Design-Adaptive Nonparametric Estimation of Conditional Quantile Derivatives

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Abstract

This paper proposes a new approach to constructing nonparametric estimators of conditional quantile functions and their derivatives with respect to conditioning variables. The new approach is intended specifically to produce estimators with biases that do not depend on the design density. This is in marked contrast to more conventional nonparametric estimators based on locally polynomial quantile regressions, the biases of which are characterised by asymptotic expansions in which the design density appears, at least at some order of approximation. The specific approach taken in this paper involves the kernel smoothing of the ratio of a preliminary nonparametric estimate of the conditional quantile function to another preliminary nonparametric estimate of the design density. Monte Carlo evidence indicates that the proposed estimators compare favourably to nonparametric estimators based on local polynomials. An empirical example exploring the relationship between individual earnings and age is also included.

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Introduction

Suppose that \{\((X_1, Y_1) : i = 1, \ldots, n\)\} is a random sample from a population of random vectors \((X, Y)\), where \(X\) and \(Y\) are both scalar valued. For a given \(\alpha \in (0, 1)\) and a given value \(x\) in the support of \(X\), let \(q(\alpha, x)\) denote the conditional \(\alpha\)-quantile of \(Y\). This paper considers the estimation of \(q(\alpha, x)\) and its derivatives with respect to \(x\). The approach taken in this paper is nonparametric, and as such is expected to be of use in applied situations when the restrictions imposed by more computationally convenient parametric approaches are not taken to be appropriate.

Direct models of conditional quantiles have taken on an important role in the statistical sciences. In general, they offer researchers the possibility of being able to engage in a systematic analysis of the effects of a set of conditioning variables on all aspects of the distribution of a response variable. In particular, a researcher by varying the quantile index \(\alpha\) can examine the specific effects of covariates on any point of the distribution of a variable of interest. Thus the differential effects of unionisation on the incomes of relatively high-wage and low-wage workers can be assessed by examining various quantiles of income \((Y)\) as functions of an indicator variable for union membership \((x)\). Similarly, in survival analysis the differential effect of some intervention \((x)\) on survival time \((Y)\) can be analysed separately for low-risk and high-risk individuals by constructing estimates of the conditional quantile function of \(Y\) given \(x\) for various quantiles.

Koenker and Bassett [1978] pioneered the application of linear models of conditional quantiles, thus generalizing conventional linear regression models identified by a median restriction. This has proven to be popular in recent years, particularly amongst econometricians (e.g., Koenker and Hallock 2001, Koenker 2005 and references cited therein). Nonparametric models of conditional quantiles, on the other hand, have an arguably longer history of discussion in the statistical literature. Relevant works in this connection include the contributions of Bhattacharya [1963], Stone [1977], Chaudhuri [1991], Fan et al [1994], Koenker et al [1994] and Yu and Jones [1998]. A comprehensive review can be found in Koenker [2005].

This paper proposes a new nonparametric approach to estimating conditional quantiles and their derivatives with respect to covariates. In particular, for a given quantile index \(\alpha \in (0, 1)\) and covariate value \(x\), the specific approach taken here involves the kernel smoothing about \(x\) of preliminary nonparametric estimates of \(q(\alpha, X_i) / g(X_i)\), where \(i \in \{1, \ldots, n\}\), and where \(g(\cdot)\) denotes the marginal density of the conditioning variable. The specific kernel function used to smooth
\{q(\alpha, X_i) / g(X_i) : i = 1, \ldots, n\} depends on which \(x\)-derivative of \(q(\alpha, x)\) is of interest for the specific empirical problem being considered.

The estimators proposed in this paper bear a passing resemblance to estimates of conditional quantile derivatives that are constructed by taking a weighted average of a nonparametric estimate of a conditional quantile derivative over the support of the conditioning variable (e.g., Chaudhuri et al. 1997, Ma and Koenker 2006, Ichimura and Lee 2010). This approach is generally appropriate for semiparametric single-index models of conditional quantiles given vector-valued covariates. A critical difference between “average derivative” estimators of this sort and the estimators proposed in this paper concerns the extent of the averaging involved over the design space. Average derivative estimators involve the computation of a global average over all sample observations and enjoy a parametric (i.e., \(n^{-1/2}\)) rate of convergence under some conditions. The estimators proposed here, on the other hand, are essentially local averages over shrinking neighbourhoods of a given point of interest \(x\) in the design space and converge at nonparametric rates.

This paper is related to the seminal contribution of Stone [1977], which considered nearest-neighbours methods in the context of locally constant nonparametric estimators, and also to Chaudhuri [1991], which introduced locally polynomial estimates incorporating uniform kernels. The class of estimator of conditional quantiles and their derivatives with respect to conditioning variables proposed below has been designed specifically for improved bias properties over the more conventional nonparametric estimators based on locally polynomial quantile regressions, which include locally constant quantile regressions as a special case. In particular, the nonparametric estimators introduced here have biases that do not depend on the design density or on any of its derivatives at any term of the corresponding asymptotic expansions. This is a property that is not shared by locally polynomial estimators of \(q(\alpha, x)\) and of its derivatives with respect to \(x\).

In this connection, note that a locally linear estimator of \(q(\alpha, x)\) constructed with a symmetric density kernel having finite moments to at least sixth order and implemented using a bandwidth \(h_q\) can in fact only be described as “weakly” asymptotically design adaptive. Note that while the largest-order terms of the corresponding asymptotic bias expansion (i.e., those terms of order \(h_q^2\)) do not involve the value of the design density \(g(x)\), the terms of next largest order (i.e., \(h_q^4\)) do in fact depend on \(g(x)\) and on the values \(g^{(1)}(x)\) and \(g^{(2)}(x)\) of its first two derivatives. This is a property that is also exhibited by the asymptotic bias of a locally linear estimator of a conditional mean function [e.g.,
Fan, 1992] implemented under the same restrictions on the kernel function. Locally linear estimators of the derivative $\frac{d}{dx} q(\alpha, x)$, on the other hand, are not even asymptotically design adaptive in the weak sense described above. In this case, $g(x)$ and its derivatives appear in every term of the corresponding asymptotic bias expansion. Further details appear below in Section 3.

The “strong” asymptotic design adaptiveness of the estimators proposed here is expected to be particularly advantageous in the construction of estimates of $q(\alpha, x)$ or its derivatives over large ranges of $x$. In this case, the estimators proposed in this paper are expected to exhibit smaller biases in small samples than estimators based on local polynomials, particularly for values of $x$ located in relatively sparse regions of the design distribution.

The remainder of this paper proceeds as follows. The class of estimator proposed here is described in somewhat intuitive terms in Section 2. The main asymptotic results are described in Section 3. Section 4 presents the results of a series of simulation experiments designed to assess the finite-sample performance of the proposed estimators, while Section 5 describes an application of the estimators developed below to models of the relationship between various quantiles of individual earnings and age. Section 6 concludes, and is followed by an appendix, which contains both a detailed discussion of the regularity conditions assumed to underlie the three theoretical results given in Section 3 and the actual proofs of those results.

2 The estimator

Consider $q(\alpha, x) = F_{Y|X=x}^{-1}(\alpha)$, i.e., the conditional $\alpha$-quantile of $Y$ given $X = x$. Let $\nu$ denote a non-negative integer, and let $D_x^{\nu}$ denote the differential operator $\frac{d^{\nu}}{dx^{\nu}}$.

This paper considers the problem of estimating $D_x^{\nu} q(\alpha, x)$ for a given $\alpha \in \mathcal{A}$ and $x \in \mathcal{X}$, where $\mathcal{A}$ denotes a compact subset of $(0, 1)$ and $\mathcal{X} \subseteq \mathbb{R}$ is a compact set contained within the interior of the support of the conditioning variable $X$. Although the focus in this paper is squarely on the case where the conditioning variable $X$ is scalar valued, it should be noted that an extension to the case of vector-valued covariates is straightforward. In practice, however, the number of dimensions in the design space that can usefully be considered is limited by the curse of dimensionality.

Let $g(\cdot)$ denote the marginal density of $X$. This paper proposes to estimate
\[ D_x^q q(\alpha, x) \] using

\[ \hat{\theta}_n(\alpha, x) \equiv \frac{1}{n} \sum_{i=1}^{n} D_x^q K_h (X_i - x) \hat{q}_n^*(\alpha, X_i), \tag{1} \]

where \( K_h(\cdot) \equiv K\left( \frac{\cdot}{h} \right) \) denotes a smoothing kernel, \( h \to 0 \) the associated bandwidth and \( \hat{q}_n^*(\alpha, x) \) denotes a particular nonparametric estimator of \( q(\alpha, x)/g(x) \) given below in (6).

In order to motivate the proposed estimator of \( D_x^q q(\alpha, x) \) given above in (1), pretend momentarily that the functional forms of both \( g(\cdot) \) and \( q(\alpha, \cdot) \) are known for each \( \alpha \in A \). In this case, one can consider an infeasible kernel “estimate” of \( D_x^q q(\alpha, x) \) given by \( D_x^q (nh)^{-1} \sum_{i=1}^{n} K(h^{-1}(X_i - x)) q(\alpha, X_i) / g(X_i) \). Provided that \( \int u^2 K(u) du < \infty \) and certain other regularity conditions are met, we have as \( n \to \infty \) the asymptotic representation

\[ D_x^q \frac{1}{nh} \sum_{i=1}^{n} K(\cdot) q(\alpha, X_i) g(X_i) \]

\[ = \frac{1}{h} E \left[ D_x^q K(\cdot) q(\alpha, X_1) \right] + o_p(1) \]

\[ = D_x^q q(\alpha, x) + o_p(1). \tag{3} \]

The design density \( g(\cdot) \) is cancelled out in (3) and as such doesn’t appear in (4). This indicates that the infeasible estimator of \( D_x^q q(\alpha, x) \) given by (2) is “design adaptive” in the sense that its bias is unaffected to any order of approximation by the value of \( g(x) \). This property stands in marked contrast to nonparametric estimators of \( D_x^q q(\alpha, x) \) based on local polynomials, which in general have biases depending on \( g(x) \) and its derivatives, at least to some order of approximation—some details appear below in Section 3.

The proposed estimator of \( D_x^q q(\alpha, x) \) given above by \( \hat{\theta}_n(\alpha, x) \) in (1) simply involves the replacement of \( q(\alpha, \cdot) \) and \( g(\cdot) \) in (2) with appropriate estimates to arrive at the preliminary estimates \( \hat{q}_n^*(\alpha, X_i) \) \((i = 1, \ldots, n)\) embedded in the definition of \( \hat{\theta}_n(\alpha, x) \). The construction of these preliminary estimates is described as follows.

Let \( \kappa(\cdot) \) denote a smoothing kernel satisfying \( \int \kappa(u) du = 1, \int u^2 \kappa(u) du < \infty \) and other conditions set out in Section 3, and let \( h_g \to 0 \) denote the associated bandwidth. Define \( \hat{g}_n(x) \equiv (nh_g)^{-1} \sum_{i=1}^{n} \kappa(h_g^{-1}(X_i - x)) \), i.e., a Rosenblatt-Parzen estimator of the design density evaluated at \( x \). For each \( i \in \{1, \ldots, n\} \),
let

\[ Y_i^* \equiv \frac{Y_i}{\hat{g}_n(X_i)}. \] (5)

Define another, possibly different, smoothing kernel \( k(\cdot) \) also satisfying \( \int k(u)du = 1 \) and certain other restrictions. Let \( h_q \to 0 \) denote the associated bandwidth. For each \( \alpha \in (0, 1) \), define the so-called “check” function \( \rho_\alpha(u) \equiv u(\alpha - 1 \{ u < 0 \}) \) as in Koenker and Bassett [1978]. For \( (\alpha, x) \in \mathcal{A} \times \mathcal{X} \), the preliminary estimate \( \hat{q}_n^*(\alpha, x) \) embedded in the definition of \( \hat{\theta}_n(\alpha, x) \) in (1) is defined to satisfy

\[
\begin{bmatrix}
\hat{q}_n^*(\alpha, x) \\
\hat{q}_n^*(\alpha, x)
\end{bmatrix}
\]

\[
\equiv \arg \min_{(q_0, q_1) \in \mathbb{R}^2} \sum_{i=1}^{n} \rho_\alpha \left( Y_i^* - q_0 - q_1 \cdot (X_i - x) \right) k \left( h_q^{-1} (X_i - x) \right) \tau_n \left( \hat{g}_n (X_i) \right),
\] (6)

where \( \tau_n(\cdot) \) denotes the trimming function given below in (7). In this connection, note that the random denominator \( \hat{g}_n(X_i) \) in (5) may be small and lead to technical difficulties with the analysis of the behaviour of the estimator given in (6). For this reason, the device of ignoring observations corresponding to overly small values of \( \hat{g}_n(X_i) \) is adopted here.

In particular, the derivation of the main results given below in Section 3 will in fact often require the finiteness of certain expectations of fractional quantities involving powers of \( g(X_1) \) in the denominator. For this reason the sort of trimming scheme described here is analytically convenient in general, if not indeed necessary in every case. In specific terms, a variant of the “smoothed” trimming scheme considered by Andrews [1995] and Ai [1997] is adopted. The trimming scheme used here involves a constant \( \zeta > 0 \), and the trimming function given by

\[
\tau_n(u) \equiv \begin{cases}
1 & , \quad u \geq 2n^{-\zeta} \\
0 & , \quad u \leq n^{-\zeta} \\
\bar{\tau}_n(u) & , \quad u \in (n^{-\zeta}, 2n^{-\zeta})
\end{cases}
\] (7)

where \( \bar{\tau}_n : (0, \infty) \to [0, 1] \) is a twice-differentiable distribution function. A suitable example in this connection would involve setting the function \( \bar{\tau}_n(u) \) to be a Beta distribution function, i.e., \( \bar{\tau}_n(u) = n^k \int_{-\infty}^{u} \sigma \left( n^k t - 1 \right) dt \), whenever \( u \in (n^{-\zeta}, 2n^{-\zeta}) \), and where for some integer \( k \):

\[
\sigma(u) = \begin{cases}
\frac{\Gamma(2(k+1))}{\Gamma^2(k+1)} w^k(1-w)^k & , \quad w \in [0, 1] \\
0 & , \quad \text{otherwise}
\end{cases}
\]
The next section of the paper presents the asymptotic properties of the proposed estimator \( \hat{\theta}_n(\alpha, x) \). For each \((\alpha, x) \in A \times \mathcal{X}\), the estimator is shown to be consistent for \( D_x^\nu q(\alpha, x) \) as well as asymptotically normal.

3 Main results

The main asymptotic results concerning the sampling behaviour of \( \hat{q}_n^*(\alpha, x) \) and \( \hat{\theta}_n(\alpha, x) \) are given as follows. Recall that the estimand is \( D_x^\nu q(\alpha, x) \), where \( D_x^\nu \) denotes the differential operator \( \frac{d^\nu}{dx^\nu} \) for some non-negative integer \( \nu \). It is assumed that \((\alpha, x) \in A \times \mathcal{X}\), where \( A \) denotes a compact subset of \((0, 1)\) and \( \mathcal{X} \) denotes a compact set contained within the interior of the support of the conditioning variable \( X \). Let \( g(\cdot) \) denote the marginal density of \( X \).

*Proposition 1.* Suppose the conditions of Assumption 1; part (3) of Assumption 2, parts (1) and (2) of Assumption 3, parts (1) and (2) of Assumption 4 and parts 1–3 of Assumption 5 given below in Appendix A hold. Then the estimator \( \hat{q}_n^*(\alpha, x) \) given above in (6) satisfies
\[
\sqrt{nh_q} \left( \hat{q}_n^*(\alpha, x) - \frac{q(\alpha, x)}{g(x)} \right) = O_p(1)
\]
for each \((\alpha, x) \in A \times \mathcal{X}\) as \( n \to \infty \).

The proof of Proposition 1 follows a generalisation of the analytical approach used by Pollard [1991], Hjørt and Pollard [1993] and Knight [1998], amongst others, to deduce the limiting distribution of \( M \)-estimators involving non-smooth criterion functions. In particular, Proposition 1 is deduced from the asymptotic behaviour of the normalised and re-centred quantile-regression objective function minimised by the preliminary estimators \( \hat{q}_n^*(\alpha, x) \) and \( \hat{q}_{n1}^*(\alpha, x) \) given above in (6).

In this connection, note that for \( \delta \equiv (\delta_0, \delta_1)^\top \); \( q_y^{(1)}(\alpha, x) \equiv \frac{d}{dx}[q(\alpha, x)/g(x)] \)
and \(q^2_b(\alpha, x) \equiv \frac{d^2}{dx^2} \left[ q(\alpha, x)/g(x) \right] \) that the function

\[
Z_n(\delta) \equiv \sum_{i=1}^{n} \left( \rho_{\alpha} \left( Y_i^* - \frac{q(\alpha, x)}{g(x)} - (X_i - x) \cdot q_g^{(1)}(\alpha, x) \right) - \frac{1}{\sqrt{nh_q}} \left( \delta_0 + h_q^{-1} (X_i - x) \cdot \delta_1 \right) \right)
\]

\[
- \rho_{\alpha} \left( Y_i^* - \frac{q(\alpha, x)}{g(x)} - (X_i - x) \cdot q_g^{(1)}(\alpha, x) \right)
\]

\[
\cdot k \left( h_q^{-1} (X_i - x) \right) \tau_n (\hat{g}_n (X_i))
\]

is convex and minimised for each \((\alpha, x) \in A \times \mathcal{X}\) at

\[
\begin{bmatrix}
\hat{\delta}_{n0}(\alpha, x) \\
\hat{\delta}_{n1}(\alpha, x)
\end{bmatrix} = \sqrt{nh_q} \begin{bmatrix}
\hat{q}^*_n(\alpha, x) - \frac{q(\alpha, x)}{g(x)} \\
h_q \left( \hat{q}^*_n(\alpha, x) - q_g^{(1)}(\alpha, x) \right)
\end{bmatrix}.
\] (8)

The asymptotic behaviour of \(\hat{q}^*_n(\alpha, x)\) can be deduced from the asymptotic behaviour of the criterion function \(Z_n(\delta)\) for each fixed \(\delta \in \mathbb{R}^2\). It is shown in the online supplement that for each fixed \(\delta\), \(Z_n(\delta)\) converges weakly to a limiting function \(Z_\infty(\delta)\) that is convex in \(\delta\). This in turn implies the uniqueness of its minimiser, and the asymptotic normality of the estimator \(\hat{\delta}_{n0}(\alpha, x)\) given above in (8).

The asymptotic normality of \(\hat{\delta}_{n0}(\alpha, x)\) for each \((\alpha, x) \in A \times \mathcal{X}\) yields the conclusion of Proposition 1. This in turn is a critical input into the following results regarding the asymptotic behaviour of \(\hat{\theta}_n(\alpha, x)\).

**Theorem 1.** Suppose the conditions of Assumptions 1–3; parts 1, 2, and 3(a) of Assumption 4 and parts 1, 2, 3 and 4(a) of Assumption 5 given below in Appendix A hold. Then the estimator \(\hat{\theta}_n(\alpha, x)\) given above in (1) satisfies

\[
\hat{\theta}_n(\alpha, x) = D^\nu_{\alpha} q(\alpha, x) + o_p(1)
\]

for each \((\alpha, x) \in A \times \mathcal{X}\) as \(n \to \infty\).

**Theorem 2.** Suppose the conditions of Assumptions 1–5 given below in Appendix A hold.
1. Define $K(\nu)(u) \equiv \frac{d^\nu}{du^\nu} K(u)$. The estimator $\hat{\theta}_n(\alpha, x)$ given above in (1) satisfies

$$
\sqrt{nh^{1+2\nu}} \left( \hat{\theta}_n(\alpha, x) - D^\nu_x q(\alpha, x) \right) \xrightarrow{d} N \left( 0, \frac{q^2(\alpha, x)}{g(x)} \int (K(\nu)(u))^2 du \right)
$$

for each $(\alpha, x) \in A \times X$.

2. In addition, provided that $\sup_{\alpha \in A} \sup_{x \in X} |D^\nu_x q(\alpha, x)| < \infty$, then $\hat{\theta}_n(\alpha, x)$ has a bias for fixed $(\alpha, x) \in A \times X$ given by

$$
E \left[ \hat{\theta}_n(\alpha, x) \right] - D^\nu_x q(\alpha, x) = \frac{h^2}{2} \int t^2 K(t_1) \, dt_1 \cdot D^{\nu+2}_x q(\alpha, x) + \frac{h^4}{24} \int t^4 K(t_1) \, dt_1 \cdot D^{\nu+4}_x q(\alpha, x) + O(h^6).
$$

**Remark 1.** The approximate bias of $\hat{\theta}_n(\alpha, x)$ in large samples is seen under the conditions of Theorem 2 not to depend on the value of the design density at $x$ nor on any of its derivatives. The estimator $\hat{\theta}_n(\alpha, x)$ tends to exhibit the greatest degree of bias when the conditional quantile function $q(\alpha, x)$ exhibits a high degree of curvature on $X$. On the other hand, if $q(\alpha, x)$ is linear in the components of the design vector, then $\hat{\theta}_n(\alpha, x)$ is unbiased.

