presence of rich type spaces, which means that bidders might get information from sources other than their own valuation. We have characterized an upper bound on Revenue extraction which is driven by the information rents that agents hold due to their private information regarding their own valuation, even after beliefs are accounted for.

If the set of possible interim beliefs satisfies a linear independence assumption (LI), the upper bound characterized above is achieved, and gives us the optimal revenue. The connection between the relaxed and the original problems comes from the fact that private information regarding beliefs can be elicited with no cost by the use of ‘bets’ that pay a given agent depending his opponents’ types. We have also characterized one optimal mechanism.

In this case, we have established a rent extraction result that generalizes the standard full extraction result of CM to richer type spaces, where agents’ beliefs about their opponents do not fully determine their private information (more specifically, their payoff relevant private information).

We provide one example to illustrate that the richness of the type space matters for Optimal Revenue extraction. We consider a class of models that entail the same marginal distributions over valuation levels, but have different distribution over interim beliefs. By introducing richness into the model we reduce the amount of revenue extracted since the residual private information of valuation generates information rents to the bidders.

Beyond this, we prove a suitable weakening of the results that would hold when condition (LI) is not satisfied. We provide an example to show that the results do not carry to the more general situation and to how things change. In the example, the result breaks down because private information regarding beliefs also generates information rents to the bidders.

All of the results would hold if agents had continuously distributed valuations for the good. Nonetheless, in this case the condition (LI) would still require that the set of possible interim beliefs is finite and linearly independent. The finiteness property would be true whenever each agent observes a signal that is informative about his opponents’, but has a finite support.

Obviously, we have restricted attention to the problem of auction design in the presence of rich type spaces, and it remains a question of interest how the insights delineated in this paper would carry over to different (and more general) problems in mechanism design. The main property of this problem that is critical for the results is the linearity of the objective of the mechanism designer with respect to the mechanism (in this case it is just a sum of
transfers). The analysis would go through in alternative situations that satisfy this property. We do observe that this is a restrictive property, which suggests that different approaches might be needed to alternative problems.

Another next step is to answer the question of whether the auctioneer could find a mechanism that overperforms the best dominant strategy mechanism in a robust way, but considering all possible type spaces with common prior (contrasting with Chung and Ely (2007) Chung and Ely (2007)'s result, that hinges on the use of non common prior type spaces).

8 Appendix

Proof of Lemma (2). All the arguments are standard. First fix a player \( i \) and a belief \( \beta_i \in \tilde{\beta}_i(T_i) \). Consider possible types \((v^k_i, \beta_i)\) and \((v^{k-1}_i, \beta_i)\), then the incentive constraint of \( t_i \) announcing to be \( t'_i \) and the reverse one imply the first inequality.

Define also
\[
\chi_i(t_i) \equiv \int \xi_i^*(t_i, t_{-i}) \, d\beta_i(t_{-i})
\]
and
\[
\tau_i(t_i) \equiv \int \pi_i^*(t_i, t_{-i}) \, d\beta_i(t_{-i})
\]
Now suppose that for some \( k \) such that \( 2 \leq k \leq K_i \),
\[
\int \left[ \xi_i^* \left( v_i^k, \beta_i, t_{-i} \right) v_i^k - \pi_i^* \left( v_i^k, \beta_i, t_{-i} \right) \right] \, d\beta_i(t_{-i})
- \int \left[ \xi_i^* \left( v_i^{k-1}, \beta_i, t_{-i} \right) v_i^k - \pi_i^* \left( v_i^{k-1}, \beta_i, t_{-i} \right) \right] \, d\beta_i(t_{-i}) = \Delta > 0.
\]
Then we could change the mechanism by (with abuse of notation) \( \pi_i^*(t_i) = \pi_i^*(t_i) + \Delta \) for all types \( t_i = (v_i, \beta_i) \) with \( v_i \geq v_i^k \), and unchanged otherwise.

