A characterization of the Extended Serial Correspondence*

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Abstract

We study the problem of assigning objects to a group of agents, when each agent reports ordinal preferences over the objects. We allow for indifference among objects. We focus on probabilistic methods, in particular, the extended serial correspondence, introduced by Katta and Sethuraman (2006). Our main result is a characterization of the extended serial correspondence in welfare terms by means of stochastic dominance efficiency, stochastic dominance no-envy and “bounded invariance,” an axiom we adapt from Bogomolnaia and Heo (2011). We also prove that an assignment matrix is selected by the extended serial correspondence if and only if it satisfies “non-wastefulness” and “ordinal fairness,” which we adapt from Kesten et al. (2011).

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Keywords: the serial rule; sd efficiency; sd no-envy; the extended serial correspondence; bounded invariance.

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1. Introduction

We study the problem of assigning objects to a group of agents, when each agent reports ordinal preferences over objects. We focus on probabilistic methods: each agent receives a probability distribution over objects whose coordinates add up to 1. An assignment matrix specifies, for each agent and each object, the probability that the agent will receive the object. A solution selects a set of assignment matrices for each preference profile. If a solution is single-valued, we call it a rule. If a solution is not necessarily single-valued, we call it a correspondence.

On the domain of strict preferences, Bogomolnaia and Moulin (2001) propose to compare lotteries on the basis of first-order stochastic dominance. Given two lotteries, say $\pi_0$ and $\pi'_0$, we say that “$\pi_0$ weakly first-order stochastically dominates $\pi'_0$ at an agent’s preference,” if (1) the probability of his receiving his most preferred object at $\pi_0$ is at least as large as the corresponding probability at $\pi'_0$, (2) the sum of the probabilities of his receiving either his most preferred object or his second most preferred object at $\pi_0$ is at least as large as the corresponding sum at $\pi'_0$, (3) and so on. If $\pi_0$ weakly first-order stochastically dominates $\pi'_0$ at an agent’s preference and $\pi_0 \neq \pi'_0$, then we say that “$\pi_0$ stochastically dominates $\pi'_0$ at his preference”. Throughout the paper, the notation “sd” stands for “stochastic dominance”.\footnote{We adopt the terminology and notation of Thomson (2010a, 2010b).} We say that an agent is “sd better off” at an assignment than at another assignment if the former first-order stochastically dominates the latter at his preference.

Let us now discuss two central properties of assignment matrices. Both are based on stochastic dominance comparisons. First is an efficiency notion. “Sd efficiency” requires that there should be no other assignment matrix at which some agent is sd better off and no agent is “sd worse off”. Next is a fairness notion. “Sd no-envy” requires that for each pair of agents, say $i$ and $j$, either agent $i$ should be sd better off at his assignment than at that of agent $j$ or the assignments of $i$ and $j$ be the same. We say that a rule satisfies sd efficiency and sd no-envy, respectively, if for each economy, it selects an assignment matrix satisfying sd efficiency and sd no-envy.

The “serial rule”, introduced by Bogomolnaia and Moulin (2001), satisfies these two properties, but it is not the only rule to do so. However, it is the only one to satisfy the two properties together with one of several invariance properties (Bogomolnaia and Heo (2011), Hashimoto and Hirata (2011), Heo (2011a), and Kesten et al. (2011)).\footnote{Heo (2011a) imposes one additional requirement called “consistency”. There are two other characterizations of the rule that do not involve any invariance-type axiom (Hashimoto and Hirata (2011) and Kesten et al. (2011)).}

The serial rule was originally defined for strict preference profiles, and it is not obvious how to adapt it to the weak preference domain. Consider, for example, a three-agent economy in
which preferences are as follows. If each agent finds several objects indifferent, we place these objects between parenthesis.

(i) the first agent prefers \((a, b)\) to \(c\),
(ii) the second agent prefers \(b\) to \((a, c)\), and
(iii) the third agent prefers \(b\) to \(a\) and prefers \(a\) to \(c\).

Note that the first agent is indifferent between \(a\) and \(b\), but the other agents prefer \(b\) to each other object. To achieve sd efficiency, the first agent has to “yield” \(b\) to the others, in exchange for receiving \(a\) instead. Then, the other agents should share \(b\) in a way that sd no-envy is not violated. Next, the second agent is indifferent between the remaining objects \(a\) and \(c\), but the other agents prefer \(a\) to \(c\). To achieve sd efficiency, again, the second agent has to yield \(a\) to the others, in exchange for receiving \(c\). Lastly, the first and the third agents should share \(a\) in a way that sd no-envy is not violated. If there are more than three agents and the structure of a preference profile is not as simple as in the above example, it is not clear who should yield which object to whom.

This problem was solved by Katta and Sethuraman (2006). For each preference profile, they introduce a sequence of flow graphs and then calculate the maximum flow of each graph. The resulting sequence of maximum flows determines a set of assignment matrices. The “extended serial correspondence” is defined through this algorithm. Due to the difficulties generated by indifference, their proposal is not single-valued. However, all assignment matrices selected by this correspondence are sd-equivalent in welfare terms. Thus, this solution is what is sometimes called “essentially single-valued”. They proved that it satisfies sd efficiency and sd no-envy. However, so far, there has been no characterization of it.\(^3\)

Our main contribution is a characterization of the extended serial correspondence (in welfare terms) by means of sd efficiency, sd no-envy, and “bounded invariance,” a natural adaptation to the domain of weak preferences of the axiom of the same name in Bogomolnaia and Heo (2011). On the strict preference domain, bounded invariance is defined as follows. Keeping each other agent's preference fixed, suppose that the preference of an agent, say agent \(i\), changes but his weak upper contour set at an object, say \(a\), remains the same and that so does his ranking over this set. Then, the requirement is that the probability of each agent receiving each object in agent \(i\)'s weak upper contour set at \(a\) should remain the same. We adapt this property to the domain of weak preferences.\(^4\)

\(^3\)On a restricted domain of preference profiles, however, Heo (2011b) shows that for each economy, an assignment matrix is selected by the extended serial correspondence if and only if it satisfies sd efficiency and sd no-envy. This result is interesting, in particular, because both axioms are “punctual” (they apply to each economy separately).

\(^4\)The current authors adapted bounded invariance from Bogomolnaia and Heo (2011) in independent papers. Yilmaz was the first to obtain the characterization (Theorem 1). He also adapted a result due to Kesten et al.
We also present an alternative representation of assignments, called “preference-decreasing consumption schedules” (Heo, 2011a and Bogomolnaia and Heo, 2011). This representation allows us to obtain a straightforward proof for another characterization of the extended serial correspondence along the lines of Kesten et al. (2011): an assignment matrix is selected by the extended serial correspondence if and only if it satisfies “non-wastefulness” and “ordinal fairness,” adapted from Kesten et al. (2011). Using this technique, we also provide a complementary proof of our main result, which reveals different interesting aspects of the extended serial correspondence. It links our result to that in Bogomolnaia and Heo (2011).