**Remark 2.** Suppose there is interest in both the conditional quantile $q(\alpha, x)$ ($\nu = 0$) and the first derivative of $q(\alpha, x)$ ($\nu = 1$) with respect to $x$. In these cases it is instructive to compare the biases of the proposed estimator $\hat{\theta}_n(\alpha, x)$ when applied to the estimation of both $q(\alpha, x)$ and $\frac{d}{dx} q(\alpha, x)$ with those of estimators of $q(\alpha, x)$ and $\frac{d}{dx} q(\alpha, x)$ based on locally linear $\alpha$-quantile regression.

Consider the locally linear $\alpha$-quantile regression estimators given by

$$
\begin{bmatrix}
\hat{q}_n(\alpha, x) \\
\hat{q}_{n1}(\alpha, x)
\end{bmatrix} = \arg \min_{(q_0, q_1) \in \mathbb{R}^2} \sum_{i=1}^n \rho_\alpha(Y_i - q_0 - q_1 \cdot (X_i - x)) k \left( h^{-1}_q (X_i - x) \right), \quad (9)
$$
where \( k(\cdot) \) and \( h_q \to 0 \) satisfy the conditions of Assumptions 4 and 5, respectively. Also suppose that the conditional density function \( f_{Y_1 \mid X_1}(\cdot) \), quantile function \( q(\alpha, x) \) and design density \( g(x) \) satisfy the various conditions of Proposition 1. A modification of the proof of Proposition 1 then yields the following asymptotic expansion for the bias of the locally linear quantile estimate \( \hat{q}_n(\alpha, x) \):

\[
E [q_n(\alpha, x)] - q(\alpha, x) = \frac{h_q^2}{2} q^{(2)}(\alpha, x) \cdot \int t^2 k(t_1) \, dt_1 \int k^2(t_1) \, dt_1 \\
+ \frac{h_q^4}{2} f_{Y_1 \mid X_1}(q(\alpha, x)) \left\{ \frac{1}{2} q^{(2)}(\alpha, x) \left( f^{(2)}_{Y_1 \mid X_1}(q(\alpha, x)) (q^{(1)}(\alpha, x))^2 \\
+ f^{(1)}_{Y_1 \mid X_1}(q(\alpha, x)) q^{(2)}(\alpha, x) \right) + \gamma_1 q^{(3)}(\alpha, x) f^{(1)}_{Y_1 \mid X_1}(q(\alpha, x)) q^{(1)}(\alpha, x) \right. \\
+ \left. \frac{1}{2} \gamma_2 q^{(4)}(\alpha, x) f_{Y_1 \mid X_1}(q(\alpha, x)) - \frac{1}{2} (q^{(2)}(\alpha, x))^2 f^{(1)}_{Y_1 \mid X_1}(q(\alpha, x)) \right. \\
+ \left. \frac{g^{(1)}(x)}{g(x)} \left[ q^{(2)}(\alpha, x) f^{(1)}_{Y_1 \mid X_1}(q(\alpha, x)) q^{(1)}(\alpha, x) + \gamma_1 q^{(3)}(\alpha, x) f_{Y_1 \mid X_1}(q(\alpha, x)) \right] \\
+ \frac{g^{(2)}(x)}{g(x)} \cdot \frac{1}{2} q^{(2)}(\alpha, x) f_{Y_1 \mid X_1}(q(\alpha, x)) \right} \cdot \int t^4 k(t_1) \int k^2(t_1) \, dt_1 \\
+ O(h_q^4),
\]

(10)

where \( q^{(j)}(\alpha, x) = \frac{\partial^j}{\partial x^j} q(\alpha, x) \) and \( \gamma_1 \in (0, 1) \) is some constant. A high degree of curvature in \( q(\alpha, x) \) on \( X \) is immediately seen to have an inflationary effect on the small-sample bias of \( \hat{q}_n(\alpha, x) \). It is also clear that the design density only starts to affect the bias of \( \hat{q}_n(\alpha, x) \) at terms of order \( h_q^4 \). Here it is seen that large values of \( g^{(1)}(x)/g(x) \) or of \( g^{(2)}(x)/g(x) \) can cause to \( \hat{q}_n(\alpha, x) \) to exhibit a relatively large bias in finite samples. On the other hand, the leading term of the asymptotic bias expansion is indeed free of \( g(x) \).

In contrast, the proposed estimate \( \hat{q}_n(\alpha, x) \) of \( q(\alpha, x) \) has a bias under the conditions of Theorem 2 given by

\[
E [\hat{q}_n(\alpha, x)] - q(\alpha, x) = \frac{h_q^2}{2} q^{(2)}(\alpha, x) \int t^2 K(t_1) \, dt_1 + \frac{h_q^4}{24} q^{(4)}(\alpha, x) \int t^4 K(t_1) \, dt_1 \\
+ O(h_q^6).
\]

(11)

From this it is clear that the finite sample bias of \( \hat{q}_n(\alpha, x) \) for \( q(\alpha, x) \) is affected by the curvature of \( q(\alpha, x) \) on \( X \) but is unaffected by the values of \( g^{(1)}(x)/g(x) \) or of
\(g^{(2)}(x)/g(x)\). It is also clear that the leading terms of the bias expansions given in (10) and (11) are asymptotically equivalent when the bandwidths used satisfy \(h_q/h \to C \in (0, \infty)\). As such, the large-sample bias of the proposed estimator \(\hat{\theta}_n(\alpha, x)\) applied to estimation of \(q(\alpha, x)\) is no larger in magnitude than that of the locally linear estimator \(\hat{q}_n(\alpha, x)\) implemented using the same bandwidth.

Now consider the bias of the locally linear derivative estimator \(\hat{q}_{n1}(\alpha, x)\) for
\[
\frac{d}{dx} q(\alpha, x). \quad \text{Under the conditions of Proposition 1 we have}
\]

\[
E[\hat{q}_{n1}(\alpha, x)] - \frac{d}{dx} q(\alpha, x) = \frac{h_n^2}{2} f_{Y_1|X_1=x}^{-1} (q(\alpha, x)) \cdot \left[ \left( \gamma_1 q^{(3)}(\alpha, x) f_{Y_1|X_1=x} (q(\alpha, x)) + \frac{q^{(2)}(\alpha, x)}{q(\alpha, x)} f_{Y_1|X_1=x} (q(\alpha, x)) \right) q^{(1)}(\alpha, x) \right] + \frac{h_n^2}{2} f_{Y_1|X_1=x}^{-1} (q(\alpha, x)) \cdot \left[ \left( \gamma_1 q^{(3)}(\alpha, x) f_{Y_1|X_1=x} (q(\alpha, x)) + \frac{q^{(2)}(\alpha, x)}{q(\alpha, x)} f_{Y_1|X_1=x} (q(\alpha, x)) \right) q^{(1)}(\alpha, x) \right] + \frac{h_n^2}{2} f_{Y_1|X_1=x}^{-1} (q(\alpha, x)) \cdot \left[ \left( \gamma_1 q^{(3)}(\alpha, x) f_{Y_1|X_1=x} (q(\alpha, x)) + \frac{q^{(2)}(\alpha, x)}{q(\alpha, x)} f_{Y_1|X_1=x} (q(\alpha, x)) \right) q^{(1)}(\alpha, x) \right] + \frac{h_n^2}{2} f_{Y_1|X_1=x}^{-1} (q(\alpha, x)) \cdot \left[ \left( \gamma_1 q^{(3)}(\alpha, x) f_{Y_1|X_1=x} (q(\alpha, x)) + \frac{q^{(2)}(\alpha, x)}{q(\alpha, x)} f_{Y_1|X_1=x} (q(\alpha, x)) \right) q^{(1)}(\alpha, x) \right] + \frac{h_n^2}{2} f_{Y_1|X_1=x}^{-1} (q(\alpha, x)) \cdot \left[ \left( \gamma_1 q^{(3)}(\alpha, x) f_{Y_1|X_1=x} (q(\alpha, x)) + \frac{q^{(2)}(\alpha, x)}{q(\alpha, x)} f_{Y_1|X_1=x} (q(\alpha, x)) \right) q^{(1)}(\alpha, x) \right] + \frac{h_n^2}{2} f_{Y_1|X_1=x}^{-1} (q(\alpha, x)) \cdot \left[ \left( \gamma_1 q^{(3)}(\alpha, x) f_{Y_1|X_1=x} (q(\alpha, x)) + \frac{q^{(2)}(\alpha, x)}{q(\alpha, x)} f_{Y_1|X_1=x} (q(\alpha, x)) \right) q^{(1)}(\alpha, x) \right]
\]

where again \( \gamma_1 \in (0, 1) \) is some constant. Here it is seen that the bias of \( \hat{q}_{n1}(\alpha, x) \) is not free of \( g(\cdot) \) at any order of approximation. In contrast, under the conditions of Theorem 2, the proposed procedure \( \theta_n(\alpha, x) \) as applied to estimation of
\[
\frac{d}{dx} q(\alpha, x) \text{ has a bias given by}
\]
\[
E \left[ \hat{\theta}_n (\alpha, x) \right] - q^{(1)}(\alpha, x)
\]
\[
= \frac{h^2}{2} q^{(3)}(\alpha, x) \int t^2 K(t) \, dt + \frac{h^4}{24} q^{(5)}(\alpha, x) \int t^4 K(t) \, dt_1
\]
\[
+ O \left( h^6 \right).
\]

Thus as was the case when applied to estimation of \(q(\alpha, x)\), the magnitude of the finite-sample bias of \(\hat{\theta}_n (\alpha, x)\) in this case largely depends on the curvature of \(q(\alpha, x)\) on \(X\) and is explicitly unaffected by the values of \(g^{(1)}(x)/g(x)\), \(g^{(2)}(x)/g(x)\) or of \(g^{(3)}(x)/g(x)\).

4 Numerical evidence

This section presents the results of simulation experiments evaluating the finite-sample performance of the proposed nonparametric estimator of \(D_x q(\alpha, x)\) relative to a more conventional nonparametric estimator based on locally linear quantile regression. In particular, the simulations consider the problem of estimating the first-order derivatives of a quantile function \(q(\alpha, x)\) with respect to \(x\) at each value of \(x\) lying within a grid of evenly spaced points.

Two simulated data-generating processes were considered. Both models involve simulations of the random vector \((X, U)\), where \(X \sim N(0.5, 1)\), \(U \sim N(0, 1)\) and \(U\) is taken to be independent of \(X\). The first data-generating process is a location-shift model given by

\[
Y = 1 + 4X - 3X^2 + U,
\]

while the second data-generating process incorporates shifts in both location and scale:

\[
Y = 1 + 4X - 3X^2 + \sin(X)U.
\]

The simulations considered the problem of estimating the first derivative of the conditional median function for each of the two models given in (14) and (15). In the case of the location-shift model described in (14), \(q(0.5, x) = \Phi^{-1}(0.5) + 1 + 4x - 3x^2\), with first derivative given by

\[
D_x q(0.5, x) = \frac{d}{dx} q(0.5, x) = 4 - 6x.
\]

\(14\)
With respect to the location-scale model appearing in (15), \( q(.5, x) = \sin(x)\Phi^{-1}(.5) + 1 + 4x - 3x^2 \), and
\[
D_x^1 q(.5, x) = \cos(x)\Phi^{-1}(.5) + 4 - 6x. \tag{17}
\]

For each value of \( x \) appearing in a dense grid of values contained in \([0, 1]\), the simulations compared the sampling behaviour of the proposed estimator \( \hat{\theta}_n(.5, x) \) to that of the estimated slope from the locally linear median regression (LLMR) procedure given above in (9) with \( \alpha = .5 \). The focus was largely on the bias exhibited by the two estimators as \( x \) ranges over the unit interval. The simulations evaluated the finite-sample accuracy of the bias approximation given above in Theorem 2, which states that in large samples the bias of \( \hat{\theta}_n(.5, x) \) depends solely on the curvature of the conditional median function \( q(.5, x) \). The bias of \( \hat{\theta}_n(.5, x) \) for \( x \in [0, 1] \) was compared with that of the LLMR estimator, which from the approximation in (12) depends in a relatively complex way on both the design density—deliberately taken to be nonuniform in these simulations—and on the curvature of \( q(.5, x) \).

The simulations also evaluated the finite-sample accuracy of the bias approximations set out in the remarks following the statement of Theorem 2. From (13) above it is noted that the bias of \( \hat{\theta}_n(.5, x) \) is approximately proportional to the third derivative \( q^{(3)}(.5, x) \) of the conditional median. This derivative is zero in the case of the location-shift model in (14), and proportional to \( \sin(x) \) in the context of the location-scale model in (15). As such, the simulated bias of \( \hat{\theta}_n(.5, x) \) in the case of the location-shift model was expected to be uniformly close to zero for all \( x \in [0, 1] \). On the other hand, the bias of \( \hat{\theta}_n(.5, x) \) in the case of the location-scale model was expected to fluctuate with \( \sin(x) \) as \( x \) ranges over \([0, 1]\).

The comparisons of the performance of \( \hat{\theta}_n(.5, x) \) to that of the corresponding LLMR estimator involved 1000 Monte Carlo replications. Simulated samples \( \{(X_i, Y_i) : i = 1, \ldots, n\} \) of sizes \( n = 100 \) and \( n = 400 \) were considered. The results for samples of size \( n = 400 \) were qualitatively similar to the results for samples with \( n = 100 \) observations, and as such are not reported here to economise on space.

The simulations considered the problem of estimating the quantities in (16) and (17) for each value of \( x \) in a grid of 100 evenly spaced points contained in \([0, 1]\). For \( S_X^2 \equiv \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n - 1) \), the estimator \( \hat{\theta}_n(.5, x) \) was implemented using the following choices of tuning parameters and kernel functions:

- **Design density estimator (i.e., \( \hat{g}_n(x) \)):** Epanechnikov kernel, bandwidth \( h_g = S_X n^{-1/5} \).
• Preliminary quantile estimator (i.e., \( \hat{q}_n^*(.5, x) \)): Third-order Gaussian kernel
  \[ k(u) = .5 (3 - u^2) \phi(u), \text{bandwidth} \ h_q = S_X n^{-1/4}. \]

• Trimming parameter (i.e., (7)): \( \zeta = 1. \) This essentially implied no trimming
  of observations for the sample sizes considered.

• Final expression of \( \hat{\theta}_n(.5, x) \) (i.e., (1)): (Second-order) Gaussian kernel,
  bandwidths \( h = cn^{-1/4}, \) where \( c \in \{.5, 1, 1.5, 2\}. \)

For each value of \( x \) considered, the performance of the resulting estimates \( \hat{\theta}_n(.5, x) \)
was compared to that of the corresponding LLMR slope estimate \( \hat{q}_{n1}(.5, x) \) implemented
using a standard (second-order) Gaussian kernel and bandwidth \( h_{qq} = S_X n^{-1/7}. \) Note that \( h_{qq} \)
is proportional to the bandwidth that minimises the asymptotic mean integrated squared error of
the LLMR slope estimate with constant weight function [e.g., Fan and Gijbels, 1996, p. 49].

Figures 1 and 2 indicate how the squared bias of the two estimators varied
with \( x \) in the context of the location-shift and location-scale models, respectively.
It is seen from Figure 1 that the bias of the LLMR estimator increased with the
distance of \( x \) from the modal value of the design variable. It is also apparent from
Figure 1 that the bias of \( \hat{\theta}_n(.5, x) \) was both insensitive to \( x \) and relatively low for
each setting of the final \( h \)-bandwidth considered. This seems to validate the bias
approximation given above in Theorem 2.

Figure 2 displays the corresponding pattern for the location-scale model. The
bias of the LLMR estimator was seen to increase as \( x \) moves away from the centre
of the unit interval. Somewhat surprisingly in light of the approximation in (13),
the bias of \( \hat{\theta}_n(.5, x) \) for each setting of \( h \) considered was seen both to be relatively
invariant to \( x \) and close to zero.

Figures 1 and 2 also suggest that the bias of \( \hat{\theta}_n(.5, x) \) may exhibit an initial
decrease before a subsequent increase as one moves from smaller to larger values
of \( h \). In general, however, the performance of \( \hat{\theta}_n(.5, x) \) in terms of bias is seen to
be excellent over several different settings of \( h \) for a fairly wide range of \( x \)-values
near the centre of \([0, 1]\).

Finally, Tables 1 and 2 display averages over the grid of \( x \)-values considered
of each estimator’s squared bias, standard deviation and root mean squared error.
Table 1 presents results for the location-shift model in (14), while Table 2 gives
the corresponding results for the location-scale model in (15). In the context of
both models, the overall performance of \( \hat{\theta}_n(.5, x) \) over the range of \( x \)-values con-
sidered is seen to be somewhat sensitive to the setting of the final \( h \)-bandwidth.
On the other hand, the overall performance of \( \hat{\theta}_n(.5, x) \) in both models appears
Figure 1: Squared bias of derivative estimators, location-shift model; $n = 100$, $\alpha = .5$
Figure 2: Squared bias of derivative estimators, location-scale model; \( n = 100, \alpha = .5 \)
Table 1: Performance of derivative estimators, average quantities over 1000 replications and over a grid of 100 evenly spaced points in $[0, 1]$, location-shift model; $n = 100$, $\alpha = .5$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Squared bias</th>
<th>Standard deviation</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Locally linear QR</td>
<td>.1328</td>
<td>.4827</td>
<td>.5961</td>
</tr>
<tr>
<td>$\hat{\theta}_{100}, h = .5 \times 100^{-1/4}$</td>
<td>.0227</td>
<td>1.6124</td>
<td>1.6194</td>
</tr>
<tr>
<td>$\hat{\theta}_{100}, h = 100^{-1/4}$</td>
<td>.0102</td>
<td>.4707</td>
<td>.4812</td>
</tr>
<tr>
<td>$\hat{\theta}_{100}, h = 1.5 \times 100^{-1/4}$</td>
<td>.0006</td>
<td>.2661</td>
<td>.2671</td>
</tr>
<tr>
<td>$\hat{\theta}_{100}, h = 2 \times 100^{-1/4}$</td>
<td>.0182</td>
<td>.2331</td>
<td>.2617</td>
</tr>
</tbody>
</table>

Table 2: Performance of derivative estimators, average quantities over 1000 replications and over a grid of 100 evenly spaced points in $[0, 1]$, location-scale model; $n = 100$, $\alpha = .5$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Squared bias</th>
<th>Standard deviation</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Locally linear QR</td>
<td>.2047</td>
<td>.3819</td>
<td>.5671</td>
</tr>
<tr>
<td>$\hat{\theta}_{100}, h = .5 \times 100^{-1/4}$</td>
<td>.0148</td>
<td>1.5601</td>
<td>1.5637</td>
</tr>
<tr>
<td>$\hat{\theta}_{100}, h = 100^{-1/4}$</td>
<td>.0088</td>
<td>.3782</td>
<td>.3883</td>
</tr>
<tr>
<td>$\hat{\theta}_{100}, h = 1.5 \times 100^{-1/4}$</td>
<td>.0030</td>
<td>.2036</td>
<td>.2104</td>
</tr>
<tr>
<td>$\hat{\theta}_{100}, h = 2 \times 100^{-1/4}$</td>
<td>.0198</td>
<td>.1972</td>
<td>.2321</td>
</tr>
</tbody>
</table>

to dominate that of the LLMR estimator for three out of the four settings of $h$ considered.

5 Empirical example: Age-earnings profiles of U.S. workers

This example considers the empirical relationship between individual earnings and age. The latter quantity is generally taken to serve as a proxy for experience of the job market. The relationship between earnings and age has been the subject of an extensive literature in labour economics since at least the time of the seminal studies of Heckman and Polachek [1974] and Mincer [1974]. In particular, Mincer [1974] found that individual rates of pay typically increase with age, but at a diminishing rate over time.

In what follows, a nonparametric approach is taken to modelling the relation-
ship between three quantiles of individual earnings and age. In order to do this, a sample of male wage earners in the United States was taken from the March 2006 supplement to the Current Population Survey. This sample was further restricted to male wage earners having a high-school diploma and no further schooling. This resulted in a sample size of $n = 4123$. The youngest worker in the sample is 15, and the oldest 80.

The methods developed in this paper were applied to estimate the derivatives of the conditional $\alpha$-quantile of the natural logarithm of earnings with respect to age for each year of age between 15 and 80. The quantiles $\alpha = .1, \alpha = .5$ and $\alpha = .9$ were considered in succession. The same set of smoothing kernels and tuning parameters used to implement the corresponding derivative estimates described above in Section 4 were also used here, with the exception of the bandwidth $h$ used in the final expression of $\hat{\theta}_n(\alpha, \text{age})$. Here the setting $h = cn^{-1/4}$ was used, where $c = 1, c = 6, c = 12, c = 18$ and $c = 24$ in succession.

The resulting estimates of the derivatives of the conditional .1-, .5- and .9-quantiles with respect to each year of age available are displayed in Figures 3, 4 and 5, respectively. The patterns displayed here provide some evidence to suggest that earnings growth is significantly affected by the first few years of experience of the labour market, but that the effect of an additional year of experience becomes largely insignificant at later stages in a worker’s career. There is also evidence to suggest that the importance of the first few years of labour-market experience increases as one moves from the 10th percentile of the earnings distribution to the 90th percentile.

