Abstract

Consider a problem in which the cost of building an irrigation canal has to be divided among a set of people. Each person has different needs. When the needs of two or more people overlap there is congestion. In problems without congestion, a unique canal serves all the people and it is enough to finance the cost of the largest need to accommodate all the other needs. In contrast, when congestion is considered, more than one canal might need to be built and each canal has to be financed.

In problems without congestion, axioms related with fairness (equal treatment of equals) and group participation constraints (no-subsidy or core constraints) are generally compatible. With congestion, we show that these two axioms are incompatible.

We define weaker axioms of fairness (equal treatment of equals per canal) and group participation constraints (no-subsidy across canals) that in conjunction with a few other axioms characterize the sequential contributions family of rules. Moreover, when we include a new axiom we characterize a subfamily of rules.

Finally, we adapt some other properties to the problem with congestion and study which of the rules we define satisfy these axioms.

JEL classification: C71, D63, D71, H41

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1 Introduction

Consider a group of farms arranged in a linear way from an irrigation ditch. The farmers have to build an irrigation canal and share the cost among themselves. Each farmer needs the canal to be long enough to reach his field. When more than one farmer use the same canal, the canal might need to be wider to transport a larger quantity of water. This congestion might increase the cost of the canal.

Without congestion, a unique canal can serve everyone. In contrast, when there is congestion, more than one canal might need to be built. The total amount to be collected is the sum of the costs of each of these canals.

Without congestion, the irrigation problem is mathematically equivalent to the airport problem. However, in our paper we refer to the irrigation problem.

Without congestion, no-subsidy is a basic property. It says that no group of farmers should pay more than the cost of building their own canal. While without congestion nearly all of the well-known rules satisfy this axiom (Thomson 2007), with congestion this property becomes very demanding. Most rules violate it. In particular, in the no-congestion case the sequential equal contributions rule is well-behaved (Aadland and Kolpin 1998, Potters and Sudhölter 1999, Dubey 1982, Chun, Kayı and Yeh 2008).

However, with congestion, this rule violates no-subsidy. Moreover, a weaker axiom than no-subsidy, no-subsidy across canals and a basic fairness axiom, equal treatment of equals, are incompatible.

To accommodate congestion, we extend the sequential equal contributions rule. This rule divides equally the cost of each segment among the people using it in each canal. We introduce a family of rules, the sequential contributions family, where for each canal and each segment the cost of serving one person is divided equally among all its users and the cost of serving two or more people is divided among users whose need is not served when this segment is built. The sequential equal contributions rule is not a member of this family. But the restriction of each member of this family to the no-congestion case coincides with the sequential equal contributions rule.

Without congestion, the sequential equal contributions rule is characterized by two basic fairness axioms. Equal treatment of equals says that people with equal parameters should pay equal amounts. Independence of at-least-as-large costs says that each person’s payment should not depend on costs larger than his (Moulin and Shenker 1992). An implication of this result is that equal treatment of equals and independence of at-least-as-large costs imply no-subsidy. With congestion, we define a weaker axiom than equal treatment of equals, namely equal treatment of equals per canal, and a natural extension of independence of at-least-as-large costs.
costs, independence of at-least-as-large lengths.\footnote{Each person’s payment should not depend on needs greater than his.} In this model, these two new axioms do not characterize a unique rule. Further, they do not imply no-subsidy across canals.

However, no-subsidy across canals, equal treatment of equals per canal, independence of at-least-as-large lengths in conjunction with strong congestion monotonicity, characterize the sequential contributions family of rules. Moreover, when we impose equality in increased congestion we characterize a subfamily of rules, the sequential conditional contributions rules.

Finally, we extend some other properties such as conditional cost additivity, population monotonicity, and first-person consistency to the problem with congestion. We study which of our rules satisfy these axioms. Even though the sequential equal contributions rule and the sequential contributions with equality in congestion rule satisfy conditional cost additivity, some members of the sequential conditional contributions family and the sequential contributions family violate it\footnote{Without congestion, Dubey (1982) characterizes the sequential equal contributions rule with equal treatment of equals, conditional cost additivity, and an auxiliary axiom.}.

We find that the sequential equal contributions rule and members of the sequential contributions family violate population monotonicity. This result is surprising since without congestion all the well-know rules satisfy this axiom (Thomson 2007). With congestion, the sequential equal contributions rule is first-person consistent, but some members of the sequential contributions family are not.\footnote{Without congestion, Chun et al. (2008) characterize the sequential equal contributions on the basis of this axiom and some other properties.}

The remainder of the paper is organized as follows. We present the model in Section 2. We describe some desiderata of rules in Section 3 and define our rules in Section 4. In Section 5 we present our results and in Section 6 we deal with the variable-population model.