This paper is organized as follows. Section 2 introduces model and properties. Section 3 presents our main result. Section 4 presents our second characterization result. The Appendix gives an alternative proof of our main result.

2. Model

Let \(A \equiv \{o_1, \ldots, o_{|A|}\}\) be a set of objects and \(N \equiv \{1, 2, \ldots, n\}\) a set of agents. We assume that \(|N| \leq |A|\). There may be multiple copies of each object. For each \(i \in N\), \(R_i\) represents agent \(i\)’s weak preferences over \(A\). For each \(i \in N\) and each pair \(a, b \in A\), we write that \(a I_i b\) if and only if \(a R_i b\) and \(b R_i a\). Define a strict preference over \(A\) such that for each pair \(a, b \in A\), \(a P_i b\) if and only if \(a R_i b\) and \(\neg(a I_i b)\). Let \(R\) be the set of weak preferences over \(A\) and \(P\) be the set of strict preferences over \(A\). For each \(i \in N\), each \(R_i \in R\), and each \(S \subseteq A\), let \(M(R_i, S)\) be the set of objects in \(S\) that agent \(i\) most prefers. For each \(T \subseteq N\), let \(M(R_T, S) \equiv \bigcup_{i \in T} M(R_i, S)\).

For each \(R \in R^N\), each \(i \in N\), and each \(a \in A\), let \(U(R_i, a) \equiv \{o \in A : o R_i a\}\) be the weak upper contour set at \(a\) of \(R_i\). Also, let \(I(R_i, a) \equiv \{o \in A : a I_i o\}\) be the indifference class of \(a\) at \(R_i\).

An economy is a list \((A, N, R)\). We fix \(A\) and \(N\). An economy is then represented by \(R\). Let \(\mathcal{R}\) be the set of all economies. An assignment matrix is an \(|N| \times |A|\) matrix \(\pi \equiv (\pi_{ia})_{i \in N, a \in A}\), where \(\pi_{ia}\) is the probability of agent \(i\) receiving \(a\), such that (i) for each \(i \in N\) and each \(a \in A\), \(\pi_{ia} \in [0, 1]\), (ii) for each \(a \in A\), \(\sum_{i \in N} \pi_{ia} \leq 1\), and (iii) for each \(i \in N\), \(\sum_{a \in A} \pi_{ia} = 1\). An assignment matrix \(\pi \in \Pi\) is deterministic if for each \(i \in N\) and each \(a \in A\), \(\pi_{ia} \in \{0, 1\}\). By the Birkhoff-von Neumann theorem (Birkhoff (1946), von Neumann (1953)), each assignment matrix can be written as a convex combination of deterministic assignment matrices. Let \(\Pi\) be the set of all assignment matrices. For each \(S \subseteq A\) and each pair \(\pi, \pi' \in \Pi\), we say that \(\pi\) and \(\pi'\) coincide on \(S\) if for each \(i \in N\) and each \(a \in S\), \(\pi_{ia} = \pi'_{ia}\). Let \(2^\Pi\) be the collection of all subsets of \(\Pi\). Let \(\emptyset\) be the empty set. A rule is a mapping \(\varphi : \mathcal{R} \to \Pi\).
A correspondence is a mapping \( \Phi : \mathcal{R} \to 2^\Pi \setminus \{\emptyset\} \). A subcorrespondence \( \Psi \) of \( \Phi \) is a correspondence such that for each \( R \in \mathcal{R}^N \), \( \Psi(R) \subseteq \Phi(R) \).

Let \( R \in \mathcal{R}^N \). Let \( \pi_i, \pi'_i \in \Pi \). For each \( i \in N \), we say that \( \pi_i \) weakly stochastically dominates \( \pi'_i \) at \( R_i \), if for each \( a \in A \), \( \sum_{b \in U(R_i, a)} \pi_{ib} \geq \sum_{b \in U(R_i, a)} \pi'_{ib} \). We write that \( \pi_i \sd R_i \pi'_i \). If there is at least one strict inequality, we write that \( \pi_i \sd W R_i \pi'_i \). Throughout the paper, the notation “sd” stands for “stochastic dominance.” For each pair of correspondences \( \Phi \) and \( \Psi \), we say that \( \Phi \) and \( \Psi \) are equivalent in welfare if for each \( R \in \mathcal{R}^N \), each \( \pi \in \Phi(R) \), each \( \pi' \in \Psi(R) \), and each \( i \in N \), \( \pi_i \sd R_i \pi'_i \) and \( \pi'_i \sd R_i \pi_i \).

The following are properties of assignment matrices based on the sd relations. First is an efficiency property. Consider an assignment matrix. There should be no other assignment at which some agent is sd better off and no agent is sd worse off. Formally, \( \pi \) is sd efficient at \( R \) if there is no \( \pi' \in \Pi \) such that for each \( i \in N \), \( \pi_i \sd R_i \pi'_i \) and for some \( i \in N \), \( \pi'_i \sd R_i \pi_i \). The corresponding property of a correspondence \( \Phi \) is as follows:

**Sd efficiency:** For each \( R \in \mathcal{R}^N \) and each \( \pi \in \Phi(R) \), \( \pi \) is sd efficient at \( R \).

The following is a fairness property. Each agent should find his assignment at least as "sd-desirable" as that of each other agent. Formally, \( \pi \) is sd envy-free at \( R \) if for each pair \( i, j \in N \), \( \pi_i \sd R_i \pi_j \). The corresponding property of a correspondence \( \Phi \) is as follows:

**Sd no-envy:** For each \( R \in \mathcal{R}^N \) and each \( \pi \in \Phi(R) \), \( \pi \) is sd envy-free at \( R \).

The last requirement is a natural adaptation of an axiom that appears in Bogomolnaia and Heo (2011) to the domain of weak preferences. Let \( a \in A \). Suppose that, keeping each other agent’s preference fixed, the preference of an agent, say agent \( i \), changes but his upper contour set at an object, say \( a \), remains the same and that so does his ranking over this set. Consider an assignment matrix selected for the resulting profile. Consider the set of objects that agent \( i \) finds at least as desirable as \( a \). We require that the probability of each agent receiving each of these objects should remain the same.\(^5\) For each pair \( R_0, R'_0 \in \mathcal{R} \) and each \( a \in A \), we write \( R_i(a) = R'_i(a) \) if \( U(R_i, a) = U(R'_i, a) \) and \( R_i|_{U(R_i, a)} = R'_i|_{U(R_i, a)} \).