6 Conclusion

This paper has introduced a new approach to constructing nonparametric estimates of conditional quantile functions and their derivatives with respect to covariates. The proposed estimators have the distinguishing feature of having biases that do not depend on the design density at any order of approximation regardless of which derivative is being estimated. More familiar nonparametric estimates based on local polynomials do not have this property. The results of simulation experiments presented in Section 4 indicate that this property may induce the proposed estimators under some conditions to dominate their locally linear counterparts in terms of mean squared error over a wide range of covariate values. The applicability of the estimators developed in this paper to actual data is illustrated by the example presented in Section 5.
Figure 3: Derivative of the .1-quantile of log earnings with respect to age, male wage earners with high-school diplomas but no further schooling; various settings of $h$
Figure 4: Derivative of the median of log earnings with respect to age, male wage earners with high-school diplomas but no further schooling; various settings of $h$
Figure 5: Derivative of the .9-quantile of log earnings with respect to age, male wage earners with high-school diplomas but no further schooling; various settings of $h$
A useful extension of the research presented here could include a systematic consideration of the issue of bandwidth choice. Another extension could involve considering the behaviour of the proposed estimator $\hat{\theta}_n(\alpha, x)$ as a stochastic process indexed by $\alpha$. This in turn would lead to design-adaptive methods of estimation and inference regarding stochastic processes of conditional quantiles and their derivatives with respect to conditioning variables.

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References


Appendix

This appendix contains detailed proofs of Proposition 3.1 and Theorems 3.2 and 3.3 above. These proofs are preceded by a discussion of appropriate regularity conditions, which follows in Appendix A. The proof of Proposition 3.1 appears in Appendix B, while those of Theorems 3.2 and 3.3 appear in Appendix C.

A Regularity Conditions

Regularity conditions are stated under which the asymptotic properties of \( \hat{\theta}_n(\alpha, x) \) are developed. In this connection, let \( q^{(1)}(\alpha, x) \equiv \frac{d}{dx}q(\alpha, x) \); \( q^{(2)}(\alpha, x) \equiv \frac{d^2}{dx^2}q(\alpha, x) \); \( g^{(1)}(x) \equiv \frac{d}{dx}g(x) \) and \( g^{(2)}(x) \equiv \frac{d^2}{dx^2}g(x) \).

Assumption 1 ((Conditional distributions)). The conditional distribution of \( Y \) given \( X = x \) is absolutely continuous with respect to Lebesgue measure for each \( x \in \mathcal{X} \). The corresponding density function \( f_{Y|X=x}(\cdot) \) satisfies

1. \( 0 < \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y|X=x}(q(\alpha, x)) < \infty \),
2. while the first derivative of \( f_{Y|X=x}(\cdot) \) satisfies \( \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} |f_{Y|X=x}^{(1)}(q(\alpha, x))| < \infty \).

Assumption 2 ((Conditional quantile functions)). \( \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} |D_x q^{(2)}(\alpha, x)| < \infty \).

3. All partial derivatives of \( q(\alpha, x) \) with respect to \( x \) exist to at least sixth order for each \( x \in \mathcal{X} \); these derivatives are all uniformly bounded over \( (\alpha, x) \in \mathcal{A} \times \mathcal{X} \).

Assumption 3 ((Design)). \( g(x) > 0 \) for all \( x \in \mathcal{X} \), and \( \sup_{x \in \mathcal{X}} g(x) \), \( \sup_{x \in \mathcal{X}} |g^{(1)}(x)| \) and \( \sup_{x \in \mathcal{X}} |g^{(2)}(x)| \) are all finite.

2. \( \frac{d^{L-1}}{dx^{L-1}} g(x) < \infty \), where \( L \geq 2 \) is as given below in Assumption 4.

3. \( \sup_{x \in \mathcal{X}} |D_x g(x)| < \infty \).

4. \( \sup_{x \in \mathcal{X}} |D_x g^{(2)}(x)| < \infty \).
Assumption 4 (Kernel functions). 1. \( \kappa : \mathbb{R} \to \mathbb{R} \) is a symmetric density function with finite moments to at least second order. In addition, \( \int u^2 \kappa^2(u) du < \infty \).

2. \( k : \mathbb{R} \to \mathbb{R} \) satisfies
   
   (a) \( \int k(u) du = 1 \).
   
   (b) For some integer \( L \geq 2 \), \( \int u^\alpha k(u) du = 0 \) for all \( \alpha \in \mathbb{Z}_+ \) with \( 1 \leq \alpha \leq L - 1 \) and \( \int u^L k(u) du \in (0, \infty) \).
   
   (c) \( \int k^4(u) du < \infty \).
   
   (d) \( k \) is Lipschitz, with \( |k(u + v) - k(u)| < c_k(u)|v| \) for some finite real-valued function \( c_k(\cdot) \) on \( \mathbb{R} \).
   
   (e) Both \( \int u^{16} k^2(u) du \) and \( \int u^{28} k^4(u) du \) are finite.

3. \( K : \mathbb{R} \to \mathbb{R} \) satisfies
   
   (a) \( K(\cdot) \) is a symmetric density function supported on \( \mathbb{R} \) with finite moments to at least sixth order.
   
   (b) \( \int \left( \frac{\partial^r}{\partial u^r} K(u) \right)^4 du < \infty \).
   
   (c) \( \left| \int u \left( \frac{\partial^r}{\partial u^r} K(u) \right)^2 du \right| \) and \( \left| \int u \left( \frac{\partial^r}{\partial u^r} K(u) \right)^4 du \right| \) are both finite.

Assumption 5 (Trimming and smoothing). 1. For some constant \( \zeta > 0 \), the trimming function \( \tau_n(u) \) satisfies

\[
\tau_n(u) = \begin{cases} 
1, & u \geq 2n^{-\zeta} \\
0, & u \leq n^{-\zeta} \\
\bar{\tau}_n(u), & u \in (n^{-\zeta}, 2n^{-\zeta}) 
\end{cases}
\]

where \( \bar{\tau}_n(u) \) is a distribution function with the form \( \bar{\tau}_n(u) = \int_{-\infty}^u n^\zeta \sigma \left( nt^\zeta - 1 \right) dt \), where \( \sigma(\cdot) \) is a differentiable density function uniformly bounded and supported on \([0, 1]\) with \( \sigma(0) = \sigma(1) = 0 \) and \( 0 < |\sigma^{(1)}(0)|, |\sigma^{(1)}(1)| < \infty \).

2. \( h_g \geq 0 \) satisfies the following:

(a) \( h_g \to 0 \) and \( nh_g \to \infty \) as \( n \to \infty \);

(b) \( n^\zeta \left( h_g^2 + 1 / \sqrt{n h_g} \right) \to 0 \) as \( n \to \infty \) where \( \zeta \) is the trimming constant given above in part (1).
3. $h_q \geq 0$ satisfies the following:

(a) $h_q \to 0$, $nh_q \to \infty$;
(b) $\sqrt{n h_q} \cdot h_q^{L-1} \to 0$ as $n \to \infty$ for $L \geq 2$ as given above in Assumption 4.

4. $h \geq 0$ satisfies the following:

(a) $h \to 0$, $nh^{1+2\nu} \to \infty$;
(b) $\sqrt{n h^{1+2\nu}} \cdot h \to 0$ and $h^{1+2\nu}/h_q \to 0$ as $n \to \infty$.

Remark 3. Part (1) of Assumption 1 is standard in the asymptotic analysis of conditional quantile estimators. The proof of Proposition 1 depends in part on a series of mean-value expansions involving the conditional density function $f_{Y|X}(\cdot)$; these expansions require the second part of Assumption 1.

Remark 4. The smoothness conditions on the conditional quantile function $q(\alpha, x)$ in part (3) of Assumption 2 and on the design density $g(x)$ in parts (1) and (2) of Assumption 3 are employed in the proof of Proposition 1. Parts (1) and (2) of Assumption 2 and parts (3) and (4) Assumption 3 are employed in the proof of the consistency of $\hat{\theta}_n(\alpha, x)$ for $D^\nu_x q(\alpha, x)$ (i.e., Theorem 1), and also for showing the asymptotic normality of $\hat{\theta}_n(\alpha, x)$ (Theorem 2).

Remark 5. Part (1) of Assumption 4, relating to the smoothing kernel used to construct the design density estimator $\hat{g}_n(x)$, is standard. Part (2) of Assumption 4 relates to the smoothing kernel used to construct the locally linear quantile estimator $\hat{q}_n^*(\alpha, x)$. Part (2) b) of Assumption 4 indicates that this kernel function will in general be of higher order. The imposition of a higher-order kernel at this stage ensures that the bias of $\hat{q}_n^*(\alpha, x)$ vanishes sufficiently quickly for the proposed estimator $\hat{\theta}_n(\alpha, x)$ to estimate the actual parameter of interest $D^\nu_x q(\alpha, x)$ with asymptotically negligible bias. Part (3) of Assumption 4 relates to the smoothing kernel employed directly in the formulation of $\hat{\theta}_n(\alpha, x)$, i.e., when smoothing over $\hat{q}_n^*(\alpha, X_i)$ in (1) above. In particular, part (3) a) of Assumption 4 is used in the consistency proof of $\hat{\theta}_n(\alpha, x)$ (Theorem 1) and also in the derivation of a certain asymptotic expansion of the bias of $\hat{\theta}_n(\alpha, x)$. In this connection, it was thought desirable that this expansion contain a sufficient number of terms to enable a useful comparison with the bias of a locally linear estimate of $D^\nu_x q(\alpha, x)$. Parts (3) a)–(3) c) of Assumption 4 are used in the proof of the asymptotic normality of $\hat{\theta}_n(\alpha, x)$ (Theorem 2). Part (3) of Assumption 4 generally encompasses the
Gaussian kernel and may include other popular kernels depending on the order of differentiation $\nu$ appearing in the parameter of interest $D_x^\nu q(\alpha, x)$.

**Remark 6.** Part (1) of Assumption 5 relates to the trimming scheme used in the construction of the preliminary quantile estimate $\hat{q}_n^*(\alpha, x)$. Its conditions are similar to those used in comparable trimming schemes appearing in Andrews [1995], Ai [1997] and Linton and Xiao [2001]. Part (2) of Assumption 5 relates to the bandwidth used to implement the design density estimator $\hat{g}_n(x)$. Its conditions are standard, apart from the restriction on the rate of convergence of the bandwidths relative to the trimming parameter described in Part (1). This last restriction ensures that the data-dependent trimming scheme used in the construction of $\hat{q}_n^*(\alpha, x)$ is in fact asymptotically equivalent for all $x \in X$ to an infeasible trimming scheme based on the true value of the design density.

**Remark 7.** Part (3) of Assumption 5 relates to the bandwidth $h_q$ used to construct the preliminary quantile estimator $\hat{q}_n^*(\alpha, x)$ given above in (6). The conditions of part (2) of Assumption 5 include a provision that $\sqrt{nh_q^3}h^L_qh^2_q \to 0$ as $n \to \infty$. This provision ensures that the asymptotic bias of $\hat{q}_n^*(\alpha, x)$ vanishes sufficiently quickly to guarantee the rate of convergence exhibited in Proposition 1 above.

**Remark 8.** Part (4) of Assumption 5 concerns the bandwidth $h$ used when smoothing over $\hat{q}_n^*(\alpha, X_i)$ in the formulation of $\hat{\theta}_n(\alpha, x)$ in (1) above. Part (4) a) of Assumption 5 is used when proving the consistency of $\hat{\theta}_n(\alpha, x)$ for each $(\alpha, x) \in A \times X$ (i.e., Theorem 1). Parts (4) a) and (4) b) of Assumption 5 are both employed when showing the asymptotic normality of $\hat{\theta}_n(\alpha, x)$ (Theorem 2). In particular, part (4) b) of Assumption 5 ensures that $\sqrt{nh^{1+2\nu}}(\hat{\theta}_n(\alpha, x) - D_x^\nu q(\alpha, x))$ is asymptotically centred about zero. The conditions of part (4) of Assumption 5 are similar to those governing $h_q$ in part (3), with the addition of a restriction on the relative rate of decay of $h$ relative to that of $h_q$. This restriction has the effect that the preliminary quantile estimates $\hat{q}_n^*(\alpha, X_i)$ appearing in the formulation of $\hat{\theta}_n(\alpha, x)$ converge sufficiently quickly to make $\hat{\theta}_n(\alpha, x)$ asymptotically indistinguishable on $A \times X$ from an infeasible estimate involving the true values of $q(\alpha, X_i)/g(X_i)$. Further details appear in the online supplement.
B Proof of Proposition 3.1

B.1 Preliminaries

The proof of Proposition 3.1 relies on a number of preliminary definitions, which for the sake of completeness are recapitulated here.

For each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\), let

\[
q_g^{(1)}(\alpha, x) \equiv \frac{d}{dx} \left[ \frac{q(\alpha, x)}{g(x)} \right]
\]

and

\[
q_g^{(2)}(\alpha, x) \equiv \frac{d^2}{dx^2} \left[ \frac{q(\alpha, x)}{g(x)} \right].
\]

Let \(\rho_\alpha(u) \equiv u(\alpha - 1(u < 0))\), and let \(Y^*_i \equiv Y_i/\hat{g}_0(X_i)\) for each \(i = 1, \ldots, n\).

Note that for \(\delta \equiv (\delta_0, \delta_1)^\top\), the function

\[
Z_n(\delta) \equiv \sum_{i=1}^n \left( \rho_\alpha \left( Y^*_i - \frac{q(\alpha, x)}{g(x)} \right) - q_g^{(1)}(\alpha, x) \cdot (X_i - x) - \frac{1}{\sqrt{nh_q}} \left[ \delta_0 + \delta_1 \cdot h_q^{-1}(X_i - x) \right] \right)
- \rho_\alpha \left( Y^*_i - \frac{q(\alpha, x)}{g(x)} - q_g^{(1)}(\alpha, x) \cdot (X_i - x) \right) k \left( h_q^{-1}(X_i - x) \right) \tau_n (\hat{g}_n(X_i))
\]

is convex and minimised for each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\) at

\[
\left[ \hat{\delta}_{n0}(\alpha, x) \quad \hat{\delta}_{n1}(\alpha, x) \right] = \sqrt{nh_q} \left[ \hat{q}_n^*(\alpha, x) - \frac{q(\alpha, x)}{g(x)} \right],
\]

where

\[
\hat{q}_n^* = \frac{\hat{g}_n(X_i)}{q_g^{(1)}(\alpha, x)} - \frac{q(\alpha, x)}{g(x)}.
\]

The conclusion of Proposition 3.1, namely that \(\sqrt{nh_q} (\hat{q}_n^* - q(\alpha, x)/g(x)) = O_p(1)\) for each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\), is deduced by first showing the convergence in distribution of the vector \(\left( \hat{\delta}_{n0}(\alpha, x), \hat{\delta}_{n1}(\alpha, x) \right)^\top\) for each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\).

The convergence in distribution of \(\left( \hat{\delta}_{n0}(\alpha, x), \hat{\delta}_{n1}(\alpha, x) \right)^\top\) is deduced from the asymptotic behaviour of the criterion function \(Z_n(\delta)\) for each \(\delta \in \mathbb{R}^2\). From “Knight’s identity” (Knight 1998; Koenker 2005, p. 121), we have

\[
Z_n(\delta) = Z_{n1}(\delta) + Z_{n2}(\delta),
\]

where for

\[
\psi_\alpha(u) \equiv \alpha - 1(u < 0),
\]

and
we have
\[
Z_{n1}(\delta) = -\frac{1}{\sqrt{nh_q}} \sum_{i=1}^{n} \left[ \delta_0 + h_q^{-1} (X_i - x) \delta_1 \right] \psi_{\alpha} (u_i^*(x)) k \left( h_q^{-1} (X_i - x) \right) \tau_n (\hat{g}_n (X_i)) ;
\]
\[
Z_{n2}(\delta) = \sum_{i=1}^{n} \int_{u_i(x)}^{v_{ni}(x)} (1 (u_i^*(x) \leq s) - 1 (u_i^*(x) \leq 0)) ds,
\]
and where
\[
u_{ni}(x) \equiv \frac{1}{\sqrt{nh_q}} \left[ \delta_0 + h_q^{-1} (X_i - x) \delta_1 \right] k \left( h_q^{-1} (X_i - x) \right) \tau_n (\hat{g}_n (X_i)) .
\]

Begin by examining the large-sample behaviour of \( Z_{n2} (\delta) \) on \( \mathcal{A} \times \mathcal{X} \) before proceeding to an analysis of \( Z_{n1}(\delta) \). To start, recall that
\[
\hat{g}_n (x) = \frac{1}{n} \sum_{i=1}^{n} \kappa_{h_g} (X_i - x) ,
\]
where \( \kappa_{h_g} (t) \equiv h_g^{-1} \kappa (h_g^{-1} t) \), and where \( \kappa : \mathbb{R} \to \mathbb{R} \) is a kernel function satisfying the conditions of part (1) of Assumption A.4, and \( h_g \) is a bandwidth satisfying \( h_g \to 0 \) with \( nh_g \to \infty \). Note that
\[
E \left[ \kappa_{h_g} (X_1 - x) \right]
= \int \kappa_{h_g} (x_1 - x) g (x_1) dx_1
= \int h_g^{-1} \kappa (h_g^{-1} (x_1 - x)) g (x_1) dx_1
= \int \kappa (t_1) g (x + h_g t_1) dt_1
= g(x) \int \kappa (t_1) dt_1 + h_g g^{(1)}(x) \int t_1 \kappa (t_1) dt_1
+ \frac{h_g^2}{2} \int t_1^2 g^{(2)} (\xi (x, t_1)) \kappa (t_1) dt_1 ,
\]
where \( \xi (x, t_1) \) is a point on the line segment between \( x \) and \( x + h_g t_1 \).
It follows that
\[
\sup_{x \in \mathcal{X}} \left| E \left[ \kappa_{h} (X_1 - x) \right] - g(x) \right| = O \left( h_g^2 \right),
\] (19)
where the various conditions on \( \kappa(\cdot) \) and the assumption that \( \sup_{x \in \mathcal{X}} \left| g^{(2)}(x) \right| < \infty \) have been invoked.

Note also that
\[
E \left[ \hat{g}_n^2(x) \right]
= E \left[ \frac{1}{n^2} \sum_{i,j} \kappa_{h} (X_i - x) \kappa_{h} (X_j - x) \right]
= \frac{1}{n} E \left[ \kappa_{h}^2 (X_1 - x) \right] + \frac{n(n - 1)}{n^2} \left( E \left[ \kappa_{h} (X_1 - x) \right] \right)^2.
\]

We have
\[
E \left[ \kappa_{h}^2 (X_1 - x) \right]
= \int \kappa_{h}^2 (x_1 - x) g(x_1) \, dx_1
= \int h_g^{-2} \kappa^2 (h_g^{-1} (x_1 - x)) g(x_1) \, dx_1
= h_g^{-1} \int \kappa^2 (t_1) g(x + h_g t_1) \, dt_1
= h_g^{-1} \left( g(x) \int \kappa^2 (t_1) \, dt_1 + h_g g^{(1)}(x) \int t_1 \kappa^2 (t_1) \, dt_1 \right.
+ \frac{h_g^2}{2} \int t_1^2 g^{(2)}(\xi(x, t_1)) \kappa^2 (t_1) \, dt_1 \biggr) .
\]

As such,
\[
E \left[ \hat{g}_n^2(x) \right]
= \frac{1}{nh_g} \int \kappa^2 (t_1) \, dt_1 + O \left( \frac{h_g^2}{nh_g} \right)
+ g^2(x) + h_g^2 g(x) \int g^{(2)}(\xi(x, t_1)) t_1^2 \kappa (t_1) \, dt_1
+ O \left( h_g^4 \right).
\]
Deduce that
\[
\begin{align*}
Var [\hat{g}_n(x)] &= E \left[ (\hat{g}_n(x) - E [\hat{g}_n(x)])^2 \right] \\
&= E \left[ (\hat{g}_n(x) - E [\kappa g (X_1 - x)])^2 \right] \\
&= E \left[ \hat{g}_n^2 (x) - (E [\kappa g (X_1 - x)])^2 \right] \\
&= g^2(x) + \frac{1}{nh_g} g(x) \int \kappa^2 (t_1) dt_1 + h_g^2 g(x) \int g^{(2)} (\xi (x, t_1)) t_1^2 \kappa (t_1) dt_1 \\
&\quad - g^2(x) - h_g^2 g(x) \int g^{(2)} (\xi (x, t_1)) t_1^2 \kappa (t_1) dt_1 + O (h_g^4) \\
&= O \left( \frac{1}{nh_g} + h_g^4 \right).
\end{align*}
\]

It follows that
\[
\sup_{x \in \mathcal{X}} |\hat{g}_n(x) - E [\kappa g (X_1 - x)]| = O_p \left( \frac{1}{\sqrt{nh_g}} + h_g^2 \right), \quad (20)
\]
where the assumptions that \(\sup_{x \in \mathcal{X}} g(x) < \infty\) and \(\sup_{x \in \mathcal{X}} |g^{(2)}(x)| < \infty\) have been invoked.