This new mechanism clearly generates more revenue. It also satisfies (PIC). First notice that in the new mechanism all the incentive compatibilities involving \( v_i^k \) and \( v_i^{k-1} \) remain unaltered, except for the one that involves \( v_i^k \) pretending to be \( v_i^{k-1} \), which now holds with
equality. Now arbitrary types \((v^k_a, \beta_i)\) and \((v^k_i, \beta_i)\) (for some \(a > 1\), then

\[
\begin{align*}
\chi_i(t_i) v^k_i - \tau_i(t_i) = \left[ \chi_i(t') v^k_i - \tau_i(t') \right] &= \\
\chi_i(t_i) v^k_i - \tau_i(t_i) = \left[ \chi_i(t^{k+a-1}_i) v^k_i - \tau_i(t^{k+a-1}_i) \right] &+ \\
\sum_{s=1}^{a-2} \left\{ \chi_i(t^{k+s+1}_i) v^k_i - \tau_i(t^{k+s+1}_i) - \left[ \chi_i(t^{k+s}_i) v^k_i - \tau_i(t^{k+s}_i) \right] \right\} &+ \\
\chi_i(t^{k+1}_i) v^k_i - \tau_i(t^{k+1}_i) - \left[ \chi_i(t^k_i) v^k_i - \tau_i(t^k_i) \right] &\leq \\
\chi_i(t_i) v^k_i - \tau_i(t_i) = \left[ \chi_i(t^{k+a-1}_i) v^k_i - \tau_i(t^{k+a-1}_i) \right] &+ \\
\sum_{s=1}^{a-2} \left\{ \chi_i(t^{k+s+1}_i) v^k_i - \tau_i(t^{k+s+1}_i) - \left[ \chi_i(t^{k+s}_i) v^k_i - \tau_i(t^{k+s}_i) \right] \right\} &+ \\
\chi_i(t^{k+1}_i) v^k_i - \tau_i(t^{k+1}_i) - \left[ \chi_i(t^k_i) v^k_i - \tau_i(t^k_i) \right] \leq 0,
\end{align*}
\]

where the first inequality comes from the first monotonicity property proved and the second one comes from all the "local" incentive compatibility constraints. A similar argument applies to deviation towards higher valuation types. Individual rationality is implied by incentive compatibility and the individual rationality constraint for type \(v^k_1\), which is not altered.

An analogous argument holds if it is the case that

\[
\int \left[ \xi^*_i(v^1_i, \beta, t_{-i}) v^1_i - \pi^*_i(v^1_i, \beta, t_{-i}) \right] d\beta_i(t_{-i}) = \Delta > 0,
\]

in which case one could increase the payments charged to all types with belief \(\beta_i\) by \(\Delta\).

The sufficiency of these conditions follow similar steps as the ones presented above. \(\square\)

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**Proof of Lemma (5).** As argued before, we can assume that the mechanism only depends on \(t\) through \(\bar{v}(t)\) and \(\bar{\beta}(t)\). Then we will represent the product allocation through a function \(q\) so that \(\xi(t) = q\left(\bar{v}(t), \bar{\beta}(t)\right)\). Using the necessary conditions stated in Corollary (1), we can rewrite the problem of maximizing expected surplus as

\[
\max_{q:V \times \beta(T) \to [0,1]^I} \sum_{v, \beta} \mathbb{P} \left( \bar{v} = v, \bar{\beta} = \beta \right) \left[ \sum_{i=1}^{l} \left[ V^q_{i v, \beta} \right] q_i(v, \beta) \right],
\]

\(^{11}\)The argument is the standard one through which virtual surplus is used in standard auction theory.
subject to
\[
\sum_{v_{-i},\beta_{-i}} q_i (v_i^k, \beta_i, v_{-i}, \beta_{-i}) \beta_i (v_{-i}, \beta_{-i}) \geq \sum_{v_{-i},\beta_{-i}} q_i (v_i^{k-1}, \beta_i, v_{-i}, \beta_{-i}) \beta_i (v_{-i}, \beta_{-i}) , \text{ for all } k \geq 2, \beta_i
\] (9)
and
\[
\sum_i q_i (v, \beta) \leq 1, \text{ for all } v, \beta.
\]

Where \( V_i^{v_i,\beta_i} \) is the virtual valuation for agent \( i \) with belief \( \beta_i \) and valuation \( v_i \). Virtual valuation is defined as follows:
\[
V_i^{v_i,\beta_i} = v_i^k - \frac{1 - F_i^{v_i,\beta_i}}{p_i^{v_i,\beta_i}} (v_i^{k+1} - v_i^k),
\]
where \( F_i^{v_i,\beta_i} \equiv \mathbb{P} (\bar{v}_i \leq v_i^k | \bar{\beta}_i = \beta_i) \) and \( p_i^{v_i,\beta_i} \equiv \mathbb{P} (\bar{v}_i = v_i^k | \bar{\beta}_i = \beta_i) \). We also define \( V_i^{v_i,\beta_i} = 0 \) whenever \( p_i^{v_i,\beta_i} = 0 \) (we introduce this for completeness, even though the exact level does not matter).