**Bounded invariance:** For each \( R \in \mathcal{R}^N \), each \( i \in N \), each \( a \in A \), and each \( R'_i \in \mathcal{R} \), if \( R_i(a) = R'_i(a) \), then for each \( \pi \in \Phi(R) \), there is \( \pi' \in \Phi(R'_i, R_{-i}) \) such that for each \( j \in N \) and each \( o \in U(R_i, a) \), \( \pi'_{jo} = \pi_{jo} \).

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\(^5\)This requirement is in the spirit of an axiom formulated for the probabilistic voting model (Gibbard, 1977). In our context, this can be rephrased as follows: keeping each other agent’ preference fixed, whenever the preference of an agent, say agent \( i \), changes but his weak upper contour set at some object, say \( a \), remains the same, the total probability of each agent receiving the objects that agent \( i \) finds at least as desirable as \( a \) should remain the same.
2.1. The Extended Serial Correspondence

Katta and Sethuraman (2006) propose an adaptation of the serial rule to profiles of (possibly) weak preferences. As readers familiar with the difficulties generated by indifference might expect, their proposal is not a rule but a correspondence. However, we should note that for each economy, each agent is indifferent between any two of the assignment matrices selected by the correspondence. Thus, this solution is “essentially single-valued”. The original formulation of Katta and Sethuraman (2011) was for the case in which there is only one copy of each object. We extend this correspondence to be applicable to situations in which there may be multiple copies of each object. Let us call this extension the extended serial correspondence (ES).

Let \( R \in \mathcal{R}^N \). Consider a graph with nodes consisting of a source node \( \alpha \), a sink node \( \beta \), agents in \( N \), and objects in \( A \). Each pair of nodes, say \( k \) and \( l \), may be connected by a directed arc \( kl \). The arc has a capacity denoted by \( c(kl) \). The maximum flow is the largest amount of the resource that can be sent from \( \alpha \) to \( \beta \), respecting the capacity constraints. The set of assignment matrices selected by the extended serial correspondence is given by an iterative computation of maximum flows in the graph constructed as follows.

Let \( A_0 = A \), \( X_0 = \emptyset \), \( \lambda^*_0 = 0 \), and for each pair of nodes, \( k \) and \( l \), \( c_0(kl) = 0 \).

For each \( m \in \mathbb{N}_+ \),

**Step m.** Let \( \lambda_m \in [0, 1 - \sum_{s=1}^{m-1} \lambda^*_s] \). Let \( A_m = A_{m-1} \setminus X_{m-1} \). Construct (i) the arc from \( \alpha \) to each agent \( i \in N_{m-1} \) and set its capacity to be \( c_m(\alpha i) + \lambda_m \), (ii) the arc from \( \alpha \) to each agent \( i \notin N_{m-1} \) and set its capacity to be \( c_m(\alpha i) \equiv c_{m-1}(\alpha i) + \lambda_m \), (iii) the arc from each agent \( i \) to each object \( o \in M(R_i, A_m) \) and set its capacity to be \( c_m(io) = \infty \), and the arc from each object \( o \) to \( \beta \) and set its capacity to be \( c_m(o \beta) \equiv 1 \). Let

\[
\lambda^*_m \equiv \min_{T \subseteq N} \frac{|M(R_T, A_m)| - \sum_{i \in T} c_m(\alpha i)}{|T|}.
\]

If \( T \subseteq N \) solves for \( \lambda^*_m \), we say that \( T \) is a “bottleneck set” at Step \( m \). Let \( N_m = \bigcup_{T \subseteq N} \{T : T \text{ is a bottleneck set at Step } m\} \) and \( X_m = \sum_{i \in N_m} M(R_i, A_m) \).

We say that at Step \( m \), the supply of each object in \( X_m \) reaches exhaustion, or simply, that \( X_m \) reaches exhaustion.

The algorithm terminates at Step \( \bar{m} \) if either \( A_{\bar{m}} = \emptyset \) or \( \sum_{s=1}^{\bar{m}} \lambda^*_s \geq 1 \). Note that \( \bar{m} \leq |A| \).

**Remark 1.** Let \( R \in \mathcal{R}^N \). For each pair \( \pi, \pi' \in ES(R) \), each \( i \in N \), each \( t \in \{1, \cdots, |A|\} \), and each \( a \in {\cal X}_t \), \( \sum_{o \in A_i} I_{i, a} \pi_i = \sum_{o \in A_i} I_{i, a} \pi'_i \).

\[^{6}\text{This implies that } \pi \text{ and } \pi' \text{ are sd-equivalent in welfare.}\]
3. Main result

Our main result is a characterization of the ES correspondence (in welfare terms) by means of \textit{sd efficiency}, \textit{sd no-envy}, and \textit{bounded invariance}. We emphasize that although ES is not single-valued, it is essentially single-valued.

Example 1. A family of rules satisfying \textit{sd efficiency}, \textit{sd no-envy}, and \textit{bounded invariance}

Consider the set of all ordered pairs consisting of an agent and an object, \(N \times A \equiv \{(i, o) : i \in N \text{ and } o \in A\}\). Let \(\chi\) be a strict ordering defined over \(N \times A\). Let \(\Phi^\chi\) be the subcorrespondence of the ES correspondence that selects, for each economy, the assignment matrix that maximizes the probability of each agent receiving each object in a lexicographic way according to \(\chi\). Let \(R \in \mathcal{R}^N\) and \((i, a)\) be the first pair according to \(\chi\). Let \(\Pi_0 \equiv \{\pi \in \text{ES}(R) : \text{for each } \pi' \in \text{ES}(R), \pi_{ia} \geq \pi'_{ia}\}\). If \(|\Pi_0| = 1\), then \(\Phi^\chi(R) = \Pi_0\). If \(|\Pi_0| > 1\), then consider the second pair according to \(\chi\), say \((j, b)\). Let \(\Pi_1 \equiv \{\pi \in \Pi_0 : \text{for each } \pi' \in \Pi_0, \pi_{jb} \geq \pi'_{jb}\}\). If \(|\Pi_1| = 1\), then \(\Phi^\chi(R) = \Pi_1\). Otherwise, continue until we obtain a single assignment matrix. It is easy to check that this rule satisfies the three properties.

Here is our main result.

Theorem 1.
(i) \(\text{ES}\) satisfies \textit{sd efficiency}, \textit{sd no-envy}, and \textit{bounded invariance}.
(ii) If a correspondence \(\Phi\) satisfies \textit{sd efficiency}, \textit{sd no-envy}, and \textit{bounded invariance}, then \(\Phi \subseteq \text{ES}\).