Combine (19) and (20) to deduce that
\[
\sup_{x \in \mathcal{X}} |\hat{g}_n(x) - g(x)| = O_p \left( h_g^2 + \frac{1}{\sqrt{nh_g}} \right). \quad (21)
\]

This deduction also relies on the assumptions that \(\sup_{x \in \mathcal{X}} g(x) < \infty\), \(\sup_{x \in \mathcal{X}} |g^{(2)}(x)| < \infty\) and \(\int t_1^2 \kappa^2 (t_1) dt_1 < \infty\).

Now consider that
\[
\tau_n (\hat{g}_n(x)) - \tau_n (g(x)) = (\hat{g}_n(x) - g(x)) \tau_n^{(1)} (g^*(x)),
\]
where \(g^*(x)\) is an intermediate point. We have
\[
\begin{align*}
&\sup_{x \in \mathcal{X}} |\tau_n (\hat{g}_n(x)) - \tau_n (g(x))| \\
&\leq \sup_{x \in \mathcal{X}} |\hat{g}_n(x) - g(x)| \cdot \sup_{u \in (n^{-\varsigma}, 2n^{-\varsigma})} n^{\varsigma} \sigma (n^{\varsigma} u - 1) \\
&= O_p \left( n^{\varsigma} \left( h_g^2 + \frac{1}{\sqrt{nh_g}} \right) \right), \quad (22)
\end{align*}
\]
by virtue of (21) and the uniform boundedness of \( \sigma(\cdot) \) on \([0, 1]\).

Returning to \( Z_{n2}(\delta) \), define

\[
\bar{Z}_{n2}(\delta) \equiv \sum_{i=1}^{n} \int_{0}^{\tilde{v}_{ni}(x)} \left( 1 (\tilde{u}_{i}^*(x) \leq s) - 1 (\tilde{u}_{i}^*(x) \leq 0) \right) ds,
\]

where

\[
\begin{align*}
\tilde{u}_{i}^*(x) &= \frac{Y_{i}}{g(X_{i})} - \frac{q(\alpha, x)}{g(x)} - (X_{i} - x) q_{g}^{(1)}(\alpha, x), \\
\tilde{v}_{ni}(x) &= \frac{1}{\sqrt{n h_q}} \left[ \delta_0 + h_{q}^{-1} (X_{i} - x) \delta_1 \right] k \left( h_{q}^{-1} (X_{i} - x) \right) \tau_n (g(X_{i})).
\end{align*}
\]

Write

\[
Z_{n2}(\delta) = \sum_{i=1}^{n} E \left[ Z_{n2i}(\delta) \mid X_{i} \right] + \sum_{i=1}^{n} (Z_{n2i}(\delta) - E \left[ Z_{n2i}(\delta) \mid X_{i} \right]),
\]

and

\[
Z_{n2}(\delta) \equiv \sum_{i=1}^{n} \int_{0}^{\tilde{v}_{ni}(x)} \left( 1 (\tilde{u}_{i}^*(x) \leq s) - 1 (\tilde{u}_{i}^*(x) \leq 0) \right) ds.
\]

Let \( F_i(\cdot) \) denote the conditional distribution function \( F_{Y_i|X_i}(\cdot) \) and \( f_i(\cdot) \) denote the corresponding conditional density function \( f_{Y_i|X_i}(\cdot) \). We note that for \( T_i \equiv h_{q}^{-1} (X_{i} - x) \) \((i \in \{1, \ldots, n\})\) and \( \xi_i \) a point on the line segment between \( x \) and
\[ X_i = x + H_q T_i \]

\[
E \left[ Z_{n2i}(\delta) \mid X_i \right] = \sum_{i=1}^{n} E \left[ Z_{n2i}(\delta) \mid X_i \right] = \sum_{i=1}^{n} \int_{0}^{v_{ni}(x)} \left( F_i \left( \frac{g(\alpha, X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) + s \right) - F_i \left( \frac{g(\alpha, X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \right) ds
\]

\[
= \frac{1}{\sqrt{n h_q}} \sum_{i=1}^{n} \int_{0}^{\tau_n(x)(X_i - x)} \left( F_i \left( \frac{g(\alpha, X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) + \frac{u}{\sqrt{n h_q}} \right) \right) du
\]

\[
= \frac{1}{n h_q} \sum_{i=1}^{n} \int_{0}^{\tau_n(x)(X_i - x)} \left( F_i \left( \frac{g(\alpha, X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \right) du + o_p(1)
\]

\[
= \frac{1}{2 n h_q} \sum_{i=1}^{n} \int_{0}^{\tau_n(x)(X_i - x)} \left( F_i \left( \frac{g(\alpha, X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \right) du + o_p(1)
\]
Similarly,

\[
\sum_{i=1}^{n} E [ \tilde{Z}_{ni}(\delta) | X_i ] = \frac{1}{2nh_q} \sum_{i=1}^{n} f_i \left( q(\alpha, X_i) - \frac{1}{2} g(X_i) (X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \\
\cdot \left[ \begin{array}{cc} \delta_0 & \delta_1 \\ \delta_0^{-1}(X_i - x) & \delta_0^{-2}(X_i - x)^2 \end{array} \right] \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \\
\cdot k^2 \left( h_q^{-1}(X_i - x) \right) \tau_n^2 \left( g(X_i) \right) + o_p(1). 
\]  

(24)
For each $\delta \in \mathbb{R}^2$, $(\alpha, x) \in A \times X$ and some $\gamma \in (0, 1)$ we have

$$
E \left[ f_1 \left( q(\alpha, X_1) - \frac{1}{2} g(X_1) (X_1 - x)^2 q^{(2)}_g(\alpha, \xi_1) \right) \right] \\
\cdot \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \left[ \begin{array}{c} h^{-1}_q(X_1 - x) \\ h^{-1}_q(X_1 - x) \end{array} \right] \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \\
k^2 (h^{-1}_q(X_1 - x)) \tau^2_n (g(X_1))
$$

$$
= \int f_1 \left( q(\alpha, x_1) - \frac{1}{2} g(X_1) (X_1 - x)^2 q^{(2)}_g(\alpha, \xi_1) \right) \\
\cdot \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \left[ \begin{array}{c} \frac{1}{t_1} \\ \frac{1}{t_1^2} \end{array} \right] \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \\
k^2 (t_1) \tau^2_n (g(x_1)) g(x_1) dx_1
$$

$$
= h_q \int f_1 \left( q(\alpha, x + h_q t_1) - \frac{h_q^2}{2} g(x + h_q t_1) t_1^2 q^{(2)}_g(\alpha, \gamma h_q t_1 + x) \right) \\
\cdot \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \left[ \begin{array}{c} \frac{1}{t_1} \\ \frac{1}{t_1^2} \end{array} \right] \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] k^2 (t_1) \tau^2_n (g(x + h_q t_1)) g(x + h_q t_1) dt_1 \\
+ O \left( h^2_q \right)
$$

$$
= h_q \left( \int f_{Y_1|X_1=x} (q(\alpha, x)) \right) \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \left[ \begin{array}{c} \frac{1}{t_1} \\ \frac{1}{t_1^2} \end{array} \right] \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] k^2 (t_1) \tau^2_n (g(x)) g(x) dt_1 \\
+ O \left( h^2_q \right)
$$

$$
= h_q f_{Y_1|X_1=x} (q(\alpha, x)) \tau^2_n (g(x)) g(x) \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \left[ \begin{array}{c} \frac{1}{t_1} \\ \frac{1}{t_1^2} \end{array} \right] k^2 (t_1) dt_1 \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] + O \left( h^2_q \right)
$$

$$
= h_q f_{Y_1|X_1=x} (q(\alpha, x)) g(x) \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \left[ \begin{array}{c} \frac{1}{t_1} \\ \frac{1}{t_1^2} \end{array} \right] k^2 (t_1) dt_1 \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] + O \left( n^{-\zeta} h_q \right) \\
+ O \left( h^2_q \right)
$$

$$
= h_q f_{Y_1|X_1=x} (q(\alpha, x)) g(x) \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \left[ \begin{array}{c} \frac{1}{t_1} \\ \frac{1}{t_1^2} \end{array} \right] k^2 (t_1) dt_1 \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] + O \left( h_q (n^{-\zeta} + h_q) \right).
$$
It follows from (24) that for each $\delta \in \mathbb{R}^2$ and each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ that

$$
\sum_{i=1}^{n} E \left[ \bar{Z}_{n2i}(\delta) \mid X_i \right] = 1 \quad \frac{1}{2} f_{Y_1|X_1=x}(q(\alpha, x)) g(x) \left[ \begin{array}{cc} \delta_0 & \delta_1 \\ \int t_1 k^2(t_1) dt_1 & \int t_1 k^2(t_1) dt_1 & \int t_1 k^2(t_1) dt_1 & \int t_1 k^2(t_1) dt_1 \\ \delta_0 & \delta_1 \end{array} \right] + o_p(1).
$$

(25)
For each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\) we have

\[
\sum_{i=1}^{n} E \left[ Z_{n2i}(\delta) \mid X_i \right] - \sum_{i=1}^{n} E \left[ \hat{Z}_{n2i}(\delta) \mid X_i \right] \\
\leq \frac{1}{2nh_q} \sum_{i=1}^{n} \sup_{x \in \mathcal{X}} \left[ \delta_0 \quad \delta_1 \right] \left[ \frac{1}{h_q^{-1}(X_i - x)} \quad \frac{h_q^{-1}(X_i - x)}{h_q^{-2}(X_i - x)^2} \right] \left[ \delta_0 \quad \delta_1 \right] k^2 \left( h_q^{-1}(X_i - x) \right) \\
\cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| f_i \left( q(\alpha, X_i) \hat{g}_n(X_i) - \frac{1}{2} \hat{g}_n(X_i) (X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \tau_n^2 \left( \hat{g}_n(X_i) \right) \\
- f_i \left( q(\alpha, X_i) - \frac{1}{2} g(X_i) (X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \tau_n^2 \left( g(X_i) \right) \right| + o_p(1) \\
= \frac{1}{2nh_q} \sum_{i=1}^{n} \left[ \delta_0 \quad \delta_1 \right] \left[ \frac{1}{T_i} \quad T_i^2 \right] \left[ \delta_0 \quad \delta_1 \right] k^2 \left( T_i \right) \\
\cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| f_{Y_i \mid X_i=x+h_qT_i} \left( q(\alpha, h_qT_i + x) \hat{g}_n(h_qT_i + x) \right) \right| \tau_n^2 \left( \hat{g}_n(h_qT_i + x) \right) \\
- f_{Y_i \mid X_i=x+h_qT_i} \left( q(\alpha, h_qT_i + x) - \frac{h_q^2}{2} g(h_qT_i + x) T_i^2 q_g^{(2)}(\alpha, h_qT_i + x) \right) \tau_n^2 \left( g(h_qT_i + x) \right) + o_p(1) \\
= \frac{1}{2nh_q} \sum_{i=1}^{n} \left[ \delta_0 \quad \delta_1 \right] \left[ \frac{1}{T_i} \quad T_i^2 \right] \left[ \delta_0 \quad \delta_1 \right] k^2 \left( T_i \right) \\
\cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| f_{Y_i \mid X_i=x+h_qT_i} \left( q(\alpha, x + h_qT_i) \hat{g}_n(x + h_qT_i) \right) \right| \tau_n^2 \left( \hat{g}_n(x + h_qT_i) \right) + O \left( h_q^2 \right) \\
- f_{Y_i \mid X_i=x+h_qT_i} \left( q(\alpha, x + h_qT_i) \right) \tau_n^2 \left( g(x + h_qT_i) \right) + O \left( h_q^2 \right) + o_p(1) \\
= \frac{1}{2nh_q} \sum_{i=1}^{n} \left[ \delta_0 \quad \delta_1 \right] \left[ \frac{1}{T_i} \quad T_i^2 \right] \left[ \delta_0 \quad \delta_1 \right] k^2 \left( T_i \right) \\
\cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| f_{Y_i \mid X_i=x} \left( q(\alpha, x) \hat{g}_n(x) \right) \right| \tau_n^2 \left( \hat{g}_n(x) \right) + O_p \left( h_q \right) \\
- f_{Y_i \mid X_i=x} \left( q(\alpha, x) \right) \tau_n^2 \left( g(x) \right) + O_p \left( h_q \right) + o_p(1).
From (21) and (22) we have

\[
\sup_{\alpha \in A} \sup_{x \in X} f_{Y_1|X_1=x} \left( \frac{q(\alpha, x)\hat{g}_n(x)}{g(x)} \right) \tau^2_n (\hat{g}_n(x)) - f_{Y_1|X_1=x} (q(\alpha, x)) \tau^2_n (g(x)) \right) \\
= \sup_{\alpha \in A} \sup_{x \in X} \left| f_{Y_1|X_1=x} (q(\alpha, x)) \tau^2_n (\hat{g}_n(x)) - f_{Y_1|X_1=x} (q(\alpha, x)) \tau^2_n (g(x)) \right| \\
+ O_p \left( h^2_g + \frac{1}{\sqrt{nh_q}} \right) \\
= O_p \left( n^{\zeta} \left( h^2_g + \frac{1}{\sqrt{nh_q}} \right) \right).
\]

It follows from (26) that

\[
\left| \sum_{i=1}^{n} E [Z_n^{2i}(\delta)|X_i] - \sum_{i=1}^{n} E [\bar{Z}_n^{2i}(\delta)|X_i] \right| \\
= O_p \left( \frac{1}{\sqrt{nh_q}} + n^{\zeta} \left( h^2_g + \frac{1}{\sqrt{nh_q}} \right) + h_q \right) \\
= o_p(1) \tag{27}
\]

for each \((\alpha, x) \in A \times \mathcal{X}\). Here the assumptions \( \sup_{x \in \mathcal{X}} g(x) < \infty \), \( \int t_1^4 k^4 (t_1) \, dt_1 < \infty \) and

\[n^{\zeta} \left( h^2_g + \frac{1}{\sqrt{nh_q}} \right) = o(1),\]

amongst others, have been invoked.

Combine (27) with (25) to deduce that for each \( \delta \in \mathbb{R}^2 \) and each \((\alpha, x) \in A \times \mathcal{X}\),

\[
\sum_{i=1}^{n} E [Z_n^{2i}(\delta)|X_i] \\
= \frac{1}{2} f_{Y_1|X_1=x} (q(\alpha, x)) \, g(x) \right[ \delta_0 \quad \delta_1 \right] \left[ \int k^2 (t_1) \, dt_1 \quad \int t_1 k^2 (t_1) \, dt_1 \right] \left[ \delta_0 \quad \delta_1 \right] \left[ \int t_1 k^2 (t_1) \, dt_1 \quad \int t_1^2 k^2 (t_1) \, dt_1 \right] \left[ \delta_0 \quad \delta_1 \right] + o_p(1).
\]
Next, note that

\[
\begin{align*}
\sum_{i=1}^{n} Var \left[ Z_{n2i}(\delta)| X_i \right] \\
\leq \frac{1}{\sqrt{nh_q}} \max_i \left( \delta_0 + h_q^{-1}(X_i - x) \delta_1 \right) k \left( h_q^{-1}(X_i - x) \right) \tau_n \left( \hat{g}_n(X_i) \right) \\
\cdot \sum_{i=1}^{n} E \left[ Z_{n2i}(\delta)| X_i \right] \\
= \frac{1}{\sqrt{nh_q}} \max_i \left( \delta_0 + h_q^{-1}(X_i - x) \delta_1 \right) k \left( h_q^{-1}(X_i - x) \right) \tau_n \left( g(X_i) \right) \\
\cdot \sum_{i=1}^{n} E \left[ \bar{Z}_{n2i}(\delta)| X_i \right] \\
+ o_p(1) \tag{28}
\end{align*}
\]

for each \((\alpha, x) \in A \times X\) by virtue of (22) and (27).

We have

\[
\begin{align*}
E \left[ \frac{1}{\sqrt{nh_q}} \left( \delta_0 + h_q^{-1}(X_1 - x) \delta_1 \right) k \left( h_q^{-1}(X_1 - x) \right) \tau_n \left( g(X_1) \right) \right] \\
= \sqrt{\frac{h_q}{n}} \int (\delta_0 + \delta_1 t_1) k(t_1) \tau_n \left( g(x + h_q t_1) \right) g(x + h_q t_1) dt_1 \\
\leq \sqrt{\frac{h_q}{n}} g(x) \delta_0 + O \left( \sqrt{\frac{h_q}{n} \cdot h_q^2} \right),
\end{align*}
\]

and

\[
Var \left[ \frac{1}{\sqrt{nh_q}} \left( \delta_0 + h_q^{-1}(X_1 - x) \delta_1 \right) k \left( h_q^{-1}(X_1 - x) \right) \tau_n \left( g(X_1) \right) \right] = O \left( \frac{1}{n} \right),
\]

so

\[
\frac{1}{\sqrt{nh_q}} \max_i \left[ \delta_0 + h_q^{-1}(X_i - x) \delta_1 \right] k \left( h_q^{-1}(X_i - x) \right) \tau_n \left( g(X_i) \right) = O_p \left( \sqrt{\frac{h_q}{n} + \frac{1}{\sqrt{n}}} \right) \\
= o_p(1).
\]
Combine this with (28) to deduce that
\[
\sum_{i=1}^{n} (Z_{n2i}(\delta) - E[Z_{n2i}(\delta)|X_i]) = o_p(1)
\]
for each \((\alpha, x) \in A \times \mathcal{X}\) and each \(\delta \in \mathbb{R}^2\).

It follows that for each \(\delta \in \mathbb{R}^2\) and \((\alpha, x) \in A \times \mathcal{X}\) we have
\[
Z_{n2}(\delta) = \frac{1}{2} f_{Y_i|X_i=x}(q(\alpha, x)) g(x) \left[ \begin{array}{c} \delta_0 \\ \delta_1 \\ \end{array} \right] \left[ \begin{array}{c} \int k^2(t_1) \, dt_1 \\ \int t_1 k^2(t_1) \, dt_1 \\ \int t_1^2 k^2(t_1) \, dt_1 \\ \end{array} \right] \left[ \begin{array}{c} \delta_0 \\ \delta_1 \\ \end{array} \right] + o_p(1).
\]

Next, consider the asymptotic behaviour of \(Z_{n1}(\delta)\) on \(A \times \mathcal{X}\). Recall the definition of \(\bar{u}_i^*(x)\) in (23) and define
\[
\bar{Z}_{n1}(\delta) = -\frac{1}{\sqrt{n h^2_q}} \sum_{i=1}^{n} \left[ \delta_0 + h^{-1}_q (X_i - x) \delta_1 \right] \psi_\alpha (\bar{u}_i^*(x)) k \left( h^{-1}_q (X_i - x) \right) \tau_n (g (X_i)).
\]

Then
\[
\left| Z_{n1}(\delta) - \bar{Z}_{n1}(\delta) \right| \\
\leq \frac{1}{\sqrt{n h^2_q}} \sum_{i=1}^{n} \sup_{x \in \mathcal{X}} \left| \left[ \delta_0 + h^{-1}_q (X_i - x) \delta_1 \right] k \left( h^{-1}_q (X_i - x) \right) \right| \\
\cdot \left| \tau_n (g (X_i)) - \tau_n (\hat{g}_n (X_i)) \right| \\
= \frac{1}{\sqrt{n h^2_q}} \sum_{i=1}^{n} \left| \delta_0 + T_i \delta_1 \right| k(T_i) \sup_{x \in \mathcal{X}} \left| \tau_n (g (x + h_q T_i)) - \tau_n (\hat{g}_n (x + h_q T_i)) \right| \\
= \frac{1}{\sqrt{n h^2_q}} \sum_{i=1}^{n} \left| \delta_0 + T_i \delta_1 \right| k(T_i) \sup_{x \in \mathcal{X}} \left| \tau_n (\hat{g}_n (x)) - \tau_n (g(x)) + O_p(h_q) \right| .
\]

From (22) we have
\[
\sup_{x \in \mathcal{X}} \left| \tau_n (\hat{g}_n (x)) - \tau_n (g(x)) \right| = O_p \left( n^\xi \left( h^2_q + \frac{1}{\sqrt{n h^2_q}} \right) \right).
\]

Assuming that \(n^\xi \left( h^2_q + 1/\sqrt{n h^2_q} \right) = o(1)\) and invoking the various conditions of Assumptions A.3 and A.4 on \(g(\cdot)\) and \(k(\cdot)\), we have
\[
\left| Z_{n1}(\delta) - \bar{Z}_{n1}(\delta) \right| = o_p(1)
\]
(30)
for each \((\alpha, x)\) \(\in \mathcal{A} \times \mathcal{X}\).

The remainder of the proof proceeds according to two steps, which will be discussed in sequence. The first step implies the convergence in distribution of \(\bar{Z}_{n1}(\delta)\) for each \(\delta \in \mathbb{R}^2\) and each \((\alpha, x)\) \(\in \mathcal{A} \times \mathcal{X}\). The second step ensures that for each \((\alpha, x)\) \(\in \mathcal{A} \times \mathcal{X}\), the estimator \(\hat{q}^*_n(\alpha, x)\) has a bias that vanishes as \(n \to \infty\). Taken together, the results of these two steps imply an asymptotic representation for \(\hat{q}^*_n(\alpha, x)\) that holds pointwise on \(\mathcal{A} \times \mathcal{X}\), and from which the \(\sqrt{nh_q}\)-consistency of \(\hat{q}^*_n(\alpha, x)\) for \(q(\alpha, x)/g(x)\) for each \((\alpha, x)\) \(\in \mathcal{A} \times \mathcal{X}\) can be deduced.