This is clearly a convex problem, and we know that the Slater condition is satisfied, so a necessary and sufficient condition for the solution is the existence of multipliers \( \mu = (\mu_i,k,\beta_i)_{i,k,\beta_i}, \phi = (\phi(v,\beta))_{v,\beta} \geq 0 \) and allocation choice \( q_i \) such that for all \( i,k < K_i \) and \( \beta_i \)
\[
V_i^{k,\beta_i} \mathbb{P} \left( \bar{v} = (v_i^k, v_{-i}), \bar{\beta} = \beta \right) + \beta_i (v_{-i}, \beta_{-i}) [\mu_i,k,\beta_i - \mu_i,k+1,\beta_i] - \phi (v, \beta) \begin{cases} \leq 0 , & \text{if } q_i \leq 0; \\ = 0 , & \text{if } q_i > 0, \end{cases}
\]
such that for \( k = K_i \),
\[
V_i^{k,\beta_i} \mathbb{P} \left( \bar{v} = (v_i^k, v_{-i}), \bar{\beta} = \beta \right) + \beta_i (v_{-i}, \beta_{-i}) \mu_i,k,\beta_i - \phi (v, \beta) \begin{cases} \leq 0 , & \text{if } q_i \leq 0; \\ = 0 , & \text{if } q_i > 0, \end{cases}
\]
also that the monotonicity condition \( \bigcirc \) is satisfied (with equality whenever \( \mu_i,k,\beta_i > 0 \) ) and finally that
\[
\phi (v, \beta) \left[ \sum_i q_i (v, \beta) - 1 \right] = 0, \text{ for all } v, \beta.
\]

Now define \( \Lambda_i,k,\beta_i = V_i^{k,\beta_i} \mathbb{P} \left( \bar{v} = (v_i^k, v_{-i}), \bar{\beta} = \beta \right) + \beta_i (v_{-i}, \beta_{-i}) [\mu_i,k,\beta_i - \mu_i,k+1,\beta_i] \), for \( k < K_i \), and \( \Lambda_i,K_i,\beta_i = V_i^{K_i,\beta_i} \mathbb{P} \left( \bar{v} = (v_i^{K_i}, v_{-i}), \bar{\beta} = \beta \right) + \beta_i (v_{-i}, \beta_{-i}) \mu_i,K_i,\beta_i \). So notice that
if the monotonicity constraints do not bind, then we have that \( \Lambda_{i,k,\beta_i}^{\Lambda_{i,k,\beta_i}} = V_{i,k,\beta_i} \), so these values reduce to multiple of the virtual valuations as defined before.

We can directly see that

\[
\phi(v, \beta) = \max \left\{ \max_i V_{i,k,\beta_i}, 0 \right\}.
\]

So that the good is allocated only to people for which \( \Lambda_{i,k,\beta_i} = \phi(v, \beta) \). So, up to a redefinition of \( \Lambda \), we know that, for each \( i, \beta_i \),

\[
\Lambda_{i,k,\beta_i} \geq \Lambda_{i,k-1,\beta_i}, \text{ for all } k \geq 2.
\]

Given the multipliers above, define allocation \( \bar{q} \) given by

\[
\bar{q}_i(v, \beta) = \begin{cases} 
\frac{1}{\# \arg \max_j \Lambda_{j,v_j,\beta_j}}, & \text{if } i \in \arg \max_j \Lambda_{j,v_j,\beta_j} \text{ and } \max_j \Lambda_{j,v_j,\beta_j} > 0; \\
0, & \text{otherwise.}
\end{cases}
\]

Where we define \( \Lambda_{j,v_j,\beta_j} = \Lambda_{j,k,\beta_j} \) whenever \( v_j = v_j^k \).

So first notice that, since \( \Lambda_{i,k,\beta_i} \) is nondecreasing in \( k \), the allocation is increasing in valuation, for any given \( v_{-i} \) and \( \beta \). Second, and most importantly, notice that \( \bar{q} \) and the original multipliers \( (\mu, \phi) \) satisfy all the necessary conditions for a maximum. Therefore this is an optimal allocation. \( \square \)

References