Proof. (i) First, ES satisfies \textit{sd efficiency} and \textit{sd no-envy} (Katta and Sethuraman, 2006). Next, we show that ES satisfies \textit{bounded invariance}. Let \(R \in \mathcal{R}^N\), \(i \in N\), and \(a \in A\). Let \(R'_i \in \mathcal{R}\) be such that \(R_i(a) = R'_i(a)\) and let \(R' \equiv (R'_i, R_{-i})\). Let \(s \in \{1, \cdots, |A|\}\) be the step of the ES algorithm at which \(a\) reaches exhaustion. It is easy to check that from Step 1 to Step \(s\), the ES algorithm works in the same way for \(R\) and \(R'\).

(ii) Let \(\Phi\) be a correspondence satisfying \textit{sd efficiency}, \textit{sd no-envy}, and \textit{bounded invariance}. Let \(m \in \{1, 2, \cdots, |A|\}\). Suppose that for each \(R \in \mathcal{R}^N\), each \(\pi \in \Phi(R)\) coincides with an assignment matrix in \(\text{ES}(R)\) on the objects that reach exhaustion before or at Step \((m - 1)\) of the ES algorithm for \(R\) (let us call this statement “the induction hypothesis”).\(^7\) We prove that \(\pi\) coincides with an assignment matrix in \(\text{ES}(R)\) on the objects that reach exhaustion before or at Step \(m\) of the ES algorithm for \(R\). Then, an induction argument on steps of the ES algorithm completes the proof.

\(^7\)If \(m = 1\), then the induction hypothesis is vacuously true.
Let \( R \in \mathcal{R}^N \), \( \pi \in \Phi(R) \), and \( \bar{X} \equiv \bigcup_{l=1}^{m-1} X_l \) be the set of objects that reach exhaustion before or at Step \((m - 1)\) of the ES algorithm for \( R \). Note that for each \( j \not\in N_m \) and each \( a \in M(R_j, A_m) \), \( \pi_{jo} = 0 \), and there is a constant \( \bar{f} > 0 \) such that for each \( j \in N_m \), each \( a \in M(R_j, A_m) \), and each \( \bar{\pi} \in ES(R) \), \( \sum_{o \in U(R_j, a)} \bar{\pi}_{jo} = \bar{f} \).

**Lemma 1.** Let \( T \subseteq N \) be a set of agents such that for each \( i \in T \), \( \sum_{o \in \bar{X} \cup M(R_T, A_m)} \pi_{io} < \bar{f} \). Then, there are \( j \in N \setminus T \) and \( a \in M(R_T, A_m) \) such that \( \pi_{ja} \geq 0 \).

**Proof.** Let \( k \in N \). If \( k \in N_m \), he is assigned the probability \( c_m(\alpha k) + \lambda_m^* \) of receiving his most preferred objects in \( A_m \). If \( k \not\in N_m \), he simply reserves this probability of receiving his most preferred objects in \( A_m \). Since \( \pi \) coincides with an assignment in \( ES(R) \) on \( \bar{X} \), for each \( k \in N \), \( \bar{f} = c_m(\alpha k) + \lambda_m^* + \sum_{o \in \bar{X}} \pi_{ko} \). Let \( i \in T \). Since \( \bar{X} \cap M(R_T, A_m) = \emptyset \), we have \( \sum_{o \in M(R_T, A_m)} \pi_{io} < c_m(\alpha i) + \lambda_m^* \). Then,

\[
\sum_{i \in T} \sum_{o \in M(R_T, A_m)} \pi_{io} \leq \sum_{i \in T} (c_m(\alpha i) + \lambda_m^*) = \sum_{i \in T} c_m(\alpha i) + |T| \lambda_m^* \leq |M(R_T, A_m)|,
\]

where the last inequality follows from the definition of \( \lambda_m^* \). By sd efficiency, there are \( j \in N \setminus T \) and \( a \in M(R_T, A_m) \) such that \( \pi_{ja} > 0 \).

We prove that for each \( i \in N \) and each \( b \in M(R_i, A_m) \), \( \sum_{o \in U(R_i, b)} \pi_{ib} \geq \bar{f} \). Then, by feasibility and from the definition of the ES correspondence, there is \( \bar{\pi} \in ES(R) \) such that for each \( l \in \{1, \ldots, m\} \), each \( i \in N_l \), and each \( o \in M(R_i, A_l) \), \( \bar{\pi}_{io} = \pi_{io} \), as desired. Suppose, by contradiction, that there are \( i \in N \) and \( b \in M(R_i, A_m) \) such that \( \sum_{o \in U(R_i, b)} \pi_{ib} < \bar{f} \). For each pair \( j, k \in N \), let \( S_{jk} \equiv M(R_j, A_m) \setminus M(R_k, A_m) \). Let \( i^0 \not\in N \) and \( S_{i^0} \) be the set of objects in \( \bar{X} \) that are ranked below \( M(R_i, A_m) \) in \( R_i \).

**Construction of \( R^1_l \):** Let \( R^1_l \in \mathcal{R} \) be such that (i) the objects that are at least as desirable as \( M(R_i, A_m) \) are ranked as in \( R_i \) and (ii) all objects in \( S_{i^0} \) are indifferent and appear just below \( M(R_i, A_m) \).

Let \( T^1 \equiv \{ i \} \). Since the ES algorithm from Step 1 to Step \((m - 1)\) works in the same way for \( R \) and \( R^1 \), each object in \( \bar{X} \) reaches exhaustion before Step \( m \) of the ES algorithm for \( R^1 \).

By the induction hypothesis, for each \( \mu \in \Phi(R^1) \) and each \( o \in S_{i^0} \), \( \mu_{io} = 0 \). By bounded invariance, there is \( \mu^1 \in \Phi(R^1) \) such that \( \sum_{o \in \bar{X} \cup M(R_i^1, A_m)} \mu^1_{io} = \sum_{o \in \bar{X} \cup M(R_i, A_m)} \pi_{io} < \bar{f} \). By

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8 Otherwise, some probability of \( M(R_T, A_m) \) is not assigned to any agent. It is easy to find an assignment matrix that stochastically dominates \( \pi \) at \( R \), in violation of sd efficiency.

9 This unusual notation is just for expositional convenience, the reason for which will become clear along the lines of the rest of the proof.

10 The rankings over the remaining objects can be chosen arbitrarily.
Lemma 1, there are \( i^1 \in N \setminus T^1 \) and \( x^1 \in M(R_{T^1}^i, A_m) \) such that \( \mu_{i^1 x^1}^1 > 0 \). By \( sd \) efficiency, for each \( o \in S_{i^1 i}, \mu_{i^1 o}^1 = 0 \).