The two steps are described as follows. In particular, they involve showing:

(1) That
\[
\frac{1}{\sqrt{nh_q}} \sum_{i=1}^{n} (\psi_\alpha (\bar{u}_i^*(x)) - E [\psi_\alpha (\bar{u}_i^*(x))|X_i]) \left[ \frac{1}{h_q^{-1} (X_i - x)} \right] k \left( h_q^{-1} (X_i - x) \right) \tau_n (g (X_i)) \xrightarrow{d} N (0, \Sigma(\alpha, x))
\]

for each \((\alpha, x)\) \(\in \mathcal{A} \times \mathcal{X}\), where \(\Sigma(\alpha, x)\) is a symmetric and positive-definite matrix for which a closed-form representation will be derived.

(2) That
\[
\sqrt{\frac{n}{h_q}} E \left[ \psi_\alpha (\bar{u}_i^*(x)) \left[ \frac{1}{h_q^{-1} (X_i - x)} \right] k \left( h_q^{-1} (X_i - x) \right) \tau_n (g (X_i)) \right] = o(1)
\]

pointwise on \(\mathcal{A} \times \mathcal{X}\). Moreover, the rate of convergence is sufficiently fast for the bias of the estimator \(\hat{q}^*_n(\alpha, x)\) to vanish in large samples.

### B.2 First step: Convergence in finite-dimensional distributions

If the conditions of Liapounov’s central limit theorem are met, we have for each \((\alpha, x)\) \(\in \mathcal{A} \times \mathcal{X}\) that
\[
\frac{1}{\sqrt{nh_q}} \sum_{i=1}^{n} (\psi_\alpha (\bar{u}_i^*(x)) - E [\psi_\alpha (\bar{u}_i^*(x))|X_i]) \left[ \frac{1}{h_q^{-1} (X_i - x)} \right] k \left( h_q^{-1} (X_i - x) \right) \tau_n (g (X_i)) \xrightarrow{d} N (0, \Sigma(\alpha, x))
\]

(31)
where

\[
\Sigma(\alpha, x) = \lim_{n \to \infty} E \left[ h_q^{-1} Var \left[ \psi_\alpha (\bar{u}_1^*(x)) \right] | X_1 \right] \left[ \begin{array}{cc}
1 & h_q^{-1} (X_1 - x) \\
\frac{1}{h_q} (X_1 - x) & h_q^{-2} (X_1 - x)^2
\end{array} \right] h_q^{-1} (X_1 - x) \tau_n^2 (g (X_1)) \right).
\]

A closed-form representation for \( \Sigma(\alpha, x) \) is first derived. Let \( T_1 \equiv h_q^{-1} (X_1 - x) \), so that \( X_1 = x + h_q T_1 \). For some \( \xi_1 \) on a line segment between \( x \) and \( X_1 \) we have

\[
\frac{q(\alpha, X_1)}{g(X_1)} = \frac{q(\alpha, x + h_q T_1)}{g(x + h_q T_1)} = \frac{q(\alpha, x)}{g(x)} + h_q T_1 q_g^{(1)} (\alpha, x) + \frac{h_q^2}{2} T_1^2 q_g^{(2)} (\alpha, \xi_1).
\]

We have

\[
E \left[ \psi_\alpha (\bar{u}_1^*(x)) | X_1 \right] = \alpha - P \left[ \frac{Y_1}{g(X_1)} \leq \frac{q(\alpha, x)}{g(x)} + (X_1 - x) q_g^{(1)} (\alpha, x) \right] \left[ X_1 \right]
\]

\[
\begin{align*}
&= \alpha - P \left[ \frac{Y_1}{g(X_1)} \leq \frac{q(\alpha, x)}{g(x)} + h_q T_1 q_g^{(1)} (\alpha, x) \right] \left[ X_1 \right] \\
&= \alpha - P \left[ \frac{Y_1}{g(X_1)} \leq \frac{q(\alpha, X_1)}{g(X_1)} - \frac{h_q^2}{2} T_1^2 q_g^{(2)} (\alpha, \xi_1) \right] \left[ X_1 \right] \\
&= \alpha - \left( F_1 (q (\alpha, X_1)) - \frac{h_q^2}{2} g (X_1) T_1^2 q_g^{(2)} (\alpha, \xi_1) f_1 (\xi_1 (\alpha, h_q)) \right) \\
&= \frac{h_q^2}{2} g (X_1) T_1^2 q_g^{(2)} (\alpha, \xi_1) f_1 (\xi_1 (\alpha, h_q))
\end{align*}
\]

for some \( \xi_1 (\alpha, h_q) \) in the interval \( \left( q (\alpha, X_1) - h_q^2 / 2 g (X_1) T_1^2 q_g^{(2)} (\alpha, \xi_1), q (\alpha, X_1) \right) \).
As such,

\[ Var \left[ \psi_\alpha (\bar{u}_1^*(x)) \right| X_1 \]
\[ = E \left[ \psi_\alpha^2 (\bar{u}_1^*(x)) \right| X_1 \]  
\[ - (E \left[ \psi_\alpha (\bar{u}_1^*(x)) \right| X_1]^2 \]
\[ = (\alpha - 1)^2 P \left[ \frac{\gamma_1}{g(X_1)} \geq \frac{\epsilon_1}{g(X_1)} - \frac{\epsilon_2}{2} q_g^{(2)}(\alpha, \xi_1) \right| X_1 \]
\[ + \alpha^2 P \left[ \frac{\gamma_1}{g(X_1)} > \frac{\epsilon_1}{g(X_1)} - \frac{\epsilon_2}{2} q_g^{(2)}(\alpha, \xi_1) \right| X_1 \]
\[ - \frac{\epsilon^2}{4} g^2(X_1) T^{(2)}_1 \left( q_g^{(2)}(\alpha, \xi_1) \right)^2 f_1^2(\zeta_1, \alpha, h_q) \]
\[ = (\alpha - 1)^2 \left( \alpha - \frac{\epsilon^2}{2} g(X_1) T^{(2)}_1 q_g^{(2)}(\alpha, \xi_1) f_1(\zeta_1, \alpha, h_q) \right) \]
\[ + \alpha^2 \left( 1 - \alpha + \frac{\epsilon^2}{2} g(X_1) T^{(2)}_1 q_g^{(2)}(\alpha, \xi_1) f_1(\zeta_1, \alpha, h_q) \right) \]
\[ - \frac{\epsilon^4}{4} g^2(X_1) T^{(2)}_1 \left( q_g^{(2)}(\alpha, \xi_1) \right)^2 f_1^2(\zeta_1, \alpha, h_q) \]
\[ = \alpha (1 - \alpha) + \frac{\epsilon^2}{2} \cdot \frac{2\alpha - 1}{2} g(X_1) T^{(2)}_1 q_g^{(2)}(\alpha, \xi_1) f_1(\zeta_1, \alpha, h_q) \]
\[ - \frac{\epsilon^4}{4} g^2(X_1) T^{(2)}_1 \left( q_g^{(2)}(\alpha, \xi_1) \right)^2 f_1^2(\zeta_1, \alpha, h_q) \].

Therefore

\[ E \left[ \epsilon_q^{-1} Var \left[ \psi_\alpha (\bar{u}_1^*(x)) \right| X_1 \right| X_1 \]
\[ = (1 - \alpha) E \left[ \epsilon_q^{-1} k^2 \left( \epsilon_q^{-1} (X_1 - x) \right) \tau_n^2(g(X_1)) \right] \]
\[ + \frac{2\alpha - 1}{2} E \left[ \epsilon_q^{-1} g(X_1) T^{(2)}_1 h_q^{(2)}(\alpha, \xi_1) f_1(\zeta_1, \alpha, h_q) \right. \]
\[ \cdot k^2 \left( \epsilon_q^{-1} (X_1 - x) \right) \tau_n^2(g(X_1)) \]
\[ - \frac{1}{4} E \left[ \epsilon_q^{-1} g^2(X_1) T^{(2)}_1 h_q^4 \left( q_g^{(2)}(\alpha, \xi_1) \right)^2 f_1^2(\zeta_1, \alpha, h_q) \right. \cdot k^2 \left( \epsilon_q^{-1} (X_1 - x) \right) \tau_n^2(g(X_1)) \]
\[ \equiv \Sigma_{111} + \Sigma_{112} + \Sigma_{113}. \]
We have for $\xi_1$ an intermediate point between $X_1$ and $x$ that

$$
\Sigma_{111} = \alpha (1 - \alpha) h_q^{-1} \int k^2 (h_q^{-1} (x_1 - x)) \tau_n^2 (g(x_1)) g(x_1) \, dx_1
$$

$$
= \alpha (1 - \alpha) \int k^2 (t_1) \tau_n^2 (g(x + h_q t_1)) g(x + h_q t_1) \, dt_1
$$

$$
= \alpha (1 - \alpha) \int k^2 (t_1) \left[ \tau_n^2 (g(x)) g(x) + h_q t_1 h_q (2 \tau_n (g(\xi_1)) \tau_n^{(1)} (g(\xi_1)) g^{(1)} (\xi_1) g(\xi_1) + \tau_n^2 (g(\xi_1)) g^{(1)} (\xi_1)) \right] \, dt_1
$$

$$
= \alpha (1 - \alpha) \tau_n^2 (g(x)) g(x) \int k^2 (t_1) \, dt_1 + O (h_q)
$$

$$
= \alpha (1 - \alpha) g(x) \int k^2 (t_1) \, dt_1 + \alpha (1 - \alpha) (\tau_n^2 (g(x)) - 1) g(x) \int k^2 (t_1) \, dt_1 + O (h_q)
$$

$$
= \alpha (1 - \alpha) g(x) \int k^2 (t_1) \, dt_1 + O (n^{-\gamma} + h_q)
$$

$$
= \alpha (1 - \alpha) g(x) \int k^2 (t_1) \, dt_1 + o(1).
$$

Now consider that for fixed $(\alpha, x) \in A \times X$ and some constants $(\gamma_1, \gamma_2) \in$
where amongst others, the conditions that

\[
\sup_{x \in \mathcal{X}} g(x) < \infty, \\
\sup_{\alpha \in \mathcal{A}, x \in \mathcal{X}} \sup_{x \in \mathcal{X}} f_{Y_1|X_1=x}(q(\alpha, x)) < \infty, \\
\sup_{\alpha \in \mathcal{A}, x \in \mathcal{X}} \sup_{x \in \mathcal{X}} |q_g^{(2)}(\alpha, x)| < \infty
\]

and

\[
\int t_1^4 k_2^2(t_1) \, dt_1 < \infty
\]
have been invoked.

Similarly, we have for fixed \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\) and some constants \((\gamma_1, \gamma_2) \in (0, 1)^2\) that

\[
\Sigma_{113} = -\frac{1}{4} h_q^{-1} \int g^3 (x_1) (x_1 - x)^4 \left| q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \right|^2 \\
\cdot f_1^2 \left( q (\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g (x_1) (x_1 - x)^2 \left| q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \right|^2 \right) \\
\cdot k^2 (h_q^{-1} (x_1 - x)) \tau_n^2 (g (x_1)) dx_1 \\
= -\frac{h_q^4}{4} \int g^3 (x + h_q t_1) t_1^4 \left( q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right)^2 \\
\cdot f_1^2 \left( q (\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right) \\
\cdot k^2 (t_1) \tau_n^2 (g (x + h_q t_1)) dt_1 \\
\leq -\frac{1}{4} \left( h_q^4 g^3 (x) \left( q_g^{(2)} (\alpha, x) \right)^2 \right) f_{Y_1 | X_1 = x} (q(\alpha, x)) \tau_n^2 (g(x)) \int t_1^4 k^2 (t_1) dt_1 \\
+ O \left( h_q^5 \right) \\
= O \left( h_q^4 \right) \\
= o(1),
\]

where amongst others, the conditions that

\[
\sup_{x \in \mathcal{X}} g(x) < \infty,
\]

\[
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1 | X_1 = x} (q(\alpha, x)) < \infty,
\]

\[
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| q_g^{(2)} (\alpha, x) \right| < \infty
\]

and

\[
\int t_1^8 k^2 (t_1) dt_1 < \infty
\]

have been invoked.
Next, consider

\[
E \left[ h_q^{-1} Var \left[ \psi_\alpha (\bar{u}_1(x)) \right] X_1 \right] k^2 (h_q^{-1} (x_1 - x)) \tau_n^2 (g (X_1)) h_q^{-1} (x_1 - x) \]
\[
= \alpha (1 - \alpha) E \left[ h_q^{-1} k^2 (h_q^{-1} (x_1 - x)) \tau_n^2 (g (X_1)) h_q^{-1} (X_1 - x) \right]
\]
\[
+ \frac{2 \alpha - 1}{2} E \left[ h_q^{-1} g (X_1) \cdot T_1^2 h_q^{-2} q_\alpha^{(2)} (\alpha, \xi_1) \cdot f_1 (\xi_1 (\alpha, h_q)) \right]
\]
\[
- \frac{1}{4} E \left[ h_q^{-1} g^2 (X_1) (h_q^{-2} T_1^2 q_\alpha^{(2)} (\alpha, \xi_1))^2 f_1^2 (\xi_1 (\alpha, h_q)) \right]
\]
\[
= \Sigma_{121} + \Sigma_{122} + \Sigma_{123}.
\]

We have

\[
\Sigma_{121} = \alpha (1 - \alpha) h_q^{-1} \int k^2 (h_q^{-1} (x_1 - x)) \tau_n^2 (g (x_1)) h_q^{-1} (x_1 - x) g (x_1) dx_1
\]
\[
= \alpha (1 - \alpha) \int k^2 (t_1) \tau_n^2 (g (x + h_q t_1)) t_1 g (x + h_q t_1) dt_1
\]
\[
= \alpha (1 - \alpha) g (x) \int t_1 k^2 (t_1) dt_1 + O \left( n^{-\epsilon} + h_q \right)
\]
\[
= \alpha (1 - \alpha) g (x) \int t_1 k^2 (t_1) dt_1 + o(1).
\]
For fixed \((\alpha, x) \in A \times X\) and some constants \((\gamma_1, \gamma_2) \in (0, 1)^2\),

\[
\Sigma_{122} = \frac{2\alpha - 1}{2} h_q^{-1} \int g^2(x_1) (x_1 - x)^2 q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x) \cdot f_1 \left( q(\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g(x_1) (x_1 - x)^2 q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x) \right)
\]

\[
+ \int_{t_1} k^2 \left( h_q^{-1} (x_1 - x) \right) \tau_n^2 (g(x_1)) \cdot h_q^{-1} (x_1 - x) dx_1
\]

\[
= \frac{2\alpha - 1}{2} h_q^2 \int g^2(x + h_q t_1) t_1^2 q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x) \cdot f_1 \left( q(\alpha, h_q t_1 + x) - \gamma_2 \cdot \frac{1}{2} g(x + h_q t_1) t_1^2 h_q^2 q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x) \right)
\]

\[
+ \int_{t_1} k^2 (t_1) \tau_n^2 (g(x + h_q t_1)) \cdot t_1 dt_1
\]

\[
\leq \sup_{x \in X} g(x) \cdot \sup_{\alpha \in A} \sup_{x \in X} |q_g^{(2)}(\alpha, x)| \cdot \sup_{\alpha \in A} \sup_{x \in X} f_{Y_1|X_1=x}(q(\alpha, x)) \cdot \frac{2\alpha - 1}{2} h_q^2 \int t_1^2 k^2 (t_1) t_1 dt_1 + O(h_q^3)
\]

\[
= O(h_q^2)
\]

\[
= o(1),
\]

where amongst others, the conditions that

\[
\sup_{x \in X} g(x) < \infty,
\]

\[
\sup_{\alpha \in A} \sup_{x \in X} f_{Y_1|X_1=x}(q(\alpha, x)) < \infty,
\]

\[
\sup_{\alpha \in A} \sup_{x \in X} |q_g^{(2)}(\alpha, x)| < \infty
\]

and

\[
\int t_1^6 k^2 (t_1) dt_1 < \infty
\]

have been invoked.

Similarly, we have for fixed \((\alpha, x) \in A \times X\) and some constants \((\gamma_1, \gamma_2) \in (0, 1)^2\),
that

\[ \sigma_{123} = -\frac{1}{4} h_q^{-1} \int g^3(x_1) (x_1 - x)^4 \left( q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x) \right)^2 \]

\[ \cdot f_1^2 \left( q(\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g(x_1) (x_1 - x)^2 q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x) \right) \]

\[ \cdot k^2 \left( h_q^{-1} (x_1 - x) \right) \tau_n^2 (g(x_1)) \cdot h_q^{-1} (x_1 - x) dx_1 \]

\[ = -\frac{1}{4} \cdot h_q^4 \int g^3(x + h_q t_1) t_1^4 \left( q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x) \right)^2 \]

\[ \cdot f_1^2 \left( q(\alpha, h_q t_1 + x) - \gamma_2 \cdot \frac{h_q^2}{2} g(x + h_q t_1) t_1^2 q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x) \right) \]

\[ \cdot k^2 (t_1) \tau_n^2 (g(x + h_q t_1)) \cdot t_1 dt_1 \]

\[ = -\frac{h_q^4}{4} \left( \int g^3(x) t_1^4 \left( q_g^{(2)}(\alpha, x) \right)^2 f_{Y_1|X_1=x}^2 (q(\alpha, x)) k^2 (t_1) \tau_n^2 (g(x)) t_1 dt_1 \right) \]

\[ + O \left( h_q^5 \right) \]

\[ \leq \sup_{x \in X} g^3(x) \cdot \sup_{\alpha \in A} \sup_{x \in X} \left| q_g^{(2)}(\alpha, x) \right| \cdot \sup_{\alpha \in A} \sup_{x \in X} f_{Y_1|X_1=x}^2 (q(\alpha, x)) \]

\[ \cdot \frac{1}{4} \left( h_q^4 \int t_1^4 k^2 (t_1) t_1 dt_1 + O \left( h_q^5 \right) \right) \]

\[ = O \left( h_q^4 \right) \]

\[ = \Theta(1), \]

where amongst others, the conditions that

\[ \sup_{x \in X} g(x) < \infty, \]

\[ \sup_{\alpha \in A} \sup_{x \in X} \left| q_g^{(2)}(\alpha, x) \right| < \infty, \]

\[ \sup_{\alpha \in A} \sup_{x \in X} \left| q_g^{(2)}(\alpha, x) \right| < \infty \]

and

\[ \int t_1^{10} k^2 (t_1) dt_1 < \infty \]

have been invoked.
Lastly, take

\[
E \left[ h_q^{-1} \text{Var} \left[ \psi_\alpha (\bar{u}^*_1(x)) \right] | X_1 \right] k^2 \left( h_q^{-1} (X_1 - x) \right) \tau_n^2 (g (X_1)) h_q^{-2} (X_1 - x)^2 \]
\[
= \alpha (1 - \alpha) E \left[ h_q^{-1} k^2 \left( h_q^{-1} (X_1 - x) \right) \tau_n^2 (g (X_1)) h_q^{-2} (X_1 - x)^2 \right] \\
+ \frac{2\alpha - 1}{2} E \left[ h_q^{-1} g (X_1) T_1^2 h_q^2 (\alpha, \xi) f_1 (\zeta_1 (\alpha, h_q)) k^2 \left( h_q^{-1} (X_1 - x) \right) \tau_n^2 (g (X_1)) \right] \\
h_q^{-2} (X_1 - x)^2 \\
- \frac{1}{4} E \left[ h_q^{-1} g^2 (X_1) T_1^4 h_q^4 (\alpha, \xi)^2 f_1^2 (\zeta_1 (\alpha, h_q)) k^2 \left( h_q^{-1} (X_1 - x) \right) \tau_n^2 (g (X_1)) \right] \\
h_q^{-2} (X_1 - x)^2 \\
\equiv \Sigma_{221} + \Sigma_{222} + \Sigma_{223}.
\]

We have

\[
\Sigma_{221} = \alpha (1 - \alpha) h_q^{-1} \int k^2 \left( h_q^{-1} (x_1 - x) \right) \tau_n^2 (g (x_1)) h_q^{-2} (x_1 - x)^2 g (x_1) \, dx_1 \\
= \alpha (1 - \alpha) \int k^2 (t_1) \tau_n^2 (g (x + h_q t_1)) t_1^2 g (x + h_q t_1) \, dt_1 \\
= \alpha (1 - \alpha) g(x) \int t_1^2 k^2 (t_1) \, dt_1 + O \left( n^{-\epsilon} + h_q \right) \\
= \alpha (1 - \alpha) g(x) \int t_1^2 k^2 (t_1) \, dt_1 + o(1).
\]
For fixed \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\) and some constants \((\gamma_1, \gamma_2) \in (0, 1)^2\),

\[
\Sigma_{222} = \frac{2\alpha - 1}{2} \cdot h_q^{-1} \int g^2(x_1)(x_1 - x)^2 q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x) \cdot f_1\left(q(\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g(x_1) (x_1 - x)^2 q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x)\right) \\
\cdot k^2 \left(h_q^{-1}(x_1 - x)\right) \tau_n^2(g(x_1)) \cdot h_q^{-2}(x_1 - x)^2 dx_1
\]

\[
= \frac{2\alpha - 1}{2} \cdot h_q^2 \int g^2(x + h_q t_1) t_1^2 q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x) \cdot f_1\left(q(\alpha, h_q t_1 + x) - \gamma_2 \cdot \frac{1}{2} g(x + h_q t_1) t_1^2 h_q^2 q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x)\right) \\
\cdot k^2(t_1) \tau_n^2(g(x + h_q t_1)) \cdot t_1^2 dt_1
\]

\[
= \frac{2\alpha - 1}{2} \left(h_q^2 \int g^2(x) t_1^4 q_g^{(2)}(\alpha, x) f_{Y_1|X_1=x}(q(\alpha, x)) k^2(t_1) \tau_n^2(g(x)) dt_1 + O\left(h_q^3\right)\right)
\]

\[
\leq \sup_{x \in \mathcal{X}} g^2(x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} |q_g^{(2)}(\alpha, x)| \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1|X_1=x}(q(\alpha, x)) \cdot \frac{2\alpha - 1}{2} \left(h_q^2 \int t_1^4 k^2(t_1) dt_1 + O\left(h_q^3\right)\right)
\]

\[
= O\left(h_q^2\right)
\]

\[
= o(1),
\]

where amongst others, the conditions that

\[
\sup_{x \in \mathcal{X}} g(x) < \infty,
\]

\[
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1|X_1=x}(q(\alpha, x)) < \infty,
\]

\[
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} |q_g^{(2)}(\alpha, x)| < \infty
\]

and

\[
\int t_1^8 k^2(t_1) dt_1 < \infty
\]

have been invoked.