\section{The Model}

Let the finite set of people be $N$. Each person $i \in N$ needs a canal of length $l_i \geq 0$. For simplicity, we order people in $N$ in terms of their needs so that $l_1 \leq l_2 \leq \cdots \leq l_n$. Let $l \equiv (l_i)_{i \in N}$. For each $i \in N$, the interval, $(l_{i-1}, l_i) \in \mathbb{R}^2_+$, is the part of the canal that has to be built for $i$ after all the people with smaller needs have been accommodated. For each $i \in N$, we refer to this interval as a segment, $s_i \equiv (l_{i-1}, l_i)$ (Figure 1.)

A cost function $C : \mathbb{R}_+ \times \{1, \ldots, n\} \to \mathbb{R}_+$ gives the cost of constructing a canal as a function of each length and the number of people whom is intended for.
Each person needs a canal of certain length, person 1 needs $l_1$, person 2 needs $l_2$, and person 3 needs $l_3$. In this case, $s_1 = (0, l_1)$. For person 2, $s_2 = (l_1, l_2)$ because this is the interval of the canal that has to be build to accommodate his need, given that person 1’s length has been built. For person 3, $s_3 = (l_2, l_3)$.

Since there is congestion, these cost functions are monotonic in the second variable but they are not necessarily monotonic in the first. That is, for each $l \in \mathbb{R}_+$ and each $k \in \{1, ..., n - 1\}$, $C(l, k) \leq C(l, k + 1)$ (Figure 2.)

The incremental cost of constructing a certain segment $s_i$ used by $k$ people is the area below the curve $C(\cdot, k)$ from $l_{i-1}$ to $l_i$. That is, $c(s_i, k) = \int_{l_{i-1}}^{l_i} C(l, k) dl$. Because of monotonicity in congestion, for each $s_i \in \mathbb{R}_+^2$ and $k \in \{1, ..., n - 1\}$, $c(s_i, k) \leq c(s_i, k+1)$. Since each $i \in N$ only needs length $l_i$, then for each $N' \subset N$, the total cost of serving $N'$ is $C(l, c, N') \equiv \sum_{i,j \in N, j \leq i} c(s_i, |N'|- |\{j \in N' | j \leq i\}| + 1)$. Let $\mathcal{C}$ be the domain of incremental cost functions (Figure 2.)

A problem is a vector of positive lengths, $l \in \mathbb{R}_+^N$ and a cost function $c \in \mathcal{C}$. Because of congestion, it might be necessary to construct more than one canal to minimize the total cost of serving $N$. However, there is no additional cost if more than one canal is built.

Let $\Pi$ be the set of all partitions over $N$, and $\pi = (\pi_k)_{k=1}^K$ be a typical element. Let the minimum cost of serving $N$, $\min_{\pi \in \Pi^N} \{ \sum_{\pi_k \in \pi} C(l, c, \pi_k) \}$ be denoted by $mc(l, c)$.

An allocation for $(l, c)$ is a vector $x \in \mathbb{R}^N$ such that for each $i \in N$, $x_i \geq 0$, and $\sum_{i \in N} x_i = mc(l, c)$. Let $X(c)$ be the set of allocations.

Let $EP(l, c) \subset \Pi$ be the set of efficient partitions, that is, the set of partitions of $N$ for which the total cost is $mc(l, c)$.

A rule, $\varphi$, recommends for each problem and each efficient partition of it an allocation of it. That is, for each $(l, c) \in \mathbb{R}_+^N \times \mathcal{C}$ and each $\pi \in EP(l, c)$, $\varphi(l, c, \pi) \in X(c)$. 

**Figure 1: Definition of length and segment.** Each person needs a canal of certain length, person 1 needs $l_1$, person 2 needs $l_2$, and person 3 needs $l_3$. In this case, $s_1 = (0, l_1)$. For person 2, $s_2 = (l_1, l_2)$ because this is the interval of the canal that has to be build to accommodate his need, given that person 1’s length has been built. For person 3, $s_3 = (l_2, l_3)$. 

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Figure 2: Cost functions with congestion for an irrigation problem. The cost of building a canal for 1, 2, and 3 is \( C(\{1, 2, 3\}) = c(s_1, 3) + c(s_2, 2) + c(s_3, 1) \).

### 3 Properties of rules

We first introduce desirable properties of rules.

First, no group of people should collectively pay more than the minimum cost of being served separately.

**No-subsidy:** For each \((l, c) \in \mathbb{R}^N_+ \times \mathcal{C}\), each \(\pi \in EP(l, c)\), and each \(N' \subseteq N\),

\[
\sum_{i \in N'} \varphi_i(l, c, \pi) \leq mc(l_{N'}, c_{N'}).
\]

When there is congestion, this property is very demanding. Most rules violate this property. We introduce a weaker version. It says that when more than one canal is built, the users of each canal should not collectively pay more than the minimum cost of being served separately. Since efficiency is embedded in the definition of a rule, this axiom says that the users of each canal should pay the cost of constructing this canal.