**Construction of \( R_i^2 \):** Let \( R_i^2 \in \mathcal{R} \) be such that (i) the objects that are at least as desirable as \( S_{i^1 i} \) are ranked as in \( R_i^1 \) and (ii) all objects in \( S_{i^1 i} \) are indifferent and appear just below \( S_{i^1 i} \).

Let \( T^2 \equiv \{ i, i^1 \} \). By bounded invariance, there is \( \mu^2 \in \Phi(R^2) \) such that for each \( j \in N \) and each \( a \in \bar{X} \cup M(R^1_i, A_m), \mu^2_{ja} = \mu^1_{ja} \). Since \( \mu^1_{i^1 x^1} = \mu^1_{i^1 x^1} > 0 \), by \( sd \) efficiency, for each \( b \in S_{i^1 i}, \mu_{i^1 b}^2 = 0 \). Thus,

\[
\sum_{o \in \bar{X} \cup M(R_{T^2}^i, A_m)} \mu_{i^1 o}^2 = \sum_{o \in \bar{X} \cup M(R_{T^2}^i, A_m) \cup S_{i^1 i}} \mu_{i^1 o}^2 = \sum_{o \in \bar{X} \cup M(R_{T^2}^i, A_m)} \mu_{i^1 o}^1 < \bar{f},
\]

where the second equality comes from bounded invariance. By \( sd \) no-envy, \( \sum_{o \in \bar{X} \cup M(R_{T^2}^i, A_m)} \mu_{i^1 o}^2 < \bar{f} \).

By Lemma 1, there are \( i^2 \in N \setminus T^2 \) and \( x^2 \in M(R_{T^2}^i, A_m) \) such that \( \mu_{i^2 x^2}^2 > 0 \).

For \( \nu \in \{ 3, \ldots, |A| - 1 \} \), let \( R_i^{\nu} \in \mathcal{R} \) be such that (i) the objects that are at least as desirable as \( S_{i^\nu-1 i} \) are ranked as in \( R_i^{\nu-1} \) and (ii) all objects in \( S_{i^\nu-1 i} \) are indifferent and appear just below \( S_{i^\nu-1 i} \). For each \( l \in \{ 2, 3, \ldots \} \), let \( T^l \equiv \{ i^1, i^2, \ldots, i^{l-1} \} \). As a hypothesis, suppose that there is \( \mu^\nu \in \Phi(R^\nu) \) such that (1) for each \( j \in T^\nu, \sum_{o \in \bar{X} \cup M(R_{T^\nu}^j, A_m)} \mu_{j o}^\nu < \bar{f} \), (2) for each \( l \in \{ 2, \ldots, \nu \} \), there are \( i^l \in N \setminus T^l \) and \( x^l \in M(R_{T^l}^i, A_m) \) such that \( \mu_{i^l x^l}^\nu > 0 \). We show that statements (1) and (2) hold for \( \nu + 1 \) as well.

**Construction of \( R_i^{\nu+1} \):** Let \( R_i^{\nu+1} \in \mathcal{R} \) be such that (i) the objects that are at least as desirable as \( S_{i^\nu-1 i} \) are ranked as in \( R_i^{\nu} \) and (ii) all objects in \( S_{i^\nu i} \) are indifferent and appear just below \( S_{i^\nu-1 i} \).

By bounded invariance, there is \( \mu^{\nu+1} \in \Phi(R^{\nu+1}) \) such that for each \( j \in N \) and each \( a \in \bar{X} \cup M(R_{T^\nu}^j, A_m), \mu_{ja}^{\nu+1} = \mu_{ja}^\nu \). Since \( \mu_{i^\nu x^\nu}^\nu = \mu_{i^\nu x^\nu}^\nu > 0 \), by \( sd \) efficiency, for each \( b \in S_{i^\nu i}, \mu_{i^\nu b}^{\nu+1} = 0 \). Thus,

\[
\sum_{o \in \bar{X} \cup M(R_{T^\nu}^{\nu+1}, A_m)} \mu_{i^\nu o}^{\nu+1} = \sum_{o \in \bar{X} \cup M(R_{T^\nu}^{\nu+1}, A_m) \cup S_{i^\nu i}} \mu_{i^\nu o}^{\nu+1} = \sum_{o \in \bar{X} \cup M(R_{T^\nu}^{\nu+1}, A_m)} \mu_{i^\nu o}^\nu < \bar{f}.
\]

By \( sd \) no-envy, for each \( j \in T^{\nu+1}, \sum_{o \in \bar{X} \cup M(R_{T^{\nu+1}}^j, A_m)} \mu_{j o}^{\nu+1} < \bar{f} \). By Lemma 1, there are \( i^{\nu+1} \in N \setminus T^{\nu+1} \) and \( x^{\nu+1} \in M(R_{T^{\nu+1}}^{\nu+1}, A_m) \) such that \( \mu_{i^{\nu+1} x^{\nu+1}}^{\nu+1} > 0 \).

By repeating this argument, we conclude that there should be \( \mu_{[N]}^\nu \in \Phi(R_{[N]}^\nu) \) such that for some \( j \in N \setminus T^{[N]} \) and some \( b \in M(R_{T^{[N]}}^j, A_m), \mu_{[N] j b}^{[N]} > 0 \). However, since \( T^{[N]} = N \), there is no such \( j \in N \setminus T^{[N]}, \) a contradiction.

**3.1. Independence of the Axioms**

We establish that the axioms listed in Theorem 1 are independent. In each case, we indicate which axiom is violated.
• **Sd efficiency**: consider the rule that for each economy, each agent, and each object, assigns probability $\frac{1}{|N|}$ to the agent receiving the object.

• **Sd no-envy**: select a strict ordering over the agents. Consider the rule that maximizes agents’ welfare lexicographically in that order.

• **Bounded invariance**: let $N = \{1, 2, \cdots, |N|\}$, $A \equiv \{a, \cdots, z\}$, and $R$ be such that (i) $a \ R_1 b \ R_1 c \ R_1 \cdots$, $a \ R_2 c \ R_2 b \ R_2 \cdots$, $b \ R_3 c \ R_3 a \ R_3 \cdots$, and (ii) the other agents rank $\{a, b, c\}$ lower than each object in $A \setminus \{a, b, c\}$ and have different most preferred objects. Let $\pi$ be such that $\pi_1 = (1/2, 1/6, 1/3, 0, \cdots, 0)$, $\pi_2 = (1/2, 0, 1/2, 0, \cdots, 0)$, $\pi_3 = (0, 5/6, 1/6, 0, \cdots, 0)$, and each other agent is assigned his most preferred object with probability 1. Let $\Phi$ be the correspondence such that $\Phi(R) = \{\pi\}$ and for each $R' \neq R$, $\Phi(R') = ES(R')$.