Similarly, we have for fixed \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\) and some constants \((\gamma_1, \gamma_2) \in \)
\((0,1)^2\) that

\[
\Sigma_{223} = -\frac{1}{4} h_q^{-1} \int g^3(x_1) (x_1 - x)^4 \left( q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x) \right)^2
\]

\[
\cdot f_1^2 \left( q(\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g(x_1) (x_1 - x)^2 q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x) \right)
\]

\[
\cdot k^2 \left( h_q^{-1} (x_1 - x) \right) \tau_n^2(g(x_1)) \cdot h_q^{-2} (x_1 - x)^2 dx_1
\]

\[
= -\frac{1}{4} h_q \int g^3(x) (x + h_q t_1) t_1^4 \left( q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x) \right)^2
\]

\[
\cdot f_1^2 \left( q(\alpha, h_q t_1 + x) - \gamma_2 \cdot \frac{h_q^2}{2} g(x + h_q t_1) t_1^2 q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x) \right)
\]

\[
\cdot k^2(t_1) \tau_n^2(g(x + h_q t_1)) \cdot t_1^2 dt_1
\]

\[
= -\frac{1}{4} \left( h_q^4 \int g^3(x) t_1^4 \left( q_g^{(2)}(\alpha, x) \right)^2 f_{Y_1|X_1=x} q(\alpha, x) k^2(t_1) \tau_n^2(g(x)) t_1^2 dt_1
\]

\[
+ O(h_q^5)
\]

\[
\leq \sup_{x \in \mathcal{X}} g^3(x) \cdot \sup_{\alpha \in \mathcal{A}, x \in \mathcal{X}} \left| q_g^{(2)}(\alpha, x) \right| \cdot \sup_{\alpha \in \mathcal{A}, x \in \mathcal{X}} f_{Y_1|X_1=x} q(\alpha, x)
\]

\[
\cdot \frac{1}{4} \left( h_q^4 \int t_1^6 k^2(t_1) dt_1 + O(h_q^5) \right)
\]

\[
= O(h_q^4)
\]

\[
= o(1),
\]

where amongst others, the conditions that

\[
\sup_{x \in \mathcal{X}} g(x) < \infty,
\]

\[
\sup_{\alpha \in \mathcal{A}, x \in \mathcal{X}} f_{Y_1|X_1=x} q(\alpha, x) < \infty,
\]

\[
\sup_{\alpha \in \mathcal{A}, x \in \mathcal{X}} \left| q_g^{(2)}(\alpha, x) \right| < \infty
\]

and

\[
\int t_1^{12} k^2(t_1) dt_1 < \infty
\]

have been invoked.

Combining results, we have for each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\) that

\[
\Sigma(\alpha, x) = \alpha(1 - \alpha) g(x) \left[ \int k^2(t_1) dt_1 \int t_1 k^2(t_1) dt_1 \right] + o(1).
\]
The next step in the proof involves the verification of the convergence in (31) by showing that Liapounov’s condition holds. In this connection, we note that for Liapounov’s condition to hold in this case it is sufficient if both

\[
(nh_q)^{-1} h_q^{-1} E \left[ (\psi_\alpha (\bar{u}_1^*(x)) - E [\psi_\alpha (\bar{u}_1^*(x)) | X_1] \right]^4 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \rightarrow 0
\] (34)

and

\[
(nh_q)^{-1} h_q^{-1} E \left[ (\psi_\alpha (\bar{u}_1^*(x)) - E [\psi_\alpha (\bar{u}_1^*(x)) | X_1] \right]^4 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \cdot [h_q^{-1} (X_1 - x)]^8 \rightarrow 0
\] (35)

hold for any \((\alpha, x) \in A \times X\).

Conditions (34) and (35) are verified in sequence.

With respect to (34), let \(T_1 = h_q^{-1} (X_1 - x)\). Let \(\xi_1\) be a point on the line segment between \(x\) and \(X_1 = x + h_q T_1\). Let \(\zeta_1 (\alpha, h_q)\) be a point in the interval

\[
\left( q (\alpha, X_1) - \frac{h_q^2}{2} g (X_1) T_1^2 q_g^{(2)} (\alpha, \xi_1), q (\alpha, X_1) \right).
\]

Define

\[
\mu_{h_q} \equiv \frac{1}{2} g (X_1) (X_1 - x)^2 q_g^{(2)} (\alpha, \xi_1) f_1 (\zeta_1 (\alpha, h_q)).
\]

Recall from (32) that \(E [\psi_\alpha (\bar{u}_1^*(x)) | X_1] = \mu_{h_q}\), and from (33) that

\[
E [\psi_\alpha^2 (\bar{u}_1^*(x)) | X_1] = \alpha (1 - \alpha) + (2 \alpha - 1) \mu_{h_q}.
\]

Similarly, it is possible to show that

\[
E [\psi_\alpha^3 (\bar{u}_1^*(x)) | X_1] = \alpha (1 - \alpha) (2 \alpha - 1) + (3 \alpha^2 - 3 \alpha + 1) \mu_{h_q},
\]

\[
E [\psi_\alpha^4 (\bar{u}_1^*(x)) | X_1] = \alpha (1 - \alpha) (3 \alpha^2 - 3 \alpha + 1) + (4 \alpha^3 - 6 \alpha^2 + 4 \alpha - 1) \mu_{h_q}.
\]

It follows that

\[
E [\psi_\alpha^4 (\bar{u}_1^*(x)) k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1))]
\]

\[
= E \left[ E [\psi_\alpha^4 (\bar{u}_1^*(x)) | X_1] k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right]
\]

\[
= \alpha (1 - \alpha) (3 \alpha^2 - 3 \alpha + 1) E \left[ k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right]
\]

\[
+ (4 \alpha^3 - 6 \alpha^2 + 4 \alpha - 1) E \left[ \mu_{h_q} k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right];
\]

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that

\[ E \left[ \psi_\alpha^3 (\bar{u}_1^* (x)) \mu_{h_q} k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]
\[ = E \left[ E \left[ \psi_\alpha^3 (\bar{u}_1^* (x)) \mid X_1 \right] \mu_{h_q} k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]
\[ = \alpha (1 - \alpha) (2 \alpha - 1) E \left[ \mu_{h_q} k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]
\[ + (3 \alpha^2 - 3 \alpha + 1) \left[ \mu_{h_q} k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] ; \]

that

\[ E \left[ \psi_\alpha^2 (\bar{u}_1^* (x)) \mu_{h_q}^2 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]
\[ = E \left[ E \left[ \psi_\alpha^2 (\bar{u}_1^* (x)) \mid X_1 \right] \mu_{h_q}^2 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]
\[ = \alpha (1 - \alpha) E \left[ \mu_{h_q}^2 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]
\[ + (2 \alpha - 1) \left[ \mu_{h_q}^2 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] ; \]

and that

\[ E \left[ \psi_\alpha (\bar{u}_1^* (x)) \mu_{h_q}^3 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]
\[ = E \left[ E \left[ \psi_\alpha (\bar{u}_1^* (x)) \mid X_1 \right] \mu_{h_q}^3 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]
\[ = E \left[ \mu_{h_q}^3 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] . \]

Note that

\[ h_q^{-1} E \left[ \left( \psi_\alpha (\bar{u}_1^* (x)) \right) - E \left[ \psi_\alpha (\bar{u}_1^* (x)) \mid X_1 \right] \right] \]
\[ = h_q^{-1} E \left[ \left( \psi_\alpha^4 (\bar{u}_1^* (x)) - 4 \psi_\alpha^3 (\bar{u}_1^* (x)) \mu_{h_q} + 6 \psi_\alpha^2 (\bar{u}_1^* (x)) \mu_{h_q}^2 - 4 \psi_\alpha (\bar{u}_1^* (x)) \mu_{h_q}^3 + \mu_{h_q}^4 \right) \right] \]
\[ \cdot k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] . \]

As such, we consider the following asymptotic representations in sequence, where the various conditions imposed on \( k(\cdot), h_q, \) and on \( g(x), f_{Y_1 \mid X_1=x} (q(\alpha, x)) \) and \( q^{(2)}(\alpha, x) \) for \((\alpha, x) \in A \times \mathcal{X}, \) amongst others, are invoked:

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For fixed $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ and some $(\gamma_1, \gamma_2) \in (0, 1)^2$, we have the following:

\begin{align*}
(1) \quad h_q^{-1} E \left[ k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] &= h_q^{-1} \int k^4 \left( h_q^{-1} (x_1 - x) \right) \tau_n^4 (g (x_1)) g (x_1) \, dx_1 \\
&= \int k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) g (x + h_q t_1) \, dt_1 \\
&= \tau_n^4 (g(x)) g(x) \int k^4 (t_1) \, dt_1 + O (h_q) \\
&= g (x) \int k^4 (t_1) \, dt_1 + \left( \tau_n^4 (g(x)) - 1 \right) g(x) \int k^4 (t_1) \, dt_1 + O (h_q) \\
&= g (x) \int k^4 (t_1) \, dt_1 + O (n^{-\epsilon} + h_q) \\
&= g (x) \int k^4 (t_1) \, dt_1 + o(1).
\end{align*}

(2)
\[ \forall q \in \mathbb{R}, \quad h_q^{-1} E \left[ \mu_h k^4 (h_q^{-1} (X_1 - x)) \tau_n^4 (g (X_1)) \right] \]
\[ = \frac{1}{2} h_q^{-1} E \left[ g (X_1) (X_1 - x) q_g^{(2)} (\alpha, \xi_1) f_1 (\xi_1 (\alpha, h_q)) \cdot k^4 (h_q^{-1} (X_1 - x)) \tau_n^4 (g (X_1)) \right] \]
\[ = \frac{1}{2} h_q^{-1} \int g^2 (x_1) (x_1 - x)^2 q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \]
\[ \cdot f_1 \left( q (\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g (x_1) (x_1 - x)^2 q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \right) \]
\[ \cdot k^4 (h_q^{-1} (x_1 - x)) \tau_n^4 (g (x_1)) \, dx_1 \]
\[ = \frac{h_q^2}{2} \int g^2 (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \]
\[ \cdot f_1 \left( q (\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right) \]
\[ \cdot k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) \, dt_1 \]
\[ = \frac{1}{2} \left( \frac{h_q^2}{2} \int g^2 (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) f_1 (q (\alpha, x + h_q t_1)) \right) \]
\[ \cdot k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) \, dt_1 \]
\[ + O \left( h_q^3 \right) \]
\[ = \frac{1}{2} \left( \frac{h_q^2}{2} \int g^2 (x) q_g^{(2)} (\alpha, x) f_{Y_{1 | X_1 = x}} (q (\alpha, x)) \tau_n^4 (g (x)) \int t_1^2 k^4 (t_1) \, dt_1 \right) \]
\[ + O \left( h_q^3 \right) \]
\[ \leq \sup_{x \in X} g^2 (x) \cdot \sup_{\alpha \in A} \sup_{x \in X} \left| q_g^{(2)} (\alpha, x) \right| \cdot \sup_{\alpha \in A} \sup_{x \in X} f_{Y_{1 | X_1 = x}} (q (\alpha, x)) \]
\[ \cdot \frac{h_q^2}{2} \left( \int t_1^2 k^4 (t_1) \, dt_1 + O \left( h_q^3 \right) \right) \]
\[ = O \left( h_q^2 \right) \]
\[ = o(1) \]
\[ h_q^{-1} E \left[ \mu_n^2 k^4 (h_q^{-1} (X_1 - x)) \right] \left[ \tau_n^4 (g (X_1)) \right] \]

\[ = \frac{1}{4} h_q^{-1} E \left[ g^2 (X_1) (X_1 - x)^4 (q_g^{(2)} (\alpha, \zeta_1))^2 f_1^2 (\zeta_1 (\alpha, h_q)) \right] \]

\[ \cdot k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \]

\[ = \frac{1}{4} h_q^{-1} \int g^3 (x_1) (x_1 - x)^4 (q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x))^2 \]

\[ \cdot f_1^2 \left( q (\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g (x_1) (x_1 - x) \right)^2 q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \]

\[ \cdot k^4 \left( h_q^{-1} (x_1 - x) \right) \tau_n^4 (g (x_1)) \, dx_1 \]

\[ = \frac{h_q^4}{4} \int g^3 (x + h_q t_1) t_1^4 (q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x))^2 \]

\[ \cdot f_1^2 \left( q (\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right) \]

\[ \cdot k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) \, dt_1 \]

\[ = \frac{1}{4} \left( h_q^4 \int g^3 (x + h_q t_1) t_1^4 (q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x))^2 f_1^2 (q (\alpha, x + h_q t_1)) \right) \]

\[ \cdot k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) \, dt_1 + O \left( h_q^5 \right) \]

\[ = \frac{1}{4} \left( h_q^4 g^3 (x) (q_g^{(2)} (\alpha, x))^2 f_1^2 \int_{Y_1 \mid X_1 = x} (q (\alpha, x)) \tau_n^4 (g (x)) \right) \int t_1^4 k^4 (t_1) \, dt_1 + O \left( h_q^5 \right) \]

\[ \leq \sup_{x \in X} g^3 (x) \cdot \sup_{\alpha \in A} \sup_{x \in X} (q_g^{(2)} (\alpha, x))^2 \cdot \sup_{\alpha \in A} \sup_{x \in X} f_1^2 \int_{Y_1 \mid X_1 = x} (q (\alpha, x)) \]

\[ \cdot \frac{1}{4} \left( h_q^4 \int t_1^4 k^4 (t_1) \, dt_1 + O \left( h_q^5 \right) \right) \]

\[ = O \left( h_q^4 \right) \]

\[ = o(1), \]
\[
\begin{align*}
&h_q^{-1} E \left[ \mu_{h_q}^3 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 \left( g (X_1) \right) \right] \\
&= \frac{1}{8} h_q^{-1} E \left[ g^3 (X_1) (X_1 - x)^6 \left( q_g^{(2)} (\alpha, \xi_1) \right)^3 f_1^3 \left( \zeta_1 (\alpha, h_q) \right) \right. \\
&\left. \cdot k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 \left( g (X_1) \right) \right] \\
&= \frac{1}{8} h_q^{-1} \int g^4 (x_1) (x_1 - x)^6 \left( q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \right)^3 \\
&\cdot f_1^3 \left( q (\alpha, x) - \gamma_2 \cdot \frac{1}{2} g (x_1) (x_1 - x)^2 q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \right) \\
&\cdot k^4 \left( h_q^{-1} (x_1 - x) \right) \tau_n^4 \left( g (x_1) \right) dx_1 \\
&= \frac{h_q^6}{8} \int g^4 (x + h_q t_1) t_1^6 \left( q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right)^3 \\
&\cdot f_1^3 \left( q (\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right) \\
&\cdot k^4 \left( t_1 \right) \tau_n^4 \left( g (x + h_q t_1) \right) dt_1 \\
&= \frac{1}{8} \left( h_q^6 \int g^4 (x + h_q t_1) t_1^6 \left( q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right)^3 f_1^3 \left( q (\alpha, x + h_q t_1) \right) \\
&\cdot k^4 \left( t_1 \right) \tau_n^4 \left( g (x + h_q t_1) \right) dt_1 \\
&+ O \left( h_q^8 \right) \right) \\
&= \frac{1}{8} \left( h_q^6 g^4 (x) \left( q_g^{(2)} (\alpha, x) \right)^3 f_1^3 \left( q (\alpha, x) \right) \int t_1^6 k^4 (t_1) dt_1 \\
&+ O \left( h_q^7 \right) \right) \\
&\leq \sup_{x \in X} g^4 (x) \cdot \sup_{\alpha \in A, x \in X} \left( q_g^{(2)} (\alpha, x) \right)^3 \cdot \sup_{\alpha \in A, x \in X} f_1^3 \left( q (\alpha, x) \right) \\
&\cdot \frac{1}{8} \left( h_q^6 \int t_1^6 k^4 (t_1) dt_1 + O \left( h_q^7 \right) \right) \\
&= O \left( h_q^6 \right) \\
&= o(1),
\end{align*}
\]

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d) 

\[
\begin{align*}
& h_q^{-1} E \left[ \mu_{h_q}^2 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \\
& = \frac{1}{16} h_q^{-1} E \left[ g^4 (X_1) (X_1 - x)^8 \left( q_g^{(2)} (\alpha, \xi_1) \right)^4 f_1^4 (\xi_1 (\alpha, h_q)) \cdot k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \\
& = \frac{1}{16} h_q^{-1} \int g^5 (x_1) (x_1 - x)^8 \left( q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \right)^4 \\
& \quad \cdot f_1^4 \left( q (\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g (x_1) (x_1 - x)^2 q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \right) \\
& \quad \cdot k^4 \left( h_q^{-1} (x_1 - x) \right) \tau_n^4 (g (x_1)) dx_1 \\
& = \frac{h_q^8}{16} \int g^5 (x + h_q t_1) t_1^8 \left( q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right)^4 \\
& \quad \cdot f_1^4 \left( q (\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right) \\
& \quad \cdot k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) dt_1 \\
& = \frac{1}{16} \left( h_q^5 \int g^5 (x + h_q t_1) t_1^8 \left( q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right)^4 f_1^4 (q (\alpha, x + h_q t_1)) \\
& \quad \cdot k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) dt_1 \\
& \quad + O (h_q^{10}) \right) \\
& = \frac{1}{16} \left( h_q^5 \int g^5 (x) \left( q_g^{(2)} (\alpha, x) \right)^4 f_1^4_{Y_1 | X_1 = x} (q(\alpha, x)) \tau_n^4 (g(x)) \int t_1^8 k^4 (t_1) dt_1 \\
& \quad + O (h_q^9) \right) \\
& \leq \sup_{x \in \mathcal{X}} g^5 (x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left( q_g^{(2)} (\alpha, x) \right)^4 \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_1^4_{Y_1 | X_1 = x} (q(\alpha, x)) \\
& \quad \cdot \frac{1}{16} \left( h_q^8 \int t_1^8 k^4 (t_1) dt_1 + O (h_q^9) \right) \\
& = O (h_q^8) \\
& = o(1).
\end{align*}
\]

Combining results, we have for each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\) that

\[
\begin{align*}
& h_q^{-1} E \left[ (\psi_\alpha (\bar{u}_1^* (x)) - E [\psi_\alpha (\bar{u}_1 (x))] | X_1) \right]^4 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \\
& \to \alpha (1 - \alpha) (3 \alpha^2 - 3 \alpha + 1) g(x) \int k^4 (t_1) dt_1.
\end{align*}
\]
As such, provided that $nh_q \to \infty$, (34) holds.