Our result can be considered as an extension of the characterization in Bogomolnaia and Heo (2011) to the weak preference domain: recall that on the strict preference domain, the extended serial correspondence coincides with the serial rule.

4. Another Characterization

We offer here another characterization of the ES correspondence along the lines of Kesten et al. (2011). The axioms are a weaker efficiency axiom than *weak sd efficiency* and a fairness axiom that we adapt from these authors. These axioms are defined for each assignment matrix. Let $\pi \in \Pi$.

**Non-wastefulness**: for each $R \in \mathcal{R}^N$, each $i \in N$ and each pair $a, b \in A$, if $\pi_{ia} > 0$ and $b \ P_i a$, then $\sum_{j \in N} \pi_{jb} = 1$.

**Ordinal Fairness**: for each $R \in \mathcal{R}^N$, each pair $i, j \in N$, and each $a \in A$, if $\pi_{ia} > 0$, then $\sum_{o \in U(R_i, a)} \pi_{jo} > \sum_{o \in U(R_i, a)} \pi_{io}$.

Before we introduce our next characterization, we present an alternative representation of assignments, introduced in Heo (2011a) and also used in Bogomolnaia and Heo (2011). This representation allows us to obtain a straightforward proof for our second characterization as well as an alternative proof of Theorem 1 (given in the Appendix).

4.1. Consumption Schedule

Let $P \in \mathcal{P}^N$ be a strict preference profile. We start from an alternative representation of assignments at $P$. Let $\pi \in \Pi$. For each $i \in N$, we represent $\pi_i$ as a process of consuming objects at unit speed over the time interval $[0, 1]$. The process is in decreasing order of $P_i$. First, consider agent $i$’s most preferred object, say $a$. Imagine that he consumes $a$ from time 0 to
Lemma 2. Let \( R \in \mathcal{R}^N \) and \( \pi \in \Pi \). For each \( \sigma \in \Sigma^N \), each agent does his best during \([0,1]\)
at \( t(\pi, P^\sigma(R)) \) if and only if \( \pi \in ES(R) \).

**Proof.** The proof is in the Appendix. \( \square \)

We are ready to present our characterization. We emphasize that *non-wastefulness* and *ordinal fairness* apply to each economy separately.

**Theorem 2.** Let \( R \in \mathcal{R}^N \). An assignment matrix is selected by ES for \( R \) if and only if it is *non-wasteful* and *ordinally fair* at \( R \).

**Proof.** Let \( R \in \mathcal{R}^N \). Suppose that \( \pi \in ES(R) \). By Lemma 2, for each \( \sigma \in \Sigma^N \), each agent does his best during \([0,1]\) at \( t(\pi, P^\sigma(R)) \), and thus \( \pi \) satisfies the two axioms. Conversely, let \( \pi \) be an assignment matrix satisfying the two axioms. Suppose, by contradiction, that \( \pi \notin ES(R) \). Then, there is \( \sigma \in \Sigma^N \) such that \( t(\pi, P^\sigma(R)) \) is as follows: there are \( i \in N, a \in N \), and \( \tau \in [0,1] \) such that agent \( i \) stops consuming some \( b \in I(R_i,a) \) at some time \( \tau \) at which \( I(R_i,a) \) has not reached exhaustion. By *non-wastefulness*, there is \( j \in N \) who consumes a positive probability of \( b \) after \( \tau \) at \( t(\pi, P^\sigma(R)) \). However, \( \sum_{o \in U(R_i,b)} \pi_{io} < \sum_{o \in U(R_j,b)} \pi_{jo} \), in violation of *ordinal fairness*. \( \square \)

**Appendix**

**A1. The Proof of Lemma 2**

To simplify the proof, we slightly modify two formulations of the ES algorithm. Keeping the other notation of Section 2.1, for each \( m \in \mathbb{N}_+ \), define \( N_m \) and \( X_m \) as follows. If \( T \subseteq N \) solves for \( \lambda_m \), we say that \( T \) is a “bottleneck set” at Step \( m \). Let \( N_m \subseteq T \) be a bottleneck set at Step \( m \) of the smallest cardinality and let \( X_m \equiv \sum_{i \in N_m} M(R_i, A_m) \).\(^{14}\)

**Proof.** (If) It directly comes from the definition of ES. 

(Only if) Let \( R \in \mathcal{R}^N \) and \( \pi \in \Pi \) be such that for each \( \sigma \in \Sigma^N \), each agent does his best during \([0,1]\) at \( t(\pi, P^\sigma(R)) \) (let us call this “the hypothesis”). Suppose, by contradiction, that \( \pi \notin ES(R) \). Let \( \bar{\pi} \in ES(R) \). Then, there are \( s \in \{1, \cdots, |A|\} \), \( i \in N \), and \( a \in A \) such that \( i \in N_s, \ a \in M(R_i, A_s) \), and \( \sum_{o \in U(R_i,a)} \pi_{io} < \sum_{o \in U(R_i,a)} \bar{\pi}_{io} \). Choose \((i^*, a^*, s^*)\) among them so that \( \sum_{o \in U(R_i,a)} \pi_{io} \) is the smallest at \((i^*, a^*)\) (denote this by (1)). If there are more than one such triple with the same smallest sum of probabilities, then choose a triple such that \( s^* \) is the largest: \( a^* \) reaches exhaustion at the last step among them in the ES algorithm for \( R \) (denote this by (2)). Let \( \bar{s} \) be the largest integer in \( \{1, \cdots, s^*\} \) such that \( \sum_{t=1}^{\bar{s}} \lambda_t < \tau^* \).\(^{15}\) Note that \( \tau^* < \sum_{t=1}^{s^*} \lambda_t \).

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\(^{14}\)This is equivalent to the definition of ES in Section 2.1, but allows us to obtain a simple proof of Lemma 2.

\(^{15}\)For each \( t \in \{1, \cdots, \bar{s}\} \), each \( n \in N_t \), and each \( x \in M(R_n, A_t) \), \( \sum_{o \in U(R_n,x)} \pi_{no} = \sum_{o \in U(R_n,x)} \bar{\pi}_{no} \).
Let \( \bar{N} \equiv \{ i \in N : \pi_{i a^*} > 0 \} \). We claim that for each pair \( j, k \in \bar{N} \), \( \sum_{a \in U(R_j, a^*)} \pi_{ja} = \sum_{a \in U(R_k, a^*)} \pi_{ka} \leq \tau^* \) (denote this statement by (3)). Otherwise, either for some pair \( j, k \in \bar{N} \), \( \sum_{a \in U(R_j, a^*)} \pi_{ja} < \sum_{a \in U(R_k, a^*)} \pi_{ka} \) or for some \( j \in \bar{N} \), \( \sum_{a \in U(R_j, a^*)} \pi_{ja} > \tau^* \). Let \( \sigma \in \Sigma^N \) be such that for each \( n \in N \), \( \sigma_n(a^*) = |A| \): the hypothesis is violated at \( t(\pi, P^\sigma(R)) \). There are two possibilities.

**Case 1:** There is \( k \in \bar{N} \) such that \( k \notin N_{s^*} \). Let \( \gamma \) be the smallest positive integer such that \( k \in N_{s^*+\gamma} \). For each \( b \in M(R_k, A_{s^*+\gamma}) \), \( b \leq R_k \) \( a^* \) and thus, \( \sum_{a \in U(R_k, b^* \gamma)} \pi_{ka} \leq \sum_{a \in U(R_k, a^*)} \pi_{ka} \). By (3), we also have \( \tau^* \geq \sum_{a \in U(R_k, a^*)} \pi_{ka} \). Altogether, \( \sum_{a \in U(R_k, b^* \gamma)} \pi_{ka} \leq \tau^* \leq \sum_{a \in U(R_k, a^*)} \pi_{ka} \). Since \( a^* \in X_{s^*} \) and \( b \in X_{s^*+\gamma} \), we have \( \sum_{a \in U(R_k, a^*)} \pi_{ka} \leq \sum_{a \in U(R_k, b^* \gamma)} \pi_{ka} \). Thus, \( \sum_{a \in U(R_k, b^* \gamma)} \pi_{ka} \leq \sum_{a \in U(R_k, b^* \gamma)} \pi_{ka} \). By (1) and (2), \( (k, b, s^* + \gamma) \) should have been chosen instead of \( (i^*, a^*, s^*) \), a contradiction.

**Case 2:** \( \bar{N} \subseteq N_{s^*} \). By feasibility, there are \( j \in N_{s^*} \) and \( c \in M(R_j, A_{s^*}) \) such that \( \sum_{a \in U(R_j, c)} \pi_{ja} > \tau^* \). Let \( \tilde{N} \equiv \{ n \in N_{s^*} : \text{for each } a \in M(R_n, A_{s^*}), \sum_{a \in U(R_n, a)} \pi_{na} = \tau^* \} \). We claim that for each \( k \in N_{s^*} \setminus \tilde{N} \) and each \( d \in \cup_{n \in \tilde{N}} M(R_n, A_{s^*}) \), \( \pi_{kd} = 0 \). Otherwise, let \( \sigma \in \Sigma^N \) be such that for each \( n \in N \), \( \sigma_n(d) = |A| \): the hypothesis is violated at \( t(\pi, P^\sigma(R)) \). Let \( \tilde{X} \equiv \cup_{n \in \tilde{N}} M(R_n, A_{s^*}) \). Altogether, \( \pi \) is such that (i) for each \( t \in \{1, \ldots, s\} \) and each \( n \in N_t \), \( \sum_{a \in M(R_n, A_{t})} \pi_{na} = \sum_{a \in M(R_n, A_{t})} \pi_{na} \); (ii) for each \( n \in \bar{N} \) and each \( a \in M(R_n, \tilde{X}) \), \( \sum_{a \in U(R_n, a^*)} \pi_{na} = \tau^* < \sum_{t=1}^{s^*} \lambda_t \); (iii) for each \( n \notin \bar{N} \) and each \( a \in \tilde{X} \), \( \pi_{ka} = 0 \). This implies that \( \tilde{X} \subseteq X_{s^*} \) reaches exhaustion at some step(s) earlier than Step \( s^* \) of the ES algorithm, contradicting the way Step \( s^* \) of the ES algorithm is defined.

**Remark 2.** Let \( \pi, \pi^\prime \in \Pi \) and \( R \in \mathcal{R}^N \). Suppose that for each \( \sigma \in \Sigma^N \), each agent does his best during \([0, 1]\) at \( t(\pi, P^\sigma(R)) \) and \( t(\pi^\prime, P^\sigma(R)) \). By Lemma 2, \( \pi \) and \( \pi^\prime \) are welfare-equivalent.

**Remark 3.** A “local” property of the profile of consumption schedules representing each ES assignment matrix follows from Lemma 2. Let \( R \in \mathcal{R}^N \), \( a \in A \), and \( \tau \in [0, 1] \). Let \( \pi \in \Pi \) be such that for each \( \sigma \in \Sigma^N \), each agent does his best during \([0, \tau]\) at \( t(\pi, P^\sigma(R)) \). Then, the following statements are equivalent:

1. (i) for each \( \sigma \in \Sigma^N \), \( a \) reaches exhaustion no later than \( \tau \) at \( t(\pi, P^\sigma(R)) \).
2. (ii) there is \( \tau \in ES(R) \) such that for each \( \sigma \in \Sigma^N \), \( a \) reaches exhaustion no later than \( \tau \) at \( t(\hat{\tau}, P^\sigma(R)) \).\(^\hspace{1em}16\)

**A2. An alternative proof of Theorem 1 (ii)**

We also provide a complementary proof of Theorem 1 (ii). This proof links our result to that in Bogomolnaia and Heo (2011). By introducing profiles of tie-breakers (Lemma 2) for each

\(^{16}\)Equivalently, \( a \) reaches exhaustion at Step \( s \) of the ES algorithm for \( R \) such that \( \sum_{t=1}^{s^*} \lambda_t \leq \tau \).
profile of weak preferences and then working with the profile of consumption schedules, we can easily adapt the arguments of the proof in Bogomolnaia and Heo (2011).

**Proof.** Let \( \Phi \) be a correspondence satisfying sd efficiency and bounded invariance. Suppose that \( \Phi \notin ES \). Then, there are \( R \in R^N \) and \( \pi \in \Phi(R) \) such that \( \pi \notin ES(R) \). By Lemma 2, there is \( \sigma \equiv (\sigma_i)_{i \in N} \in \Sigma^N \) such that \( t(\pi, P^\sigma(R)) \) is as follows: there are \( a \in A, \tau \in [0, 1], i \in N, \) and \( x \in I(R_i, a) \) such that \( x \) has not reached exhaustion at \( \tau \) but agent \( i \) stops consuming from \( I(R_i, a) \) at \( \tau \). Let \( R \in R^N \) and \( \sigma^* \in \Sigma^N \) be such that the associated \( \tau \) is the smallest (denote this smallest \( \tau \) by \( \tau^* \)). By sd efficiency, there is \( j \in N \setminus \{i\} \) who is assigned a positive probability of receiving \( x \).