Now consider (35). We have

$$h_q^{-1}E\left[\left(\psi_\alpha(\bar{u}_1^*(x)) - E[\psi_\alpha(\bar{u}_1^*(x)|X_1]\right]^4 k^4 \left(h_q^{-1} (X_1 - x)\right) \tau_n^4 \left(g(X_1)\right) \cdot \left[h_q^{-1} (X_1 - x)\right]^8\right]$$

$$= h_q^{-1}E\left[\left(\psi_\alpha^4(\bar{u}_1^*(x)) - 4\psi_\alpha^3(\bar{u}_1^*(x)) \mu_{h_q} + 6\psi_\alpha^2(\bar{u}_1^*(x)) \mu_{h_q}^2 - 4\psi_\alpha(\bar{u}_1^*(x)) \mu_{h_q}^3 + \mu_{h_q}^4\right) \cdot k^4 \left(h_q^{-1} (X_1 - x)\right) \tau_n^4 \left(g(X_1)\right) \cdot \left[h_q^{-1} (X_1 - x)\right]^8\right].$$

As such, we consider the following asymptotic representations in sequence, where the various conditions imposed on $k(\cdot)$, $h_q$, and on $g(x)$, $f_{X_1|X_1=x}(q(\alpha, x))$ and $q^{(2)}(\alpha, x)$ for $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$, amongst others, are invoked:

(1)

$$h_q^{-1}E\left[\left(h_q^{-1} (X_1 - x)\right)^8 k^4 \left(h_q^{-1} (X_1 - x)\right) \tau_n^4 \left(g(X_1)\right)\right]$$

$$= \int t_1^8 k^4(t_1) \tau_n^4(g(x + h_q t_1)) g(x + h_q t_1) \, dt_1$$

$$= \tau_n^4(g(x)) g(x) \int t_1^8 k^4(t_1) \, dt_1 + O(h_q)$$

$$= g(x) \int t_1^8 k^4(t_1) \, dt_1 + (\tau_n^4(g(x)) - 1) \, g(x) \int t_1^8 k^4(t_1) \, dt_1 + O(h_q)$$

$$= g(x) \int t_1^8 k^4(t_1) \, dt_1 + O\left(n^{-\gamma} + h_q\right)$$

$$= g(x) \int t_1^8 k^4(t_1) \, dt_1 + o(1).$$

(2) For fixed $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ and some $(\gamma_1, \gamma_2) \in (0, 1)^2$, we have the following:
\( h_q^{-1} E \left[ \left[ h_q^{-1} (X_1 - x) \right]^{10} \mu_{h_q} k^4 (h_q^{-1} (X_1 - x)) \tau_n^4 (g (X_1)) \right] \)

\[
= \frac{1}{2} h_q^{-1} E \left[ \left[ h_q^{-1} (X_1 - x) \right]^{10} g (X_1) q_g^{(2)} (\alpha, \xi_1) \cdot f_1 (\zeta_1 (\alpha, h_q)) k^4 (h_q^{-1} (X_1 - x)) \tau_n^4 (g (X_1)) \right] 
\]

\[
= \frac{h_q^2}{2} \int t_1^{10} g^2 (x + h_q t_1) q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) 
\cdot f_1 \left( q (\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right) 
\cdot k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) \, dt_1 
\]

\[
= \frac{1}{2} \left( h_q^2 g^2 (x) q_g^{(2)} (\alpha, x) f_{Y_1 \mid X_1 = x} (q (\alpha, x)) \tau_n^4 (g (x)) \int t_1^{10} k^4 (t_1) \, dt_1 
+ O \left( h_q^3 \right) \right) 
\]

\[
\leq \sup_{x \in \tilde{X}} g^2 (x) \cdot \sup_{\alpha \in A} \sup_{x \in \tilde{X}} |q_g^{(2)} (\alpha, x)| \cdot \sup_{\alpha \in A} \sup_{x \in \tilde{X}} f_{Y_1 \mid X_1 = x} (q (\alpha, x)) 
\cdot \frac{1}{2} \left( h_q^2 \int t_1^{10} k^4 (t_1) \, dt_1 + O \left( h_q^3 \right) \right) 
\]

\[
= O \left( h_q^2 \right) 
\]

\[
= o(1), 
\]
b) 

\[ h_q^{-1} \mathbb{E} \left[ \left( h_q^{-1} (X_1 - x) \right)^8 \mu_{h_q}^2 k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]

\[ = \frac{1}{4} h_q^{-1} \mathbb{E} \left[ \left( h_q^{-1} (X_1 - x) \right)^{12} \left( q_g^{(2)} (\alpha, \xi_1) \right)^2 \cdot f_1^2 (\zeta_1 (\alpha, h_q)) k^4 \left( h_q^{-1} (X_1 - x) \right) \tau_n^4 (g (X_1)) \right] \]

\[ = \frac{h_q^4}{4} \int t_1^{12} g^3 (x + h_q t_1) \left( q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right)^2 \cdot f_1^2 \left( q (\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right) \cdot k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) dt_1 \]

\[ = \frac{1}{4} \left( h_q^4 g^3 (x) \left( q_g^{(2)} (\alpha, x) \right)^2 f_1^2 \int_{Y_1 | X_1 = x} (q (\alpha, x)) \tau_n^4 (g (x)) \int t_1^{12} k^4 (t_1) dt_1 \right. \]

\[ + O \left( h_q^3 \right) \]

\[ \leq \sup_{x \in \mathcal{X}} g^3 (x) \cdot \sup_{\alpha \in A} \sup_{x \in \mathcal{X}} \left( q_g^{(2)} (\alpha, x) \right)^2 \cdot \sup_{\alpha \in A} \sup_{x \in \mathcal{X}} f_1^2 \int_{Y_1 | X_1 = x} (q (\alpha, x)) \]

\[ \cdot \frac{1}{4} \left( h_q^4 \int t_1^{12} k^4 (t_1) dt_1 + O \left( h_q^3 \right) \right) \]

\[ = O \left( h_q^4 \right) \]

\[ = o(1), \]
\[ h_q^{-1} E \left( \left[ h_q^{-1} (X_1 - x) \right]^8 \mu h_q k^4 (h_q^{-1} (X_1 - x)) \tau_n^4 (g (X_1)) \right) \]

\[ = \frac{1}{8} h_q^{-1} E \left[ g^3 (X_1) (X_1 - x)^6 \left( q_g^{(2)} (\alpha, \xi_1) \right)^3 \cdot f_1^3 (\xi_1 (\alpha, h_q)) k^4 (h_q^{-1} (X_1 - x)) \tau_n^4 (g (X_1)) \left[ h_q^{-1} (X_1 - x) \right]^8 \right] \]

\[ = \frac{h_q^6}{8} \int t_1^{14} g^4 (x + h_q t_1) \left( q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right)^3 \cdot f_1^3 \left( q (\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right) \cdot k^4 (t_1) \tau_n^4 (g (x + h_q t_1)) dt_1 \]

\[ = \frac{1}{8} \left( h_q^6 g^4 (x) \left( q_g^{(2)} (\alpha, x) \right)^3 f_1^3 |X_1 = x (q (\alpha, x)) \tau_n^4 (g (x)) \right) \int t_1^{14} k^4 (t_1) dt_1 \]

\[ + O \left( h_q^7 \right) \]

\[ \leq \sup_{x \in \mathcal{X}} g^4 (x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| q_g^{(2)} (\alpha, x) \right|^3 \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_1^3 |X_1 = x (q (\alpha, x)) \]

\[ \cdot \frac{1}{8} \left( h_q^6 \int t_1^{14} k^4 (t_1) dt_1 + O \left( h_q^7 \right) \right) \]

\[ = O \left( h_q^6 \right) \]

\[ = o(1), \]
\[ h_q^{-1} E \left[ \left[ h_q^{-1}(X_1 - x) \right]^8 (\psi_\alpha(\bar{u}_1(x)) - E[\psi_\alpha(\bar{u}_1(x))|X_1])^4 k^4 \left( h_q^{-1}(X_1 - x) \right)^{\tau_n}(g(X_1)) \right] \]

\[ = \frac{1}{16} h_q^{-1} E \left[ \left[ h_q^{-1}(X_1 - x) \right]^8 g^4(X_1) \left( X_1 - x \right)^8 (q_{\psi_\alpha}^{(2)}(\alpha, \xi_1))^4 \cdot f_1(t_1 \xi_1, h_q) k^4 \left( h_q^{-1}(X_1 - x) \right)^{\tau_n}(g(X_1)) \right] \]

\[ = \frac{h_q^8}{16} \int t_1^{16} g^5(x + h_q t_1) \left( q_{\psi_\alpha}^{(2)}(\alpha, \gamma h_q t_1 + x) \right)^4 \cdot f_1(t_1 \xi_1, h_q) k^4(t_1) \tau_n(g(x)) \int t_1^{16} k^4(t_1) dt_1 \]

\[ + O(h_q^9) \]

\[ \leq \sup_{x \in \mathcal{X}} g^5(x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} (q_{\psi_\alpha}^{(2)}(\alpha, x))^4 \cdot \sup_{\alpha \in \mathcal{A}} f_{Y|X_1=x}^4(q(\alpha, x)) \cdot \frac{1}{16} h_q^8 \int t_1^{16} k^4(t_1) dt_1 + O(h_q^9) \]

\[ = O(h_q^8) \]

\[ = o(1) \]

Combining results, we have for each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\) that

\[ h_q^{-1} E \left[ \left[ h_q^{-1}(X_1 - x) \right]^8 (\psi_\alpha(\bar{u}_1(x)) - E[\psi_\alpha(\bar{u}_1(x))|X_1])^4 k^4 \left( h_q^{-1}(X_1 - x) \right)^{\tau_n}(g(X_1)) \right] \]

\[ \to \alpha(1 - \alpha) \left( 3\alpha^2 - 3\alpha + 1 \right) g(x) \int t_1^{\delta} k^4(t_1) dt_1. \]

As such, provided that \(nh_q \to \infty\), (35) holds.

It follows that Liapounov’s condition holds and the convergence in (31) holds for each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\).

### B.3 Second step: Asymptotic unbiasedness of \(q_n^*(\alpha, x)\)

Consider

\[ E \left[ \left[ \psi_\alpha(\bar{u}_1(x)) k \left( h_q^{-1}(X_1 - x) \right)^{\tau_n}(g(X_1)) \right] \cdot \left[ h_q^{-1}(X_1 - x) \psi_\alpha(\bar{u}_1(x)) \left( h_q^{-1}(X_1 - x) \right)^{\tau_n}(g(X_1)) \right] \right]. \]
As before, let $T_1 \equiv h_q^{-1}(X_1 - x)$, so $X_1 = x + h_q T_1$. Let $\xi_1$ be a point on the line segment between $x$ and $X_1$. Let $\zeta_1(\alpha, h_q)$ be a point in the interval

$$\left( q(\alpha, X_1) - \frac{1}{2} g(X_1)(X_1 - x)^2 q^{(2)}_g(\alpha, \xi_1), q(\alpha, X_1) \right).$$
Let \((\gamma_1, \gamma_2) \in (0, 1)^2\) be suitable constants. Assume that \(g(\cdot)\) is at least \((L - 1)\)-times continuously differentiable on \(X\). We have the following:

\[
E \left[ \psi_\alpha (u_1(x)) k \left( h_q^{-1} (X_1 - x) \right) \tau_n (g (X_1)) \right] = E \left[ E \left[ \psi_\alpha (u_1(x)) \mid X_1 \right] k \left( h_q^{-1} (X_1 - x) \right) \tau_n (g (X_1)) \right] = \frac{1}{2} E \left[ g (X_1) (X_1 - x)^2 q_g^{(2)} (\alpha, \xi_1) f_1 (\xi_1 (\alpha, h_q)) k \left( h_q^{-1} (X_1 - x) \right) \tau_n (g (X_1)) \right]
\]

\[
= \frac{1}{2} \int g^2 (x_1) (x_1 - x)^2 q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) f_1 \left( q (\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g (x_1) (x_1 - x)^2 q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \right) \cdot k \left( h_q^{-1} (x_1 - x) \right) \tau_n (g (x)) dx_1
\]

\[
= \frac{1}{2} h_q^3 \int g^2 (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) f_1 \left( q (\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g (x + h_q t_1) t_1^2 q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \right) \cdot k (t_1) \tau_n (g (x + h_q t_1)) dt_1
\]

\[
= \frac{1}{2} h_q \left( f_{Y_1 | X_1 = x} (q (\alpha, x)) \tau_n (g (x)) q_g^{(2)} (\alpha, x) \cdot h_q^2 \int t_1^2 k (t_1) g^2 (x + h_q t_1) dt_1 + O (h_q^3) \right)
\]

\[
= \frac{1}{2} h_q \left\{ f_{Y_1 | X_1 = x} (q (\alpha, x)) \tau_n (g (x)) q_g^{(2)} (\alpha, x) \left[ \frac{D_x^{L-2} g^2 (x)}{(L - 2)!} \cdot h_q^L \int t_1^L k (t_1) dt_1 + \int_0^1 (1 - u)^{L-2} \int D_x^{L-1} g^2 (x + u h_q t_1) t_1^L k (t_1) dt_1 du \right] + O (h_q^3) \right\}
\]

\[
= \frac{1}{2} h_q \left[ f_{Y_1 | X_1 = x} (q (\alpha, x)) q_g^{(2)} (\alpha, x) \frac{D_x^{L-2} g^2 (x)}{(L - 2)!} \cdot h_q^L \int t_1^L k (t_1) dt_1 \right. + O (n^{-\xi} h_q^L) + O (h_q^{L+1}) + O (h_q^3) \right]
\]

\[
= \frac{1}{2} h_q f_{Y_1 | X_1 = x} (q (\alpha, x)) q_g^{(2)} (\alpha, x) \frac{D_x^{L-2} g^2 (x)}{(L - 2)!} \cdot h_q^L \int t_1^L k (t_1) dt_1 + O \left( (n^{-\xi} + h_q^{L+1}) \right)
\]

\[
= \frac{1}{2} f_{Y_1 | X_1 = x} (q (\alpha, x)) q_g^{(2)} (\alpha, x) \frac{D_x^{L-2} g^2 (x)}{(L - 2)!} \cdot h_q^L \int t_1^L k (t_1) dt_1 + o (h_q^{L+1}).
\]
Similarly, if \( g(\cdot) \) is at least \((L - 2)\)-times continuously differentiable on \( \mathcal{X} \), the following holds:

\[
E \left[ h_q^{-1} (X_1 - x) \psi_\alpha (\bar{u}_1(x)) k \left( h_q^{-1} (X_1 - x) \right) \tau_n \left( g \left( X_1 \right) \right) \right] \\
= \frac{1}{2} \int h_q^{-1} (x_1 - x)^3 g^2 (x_1) q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \\
\cdot f_1 \left( q (\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g (x_1) \right) (x_1 - x)^2 q_g^{(2)} (\alpha, \gamma_1 (x_1 - x) + x) \\
\cdot k \left( h_q^{-1} (x_1 - x) \right) \tau_n \left( g \left( x_1 \right) \right) dx_1 \\
= \frac{1}{2} h_q \left( h_q^2 \int t_1^3 g^2 (x + h_q t_1) q_g^{(2)} (\alpha, \gamma_1 h_q t_1 + x) \\
\cdot f_1 \left( q (\alpha, x + h_q t_1) \right) k (t_1) \tau_n \left( g \left( x + h_q t_1 \right) \right) dt_1 \\
+ O \left( h_q^4 \right) \right) \\
= \frac{1}{2} \left\{ f_{Y_1 | X_1 = x} \left( q(\alpha, x) \right) \tau_n \left( g(x) \right) \\
\cdot \left[ \frac{D_{x}^{L-3} g^2(x)}{(L - 3)!} \cdot q_g^{(2)}(\alpha, x) \cdot h_q^{L-1} \int t_1^L k (t_1) dt_1 \\
+ h_q^2 \int_0^1 (1 - u)^{L-3} \int D_{x}^{L-2} q_g^{(2)}(\alpha, x) g^2 \left( x + u h_q t_1 \right) t_1^L k (t_1) dt_1 du \right] \\
+ O \left( h_q^3 \right) \right\} \\
= \frac{1}{2} h_q \left[ f_{Y_1 | X_1 = x} (q(\alpha, x)) \frac{D_{x}^{L-3} g^2(x)}{(L - 3)!} \cdot q_g^{(2)}(\alpha, x) \cdot h_q^{L-1} \int t_1^L k (t_1) dt_1 \\
+ O \left( n^{-L} + h_q^L \right) + O \left( h_q^L \right) \right] \\
= \frac{1}{2} h_q f_{Y_1 | X_1 = x} (q(\alpha, x)) \frac{D_{x}^{L-3} g^2(x)}{(L - 3)!} \cdot q_g^{(2)}(\alpha, x) \cdot h_q^{L-1} \int t_1^L k (t_1) dt_1 \\
+ O \left( \left( n^{-L} + h_q \right) h_q^L \right) \\
= \frac{1}{2} h_q^L f_{Y_1 | X_1 = x} (q(\alpha, x)) \frac{D_{x}^{L-3} g^2(x)}{(L - 3)!} \cdot q_g^{(2)}(\alpha, x) \int t_1^L k (t_1) dt_1 \\
+ o \left( h_q^L \right) \right). 
\]
Combine (36) with (37) to deduce that

\[
\sqrt{n} \frac{\psi_\alpha (\bar{u}_\alpha^* (x)) k (h_{q}^{-1} (X_1 - x)) \tau_n (g (X_1))}{h_{q}^{-1} (X_1 - x) \psi_\alpha (\bar{u}_\alpha^* (x)) k (h_{q}^{-1} (X_1 - x)) \tau_n (g (X_1))} \approx \frac{1}{2} \sqrt{n h_q f_{\mathcal{Y_1} | X_1 = x} (q (\alpha, x))} \left[ \begin{array}{c} h_q^L \cdot q_g^2 (\alpha, x) \frac{D_{q}^{L-2} q^2 (x)}{(L-2)!} \int t_1^2 k (t_1) \, dt_1 \\ h_q^{L-1} \cdot q_g^{(2)} (\alpha, x) \frac{D_{q}^{L-3} q^2 (x)}{(L-3)!} \int t_1^2 k (t_1) \, dt_1 \end{array} \right]
\]

= \o(1),

where the condition

\[
\sqrt{n h_q} \cdot h_q^{L-1} = \sqrt{n h_q}^{L-\frac{1}{2}} \rightarrow 0.
\]

has been invoked.

**B.4 Conclusion**

Combine the results of the two steps above with (30) to deduce that for each \( \delta \in \mathbb{R}^2 \) and each \((\alpha, x) \in A \times \mathcal{X}\),

\[
Z_{n_1} (\delta) \overset{d}{\rightarrow} - \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] W (\alpha, x),
\]

where

\[
W (\alpha, x) \sim N \left( 0, \alpha (1 - \alpha) g (x) \left[ \begin{array}{c} \int k^2 (t_1) \, dt_1 \\ \int t_1 k^2 (t_1) \, dt_1 \end{array} \right] \right).
\]

Combine this result with (18) and (29) to deduce that

\[
Z_n (\delta) \overset{d}{\rightarrow} - \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] W (\alpha, x) + \frac{1}{2} f_{\mathcal{Y_1} | X_1 = x} (q (\alpha, x)) g(x) \cdot \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \left[ \begin{array}{c} \int k^2 (t_1) \, dt_1 \\ \int t_1 k^2 (t_1) \, dt_1 \end{array} \right] \left[ \begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] = Z_\infty (\delta)
\]

for each \( \delta \in \mathbb{R}^2 \) and each \((\alpha, x) \in A \times \mathcal{X}\).

Note that the convexity of \( Z_\infty (\delta) \) implies the uniqueness of its minimiser. In addition, by the “arg min continuous mapping theorem” (e.g., Pollard 1991, Hjørt and Pollard 1993, Knight 1998) we have

\[
\arg \min \, Z_n (\delta) \overset{d}{\rightarrow} \arg \min \, Z_\infty (\delta) \quad (38)
\]

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for each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\).

The convergence in (38) implies that for each \((\alpha, x) \in \mathcal{A} \times \mathcal{X}\),

\[
\begin{bmatrix}
\hat{\delta}_{n0}(\alpha, x) \\
\hat{\delta}_{n1}(\alpha, x)
\end{bmatrix} \mathcal{W}^{1/2} \begin{bmatrix}
g_{q_n^*(\alpha, x)} - \frac{q(\alpha, x)}{g(x)} \\
h_q \left( g_{q_n^*(\alpha, x)} - q_1(\alpha, x) \right)
\end{bmatrix} \xrightarrow{d} f_{Y_1|X_1=x}^{-1} (q(\alpha, x)) g^{-1}(x)
\]

\[
\int \kappa^2(t_1) \, dt_1 \int t_1 \kappa^2(t_1) \, dt_1 \int t_1^2 \kappa^2(t_1) \, dt_1^{-1} \mathcal{W}(\alpha, x).
\]

Proposition 3.1 is immediate.

\section{Proofs of Theorems 3.2 and 3.3}

\subsection{Preliminaries}

The proofs of Theorems 3.2 and 3.3 involve the use of the following conceptual device.

Let \(\tilde{\tau}_n(\cdot)\) denote a trimming function satisfying all the conditions of part (1) of Assumption A.5. In particular, define

\[
\tilde{\tau}_n(u) = \begin{cases} 
1, & u \geq 2n^{-\zeta_1} \\
0, & u \leq n^{-\zeta_1} \\
\tilde{\tau}_{n1}(u), & u \in (n^{-\zeta_1}, 2n^{-\zeta_1})
\end{cases}
\]

where \(\zeta_1 > 0\) is some constant, and where \(\tilde{\tau}_{n1}(u)\) is a twice differentiable distribution function with the form \(\tilde{\tau}_{n1}(u) = \int_{-\infty}^{u} n^{-\zeta_1} \sigma \left(n^{\zeta_1} t - 1\right) \, dt\), where \(\sigma(\cdot)\) is a differentiable density function uniformly bounded and supported on \([0, 1]\) with \(\sigma(0) = \sigma(1) = 0\), and with \(0 < \left|\sigma^{(1)}(0)\right| < \left|\sigma^{(1)}(1)\right| < \infty\).