Let \( \tilde{N} \equiv \{ k \in N : \pi_{kk} > 0 \} \) and \( \tilde{S} \equiv \bigcup_{k \in \tilde{N}} \{ o \in A : x P_o R_k x \} \). By sd efficiency, for each \( s \in \tilde{S}, \pi_{is} = 0 \). Now, let \( R_i' \in R \) be such that \( R_i(x) = R_i'(x) \), and all objects in \( \tilde{S} \) are indifferent for agent \( i \) and appear just below \( x \). Let \( \Sigma \equiv (R_i', R_i \cdot i) \).

By bounded invariance, there is \( \pi' \in \Phi(R') \) such that for each \( k \in N \) and each \( o \in U(R_i, x) \), \( \pi'_{ko} = \pi_{ko} \) (thus, \( \sum_{o \in U(R_i', x) \cup \tilde{S}} \pi'_{io} = \tau^* \)). Thus, for each \( k \in \tilde{N}, \pi_{kk} > 0 \). By sd efficiency, for each \( o \in \tilde{S}, \pi'_{io} = 0 \).

Next, we claim that there is \( \tilde{\sigma} \in \Sigma^N \) such that \( I(R_i', x) \) has not reached exhaustion by \( \tau^* \) at \( t(\pi', P^\sigma(R')) \). Suppose otherwise. For each \( \sigma \in \Sigma^N \), \( t(\pi', P^\sigma(R')) \) is such that \( I(R_i', x) \) reaches exhaustion no later than \( \tau^* \). Recall that \( \tau^* \) is chosen so that for each \( R'' \in R^N \), each \( \sigma'' \in \Sigma^N \), and each \( \pi'' \in \Phi(R'') \), each agent does his best during \([0, \tau^*]\) at \( t(\pi'', P^{\sigma''}(R'')) \) (denote this statement by (†)). Thus, for each \( \sigma \in \Sigma^N \), each agent does his best during \([0, \tau^*]\) at \( t(\pi', P^\sigma(R')) \). By Remark 3, there is \( \tilde{\pi}' \in ES(R') \) such that for each \( \sigma \in \Sigma^N \), \( I(R_i', x) \) reaches exhaustion no later than \( \tau^* \) at \( t(\tilde{\pi}', P^\sigma(R')) \). Let \( s \) be the step of the ES algorithm for \( R' \) at which \( I(R_i', x) \) reaches exhaustion. As shown in the proof of Theorem 1 (i), Step 1 to Step \( s \) of the ES algorithm are the same for \( R \) and \( R' \). Thus, \( I(R_i, x) (=I(R_i', x)) \) also reaches exhaustion at Step \( s \), and for each \( \tilde{\pi} \in ES(R) \) and each \( \sigma \in \Sigma^N \), \( I(R_i, x) \) reaches exhaustion no later than \( \tau^* \) at \( t(\tilde{\pi}, P^\sigma(R)) \). By (†) and Remark 3, we obtain that for each \( \sigma \in \Sigma^N \), \( I(R_i, x) \) reaches exhaustion no later than \( \tau^* \) at \( t(\pi, P^{\sigma}(R)) \), a contradiction to the fact that \( x \) has not reached exhaustion by \( \tau^* \) at \( t(\pi, P^{\sigma}(R)) \).

By sd efficiency, at \( t(\pi', P^\sigma(R')) \), there is an agent \( l \in N \) who consumes a positive probability of some object \( c \in I(R_i', x) \) after \( \tau^* \). Thus, \( \sum_{o \in U(R_i', x) \cup S} \pi'_{io} < \sum_{o \in U(R_i, x)} \pi_{io} \). Since \( U(R_i', x) \cup S \supseteq U(R_i', c) \), we obtain a violation of sd no-envy. □

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17 That is, \( I(R_i, a) \) reaches exhaustion after \( \tau^* \) at \( t(\pi, P^{\sigma^*}(R)) \). If there is no such agent \( j \), then there is \( \delta > 0 \) such that by increasing \( \delta \) the probability of agent \( i \) receiving \( x \), decreasing by \( \delta \) the total probability of his receiving the objects to which he prefers \( x \), and keeping each other agent’s assignment the same, we make agent \( i \) sd-better off without sd-hurting any other agent.
A3. A Remark on Theorem 1 (ii)

Theorem 1 (ii) remains true if we impose the following two axioms. First is a weakening of \textit{sd efficiency} (Bogomolnaia and Heo, 2011, Hashimoto and Hirata, 2011). Let $R \in \mathcal{R}^N$ and $\pi \in \Pi$. We say that $\pi$ is \textbf{weakly sd efficient at $R$} if there is no $\pi' \in \Pi$ such that (i) for each $i \in N$, $\pi'_i R^\text{wsd}_i \pi_i$, (ii) for some $i \in N$, $\pi'_i R^\text{sd}_i \pi_i$, and (iii) $|\{i \in N : \pi'_i R^\text{sd}_i \pi_i\}| \leq 2$. The corresponding property of a correspondence $\Phi$ is as follows:

\textbf{Weak sd efficiency:} For each $R \in \mathcal{R}^N$ and each $\pi \in \Phi(R)$, $\pi$ is weakly sd efficient at $R$.

Second is an invariance notion that is weaker than \textit{bounded invariance}.

\textbf{Weak bounded invariance:} For each $R \in \mathcal{R}^N$, each $i \in N$, each $a \in A$, and each $R'_i \in \mathcal{R}$, if $R_i(a) = R'_i(a)$, then for each $\pi \in \Phi(R)$, there is $\pi' \in \Phi(R'_i, R_{-i})$ such that (i) for each $\sum_{o \in U(R_i,a)} \pi_{io} = \sum_{o \in U(R'_i,a)} \pi'_{io}$ and (ii) for each $j \in N$, if $\pi_{ja} > 0$, then $\pi'_{ja} > 0$.

Theorem 1 (ii) is now stated as follows: if a correspondence $\Phi$ satisfies weak sd efficiency, sd no-envy, and weak bounded invariance, then $\Phi \subseteq \mathcal{ES}$. The proof of Theorem 1 still works with the following modifications. In the construction of $R^*_2$: by bounded invariance, there is $\mu^2 \in \Phi(R^*_2, R_{-i})$ such that $\mu^2_{1x^1} > 0$ and for each $a \in X \cup M(R^*_1, A_m)$, $\mu^2_{ia} = \mu^1_{ia}$. In the construction of $R^\nu_{i+1}$: by bounded invariance, there is $\mu^{\nu+1} \in \Phi(R^\nu_{i+1}, R_{-i})$ such that $\mu^{\nu+1}_{v^*x^*} > 0$ and for each $a \in X \cup M(R^\nu_{T^*}, A_m)$, $\mu^{\nu+1}_{ia} = \mu^{\nu}_{ia}$.

References