It should be emphasised that \(\tilde{\tau}_n(\cdot)\) and the associated trimming parameter \(\zeta_1\) are purely conceptual—their role is simply to enable calculations involving the second and fourth moments of the empirically infeasible quantity \(\hat{\theta}_n(\alpha, x)\), which is given as follows:

\[
\hat{\theta}_n(\alpha, x) \equiv \frac{1}{n} \sum_{i=1}^{n} D_x^n K_h (X_i - x) \frac{q(\alpha, X_i)}{g(X_i)} \tilde{\tau}_n(g(X_i)) .
\]

(39)

In what follows—particularly in the proof of Theorem 3.3 below—it is assumed that the trimming parameter \(\zeta_1\) is set so as to make any residual trimming effects associated with the moments of \(\hat{\theta}_n(\alpha, x)\) asymptotically negligible.
C.2 Proof of Theorem 3.2

Begin by considering $|\hat{\theta}_n(\alpha, x) - \bar{\theta}_n(\alpha, x)|$. We have

$$
|\hat{\theta}_n(\alpha, x) - \bar{\theta}_n(\alpha, x)|
\leq \frac{1}{n} \sum_{i=1}^{n} \left( \sup_{x' \in \mathcal{X}} |D_{x_i}^h K_h (X_i - x')| \right) \cdot \left| \hat{q}^*_n (\alpha, X_i) - \frac{q(\alpha, X_i)}{g(X_i)} \tilde{\tau}_n (g(X_i)) \right|
\leq \frac{1}{nh} \sum_{i=1}^{n} \left( \sup_{x' \in \mathcal{X}} |D_{x_i}^h K_h (h^{-1}(X_i - x'))| \right)
\cdot \left| \hat{q}^*_n (\alpha, x + hT_i) - \frac{q(\alpha, x + hT_i)}{g(x + hT_i)} \tilde{\tau}_n (g(x + hT_i)) \right|
= \frac{1}{nh} \sum_{i=1}^{n} \left( \sup_{x' \in \mathcal{X}} |D_{x_i}^h K_h (h^{-1}(X_i - x'))| \right)
\cdot \left| \frac{1}{\sqrt{nh_q}} \cdot \sqrt{nh_q} \left( \hat{q}^*_n (\alpha, x + hT_i) - \frac{q(\alpha, x + hT_i)}{g(x + hT_i)} \right) + \frac{q(\alpha, x + hT_i)}{g(x + hT_i)} - \frac{q(\alpha, x + hT_i)}{g(x + hT_i)} \tilde{\tau}_n (g(x + hT_i)) \right|
= O_p \left( \frac{1}{\sqrt{nh_q}} \right) + O(n^{-\zeta_1}), \quad (40)
$$

where the result of Proposition 3.1 that $\sqrt{nh_q} (q^*_n (\alpha, x) - q(\alpha, x)/g(x)) = O_p(1)$ for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ has been exploited, along with the various assumptions on $g(\cdot), K(\cdot), q(\cdot, \cdot)$ and $h$. 

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Next, note that
\[ E \left[ \bar{\theta}_n(\alpha, x) \right] = D_x^\nu \int h^{-1}K(h^{-1}(x_1 - x)) q(\alpha, x_1) \, dx_1 \]

\[ = D_x^\nu \int_{x_1: g(x_1) \geq 2n^{-\zeta_1}} h^{-1}K(h^{-1}(x_1 - x)) q(\alpha, x_1) \, dx_1 \]

\[ + D_x^\nu \int_{x_1: n^{-\zeta_1} < g(x_1) < 2n^{-\zeta_1}} h^{-1}K(h^{-1}(x_1 - x)) q(\alpha, x_1) \tilde{\tau}_n(g(x_1)) \, dx_1 \]

\[ = D_x^\nu \int h^{-1}K(h^{-1}(x_1 - x)) q(\alpha, x_1) \, dx_1 - D_x^\nu \int_{g(x_1) < 2n^{-\zeta_1}} h^{-1}K(h^{-1}(x_1 - x)) q(\alpha, x_1) \, dx_1 \]

\[ + D_x^\nu \int_{n^{-\zeta_1} < g(x_1) < 2n^{-\zeta_1}} h^{-1}K(h^{-1}(x_1 - x)) q(\alpha, x_1) \tilde{\tau}_n(g(x_1)) \, dx_1 \]

\[ = D_x^\nu \int h^{-1}K(h^{-1}(x_1 - x)) q(\alpha, x_1) \, dx_1 + O(n^{-\zeta_1}) \]

\[ = D_x^\nu \int K(t_1) q(\alpha, x + ht_1) \, dt_1 + O(n^{-\zeta_1}) \]

\[ = D_x^\nu \int K(t_1) \left[ \sum_{\tau=0}^5 \frac{D_\tau q(\alpha, x)}{\tau!} (ht_1)^\tau + \frac{6}{6!} \int_0^1 (1 - u)^5 D_x^6 q(\alpha, x + uht_1) \, du \right] \, dt_1 \]

\[ + O(n^{-\zeta_1}) \]

\[ = D_x^\nu q(\alpha, x) + \frac{h^2}{2} q^{(2)}(\alpha, x) D_x^\nu \int t_1^2 K(t_1) \, dt_1 + \frac{h^4}{4!} D_x^{\nu+4} q(\alpha, x) \int t_1^4 K(t_1) \, dt_1 \]

\[ + O(h^6) + O(n^{-\zeta_1}) . \quad (41) \]

Invoking the condition that \( \sup_{\alpha \in \mathcal{A}} \sup_{x \in X} |D_x^\nu q(\alpha, x)| < \infty \) for all \( \tau \in \{2, 3, \ldots, 6\} \), we have

\[ |E \left[ \bar{\theta}_n(\alpha, x) \right] - D_x^\nu q(\alpha, x)| = O \left( h^2 + n^{-\zeta_1} \right) . \quad (42) \]
In addition, note that

\[
E \left[ \frac{1}{n^2} \sum_{i,j} D_x^\nu K_h (X_i - x) q(\alpha, X_i) \frac{g(\alpha, X_i)}{g(X_i)} \frac{\tau_n (g(X_i))}{h} \right] = 1
\]

\[
E \left[ \frac{1}{n^2} \sum_{i,j} D_x^\nu K_h (X_i - x) q(\alpha, X_i) \frac{g(\alpha, X_i)}{g(X_i)} \frac{\tau_n (g(X_i))}{h} \right] = 1
\]

\[
\frac{1}{n} E \left[ (D_x^\nu K_h (X_1 - x))^2 \frac{g^2(\alpha, X_1)}{g^2(X_1)} \frac{\tau_n^2 (g(X_1))}{h} \right] = \frac{n(n-1)}{n^2} \left( E \left[ D_x^\nu K_h (X_1 - x) q(\alpha, X_1) \frac{g(\alpha, X_1)}{g(X_1)} \frac{\tau_n (g(X_1))}{h} \right] \right)^2.
\]

Note also that

\[
E \left[ (D_x^\nu K_h (X_1 - x))^2 \frac{g^2(\alpha, X_1)}{g^2(X_1)} \frac{\tau_n^2 (g(X_1))}{h} \right] = \int h^{-2} (D_x^\nu K (h^{-1} (x_1 - x)))^2 \frac{g^2(\alpha, x_1)}{g(x_1)} \frac{\tau_n^2 (g(x_1))}{h} \, dx_1
\]

\[
= h^{-1-2\nu} \int \left( \frac{d^\nu}{dt^1} K(t_1) \right)^2 \frac{g^2(\alpha, x + ht_1)}{g(x + ht_1)} \frac{\tau_n^2 (g(x + ht_1))}{h} \, dt_1
\]

\[
= h^{-1-2\nu} \left[ \frac{q^2(\alpha, x)}{g(x)} \frac{\tau_n^2 (g(x))}{h} \int \left( \frac{d^\nu}{dt^1} K(t_1) \right)^2 \, dt_1 + O(h) \right]
\]

\[
= h^{-1-2\nu} \left[ \frac{q^2(\alpha, x)}{g(x)} \int \left( \frac{d^\nu}{dt^1} K(t_1) \right)^2 \, dt_1 + O(n^{-\xi_1} + h) \right].
\]

(43)

It follows that

\[
E \left[ \frac{1}{n} \frac{\tau_n (g(x))}{h} \int \left( \frac{d^\nu}{dt^1} K (t_1) \right)^2 \, dt_1 + O \left( \frac{n^{-\xi_1} + h}{n h^{1+2\nu}} \right) \right]
\]

\[
= (D_x^\nu q(\alpha, x))^2 + D_x^\nu q(\alpha, x) \cdot D_x^\nu q^{(2)}(\alpha, x) \cdot h^2 \int t_1^2 K(t_1) \, dt_1
\]

\[
+ O \left( h^4 + n^{-\xi_1} \right).
\]
Deduce that

\[
\text{Var} \left[ \bar{\theta}_n(\alpha, x) \right] = E \left[ (\hat{\theta}_n - E[\bar{\theta}_n(\alpha, x)])^2 \right] = E \left[ \bar{\theta}_n^2(\alpha, x) \right] - (E[\bar{\theta}_n(\alpha, x)])^2
\]

\[
= (D_x^\nu q(\alpha, x))^2 + \frac{1}{nh^{1+2\nu}} \cdot \frac{q^2(\alpha, x)}{g(x)} \int \left( \frac{d^\nu}{dt^\nu_1} K(t_1) \right)^2 dt_1
\]

\[
+ D_x^\nu q(\alpha, x) \cdot D_x^{\nu(2)}(\alpha, x) \cdot h^2 \int t_1^2 K(t_1) dt_1 + o \left( \frac{1}{nh^{1+2\nu}} \right)
\]

\[
- (D_x^\nu q(\alpha, x))^2 - D_x^\nu q(\alpha, x) \cdot D_x^{\nu(2)}(\alpha, x) \cdot h^2 \int t_1^2 K(t_1) dt_1
\]

\[
+ O \left( h^4 + n^{-\xi_1} \right)
\]

\[
= O \left( \frac{1}{nh^{1+2\nu}} + h^4 + n^{-\xi_1} \right).
\]

Therefore for each \((\alpha, x) \in A \times \mathcal{X}\) we have

\[
|\bar{\theta}_n(\alpha, x) - E[\bar{\theta}_n(\alpha, x)]| = O_p \left( \frac{1}{\sqrt{nh^{1+2\nu}}} + h^2 + n^{-\xi_1/2} \right), \quad (44)
\]

where the conditions that \(\sup_{\alpha \in A} \sup_{x \in \mathcal{X}} |g^{(j)}(\alpha, x)| < \infty\) for all \(j \in \{2, \ldots, 6\}\);

\(\sup_{x \in \mathcal{X}} g(x) < \infty\); and \(\int \left( \frac{d^\nu}{dt^\nu_1} K(t_1) \right)^2 dt_1 < \infty\) have been invoked.

Combine (42) and (44) via the triangle inequality to deduce that

\[
|\bar{\theta}_n(\alpha, x) - D_x^\nu q(\alpha, x)| = O_p \left( h^2 + \frac{1}{\sqrt{nh^{1+2\nu}}} + n^{-\xi_1/2} \right). \quad (45)
\]

Invoking the condition that \(nh^{1+2\nu} \to \infty\) as \(n \to \infty\), combine (40) with (45) to deduce that

\[
|\hat{\theta}_n(\alpha, x) - D_x^\nu q(\alpha, x)| = o_p(1).
\]

C.3 Proof of Theorem 3.3

Recall the definition of \(\hat{\theta}_n(\alpha, x)\) given above in (39) and begin by showing the asymptotic equivalence of \(\hat{\theta}_n(\alpha, x)\) and \(\hat{\theta}_n(\alpha, x)\) for each \((\alpha, x) \in A \times \mathcal{X}\).
similar derivation to (40) above yields

\[
\sqrt{nh^{1+2\nu}} \left( \tilde{\theta}_n(\alpha, x) - \bar{\theta}_n(\alpha, x) \right)
\]

\[
\leq \frac{1}{nh} \sum_{i=1}^{n} \left( \sup_{x' \in \mathcal{X}} \left| D_x^\nu K \left( h^{-1} (X_i - x') \right) \right| \right)
\cdot \sqrt{nh^{1+2\nu}} \left( \frac{1}{\sqrt{nh_q}} \sqrt{nh_q} \left( \hat{q}^*_n (\alpha, x + hT_i) - \frac{q(\alpha, x + hT_i)}{g(x + hT_i)} \right) \right)
\]\n
\[
+ \frac{1}{nh} \sum_{i=1}^{n} \left( \sup_{x' \in \mathcal{X}} \left| D_x^\nu K \left( h^{-1} (X_i - x') \right) \right| \right)
\cdot \sqrt{nh^{1+2\nu}} \left( \frac{q(\alpha, x + hT_i)}{g(x + hT_i)} \left( 1 - \tilde{\tau}_n (g(x + hT_i)) \right) \right)
\]

\[
= o_p(1), \quad (46)
\]

where the condition that \( h^{1+2\nu}/h_q = o(1) \) has been invoked, which in turn implies the convergence \( |\hat{q}^*_n(\alpha, x) - q(\alpha, x)/g(x)| = o_p \left( (nh^{1+2\nu})^{-1/2} \right) \). An assumption that the conceptual trimming parameter \( \zeta \) is sufficiently large so as to make

\[
\sqrt{nh^{1+2\nu}} \eta = o(1) \quad (47)
\]

has also been invoked.

The asymptotic normality of \( \sqrt{nh^{1+2\nu}} (\tilde{\theta}_n(\alpha, x) - E [D_x^\nu K_h (X_1 - x)] \cdot (q(\alpha, X_1)/g(X_1)) \cdot \tilde{\tau}_n (g(X_1))) \) is then proved for fixed \( (\alpha, x) \in \mathcal{A} \times \mathcal{X} \). In order to show this, it is sufficient that

\[
h^{1+2\nu} \text{Var} \left[ D_x^\nu K_h (X_1 - x) \frac{q(\alpha, X_1)}{g(X_1)} \cdot \tilde{\tau}_n (g(X_1)) \right]
\]

\[
\to \Omega(\alpha, x)
\]

for some \( \Omega(\alpha, x) \in (0, \infty) \) along with

\[
\frac{h^{2+4\nu}}{n} E \left( (D_x^\nu K_h (X_1 - x))^4 \frac{q^4(\alpha, X_1)}{g^4(X_1)} \cdot \tilde{\tau}_n^4 (g(X_1)) \right)
\]

\[
= o(1).
\]

Note from (43) that

\[
E \left[ (D_x^\nu K_h (X_1 - x))^2 \frac{q^2(\alpha, X_1)}{g^2(X_1)} \tilde{\tau}_n^2 (g(X_1)) \right]
\]

\[
= h^{-1-2\nu} \left[ \frac{q^2(\alpha, x)}{g(x)} \int \left( \frac{d^\nu}{dt^3} K(t_1) \right)^2 dt_1 + O \left( n^{-\zeta_1} + h \right) \right].
\]
Therefore

\[ h^{1+2\nu} E \left[ (D_x^\nu K_h (X_1 - x))^2 \frac{q^2 (\alpha, X_1)}{g^2 (X_1)} \tau_n (g (X_1)) \right] \]

\[ = \frac{q^2 (\alpha, x)}{g(x)} \int \left( \frac{d^{\nu}}{dt_1^{\nu}} K (t_1) \right)^2 dt_1 + O \left( n^{-\zeta_1} + h \right), \]

where the conditions that \( \sup_{\alpha \in A} \sup_{x \in \mathcal{X}} | q^{(j)} (\alpha, x) | < \infty \) for \( j \in \{0, 1\}; \sup_{x \in \mathcal{X}} | g^{(j)} (x) | < \infty \) for \( j \in \{0, 1\}; \left| \int t_1 \left( \frac{d}{dt_1} K (t_1) \right)^2 dt_1 \right| < \infty \) and \( \int \left( \frac{d}{dt_1} K (t_1) \right)^2 dt_1 < \infty \) have all been invoked.

As such, if in addition we invoke the condition that \( \sup_{\alpha \in A} \sup_{x \in \mathcal{X}} | D_x^\nu q (\alpha, x) | < \infty \), we have

\[ h^{1+2\nu} Var \left[ D_x^\nu K_h (X_1 - x) \frac{q (\alpha, X_1)}{g (X_1)} \tau_n (g (X_1)) \right] \]

\[ = h^{1+2\nu} \left[ E \left[ (D_x^\nu K_h (X_1 - x))^2 \frac{q^2 (\alpha, X_1)}{g^2 (X_1)} \tau_n (g (X_1)) \right] \right. \]

\[ \left. - \left( E \left[ D_x^\nu K_h (X_1 - x) \frac{q (\alpha, X_1)}{g (X_1)} \tau_n (g (X_1)) \right] \right)^2 \right] \]

\[ = \frac{q^2 (\alpha, x)}{g(x)} \int \left( \frac{d^{\nu}}{dt_1^{\nu}} K (t_1) \right)^2 dt_1 + O \left( n^{-\zeta_1} + h \right) \]

\[-h^{1+2\nu} \left[ (D_x^\nu q (\alpha, x))^2 - D_x^\nu q (\alpha, x) \cdot D_x^\nu q (2) (\alpha, x) \cdot h^2 \int t_1^2 K (t_1) dt_1 \right. \]

\[ \left. + O \left( h^4 + n^{-\zeta_1} \right) \right] \]

\[ = \frac{q^2 (\alpha, x)}{g(x)} \int \left( \frac{d^{\nu}}{dt_1^{\nu}} K (t_1) \right)^2 dt_1 + o(1) \]

\[ \equiv \Omega (\alpha, x) + o(1). \]
Next, note that

\[
\frac{h^{2+4\nu}}{n} E \left[ (D_x^\nu K_h (X_1 - x))^4 \frac{q^4 (\alpha, X_1)}{g^3 (X_1)} \tilde{\tau}_n (g (X_1)) \right] \\
= \frac{h^{2+4\nu}}{n} \int (D_x^\nu K_h (x_1 - x))^4 \frac{q^4 (\alpha, x_1)}{g^3 (x_1)} \tilde{\tau}_n (g (x_1)) \, dx_1 \\
= \frac{n^{-1} h^{-2}}{n} \int \left( \frac{d^\nu}{dh^{-1} (x_1 - x)} K (h^{-1} (x_1 - x)) \right)^4 \frac{q^4 (\alpha, x_1)}{g^3 (x_1)} \tilde{\tau}_n (g (x_1)) \, dx_1 \\
= \frac{n^{-1} h^{-1}}{n} \int \left( \frac{d^\nu}{dt_1} K (t_1) \right)^4 \frac{q^4 (\alpha, x + Ht_1)}{g (x + Ht_1)} \tilde{\tau}_n (g (x + Ht_1)) \, dt_1 \\
= \frac{n^{-1} h^{-1}}{n} \left[ \frac{q^4 (\alpha, x)}{g^3 (x)} \tilde{\tau}_n (g (x)) \int \left( \frac{d^\nu}{dt_1} K (t_1) \right)^4 \, dt_1 + O (H) \right] \\
= \frac{n^{-1} h^{-1}}{n} \left[ \frac{q^4 (\alpha, x)}{g^3 (x)} \int \left( \frac{d^\nu}{dt_1} K (t_1) \right)^4 \, dt_1 + O \left( n^{-\zeta_1} + h \right) \right] \\
= o(1),
\]

where the additional conditions that \( \int t_1 \left( \frac{d^\nu}{dt_1} K (t_1) \right)^4 \, dt_1 < \infty \) and \( \int \left( \frac{d^\nu}{dt_1} K (t_1) \right)^4 \, dt_1 < \infty \) have been invoked.

Conclude via Liapounov’s theorem that

\[
\sqrt{n h^{1+2\nu}} \left( \hat{\theta}_n (\alpha, x) - E \left[ D_x^\nu K_h (X_1 - x) \frac{q (\alpha, X_1)}{g (X_1)} \tilde{\tau}_n (g (X_1)) \right] \right) \overset{d}{\to} N (0, \Omega (\alpha, x)).
\]

Finally, given the assumptions that \( \sqrt{n h^{1+2\nu}} \cdot h = o(1), \sup_{\alpha \in A} \sup_{x \in X} \left| D_x^\nu q^{(2)} (\alpha, x) \right| < \infty \) and the condition (47) on the conceptual trimming parameter \( \zeta_1 \), we have the convergence

\[
\sqrt{n h^{1+2\nu}} \left( E \left[ D_x^\nu K_h (X_1 - x) \frac{q (\alpha, X_1)}{g (X_1)} \tilde{\tau}_n (g (X_1)) \right] - D_x^\nu q (\alpha, x) \right) \\
= O \left( \sqrt{n h^{1+2\nu}} \left( h^2 + n^{-\zeta_1} \right) \right) \\
= o(1),
\]

(49)
from which it is deduced that the bias of $\tilde{\theta}_n(\alpha, x)$ vanishes asymptotically.

Combine (46), (48) and (49) and deduce that

$$\sqrt{n} h^{1+2\nu} \left( \tilde{\theta}_n(\alpha, x) - D_x^* q(\alpha, x) \right) \xrightarrow{d} N(0, \Omega(\alpha, x))$$

for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$.

The second part of Theorem 3.3 follows from (41) and the conditions imposed on $\tilde{\tau}_n(\cdot)$ and $\zeta_1$. 

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