**No-subsidy across canals:** For each \((l, c) \in \mathbb{R}^N_+ \times \mathcal{C}\), each \(\pi \equiv \{\pi_k\}_{k=1}^K \in EP(l, c)\), and each \(k \in \{1, ..., K\}\),

\[
\sum_{i \in \pi_k} \varphi_i(l, c, \pi) = C(l, c, \pi_k).
\]

Note that when the efficient partition is the trivial partition, \(\pi = \{N\}\), *no-subsidy across canals* is implied by the definition of an allocation.

The next basic fairness axiom says that people whose needs are equal should pay equal amounts.
Equal treatment of equals: For each \((l, c) \in \mathbb{R}^N_+ \times C\), each \(\pi \in EP(l, c)\), and each \(\{i, j\} \subseteq N\) with \(l_i = l_j\),

\[\varphi_i(l, c, \pi) = \varphi_j(l, c, \pi)\].

We show in Proposition \[\square\] in contrast with the no-congestion case, that equal treatment of equals and no-subsidy across canals are incompatible. Therefore, we introduce a natural weakening of equal treatment of equals. It says that people who share a canal and whose needs are equal should pay equal amounts.

Equal treatment of equals per canal: For each \((l, c) \in \mathbb{R}^N_+ \times C\), each \(\pi \equiv \{\pi_k\}_{k=1}^K \in EP(l, c)\), each \(k \in \{1, ..., K\}\), and each \(\{i, j\} \subseteq \pi_k\) with \(l_i = l_j\),

\[\varphi_i(l, c, \pi) = \varphi_j(l, c, \pi)\].

The next axiom pertains to situations in which some people’s need might increase. Consider a person whose need does not change, and moreover, the need of each person before him does not change either. The axiom says that the amount this person pays should not change.

Independence of at-least-as-large lengths: For each pair \((l, c) \in \mathbb{R}^N_+ \times C\) such that \(c' = c\), each \(\pi \in EP(l, c)\) such that there is \(\pi' \in EP(l', c')\) for which \(\pi' = \pi\), if

1. \(l' \geq l\) and
2. for each \(i \in N\) such that for each \(j \leq i\), \(l'_j = l_j\),

then, for each \(i \in N\),

\[\varphi_i(l, c, \pi) = \varphi_i(l', c', \pi').\]

In the problem without congestion, there is an axiom with a similar spirit than the previous one. It says that what a person pays does not depend on costs higher than his own. This axiom is called independence of at-least-as-large costs. Without congestion, these two axioms are equivalent.

The next axiom pertains to situations in which the cost of serving two or more people in a segment increases. It says that nobody should benefit. Note that this axiom says nothing when the cost of serving a single person increases.

Congestion monotonicity: For each pair \((l, c) \in \mathbb{R}^N_+ \times C\) such that \(l' = l\) and for each \(\pi \in EP(l, c)\) such that there is \(\pi' \in EP(l', c')\) for which \(\pi' = \pi\), if there is \(i \in N\) such that

1. \(c'(s_i, 1) = c(s_i, 1)\),
2. there is \( k \in \{2, \ldots, n\} \) such that \( c'(s_i, k) \geq c(s_i, k) \), and

3. for each \( j \in N \setminus \{i\} \) and each \( k \in \{1, \ldots, n\} \), \( c'(s_j, k) = c(s_j, k) \),

then, for each \( i \in N \),

\[
\varphi_i(l, c, \pi) \leq \varphi_i(l', c', \pi').
\]

Next we consider a stronger version than congestion monotonicity. As with congestion monotonicity, this axiom pertains to situations in which the cost of serving two or more people in a segment increases. Let \( s_i \) be this segment. It says two things. First, each person that do not use \( s_{i+1} \) pay the same. Second, nobody should benefit. Again, this property says nothing when the cost of serving a single person increases.

**Strong congestion monotonicity:** For each pair \( (l, c) \) and \( (l', c') \in \mathbb{R}^N_+ \times \mathcal{C} \) such that \( l' = l \) and for each \( \pi \in EP(l, c) \) such that there is \( \pi' \in EP(l', c') \) for which \( \pi' = \pi \), if there is \( i \in N \) such that

1. \( c'(s_i, 1) = c(s_i, 1) \),

2. there is \( k \in \{2, \ldots, n\} \) such that \( c'(s_i, k) \geq c(s_i, k) \), and

3. for each \( j \in N \setminus \{i\} \) and each \( k \in \{1, \ldots, n\} \), \( c'(s_j, k) = c(s_j, k) \),

then,

1. for each \( j \leq i \), \( \varphi_j(l, c, \pi) = \varphi_j(l', c', \pi') \), and

2. for each \( j > i \), \( \varphi_j(l, c, \pi) \leq \varphi_j(l', c', \pi') \).

The next axiom pertains to situations in which the cost of serving two or more people in a segment increases. It says that the people who pay for the increase should divide this increase equally among themselves.

**Equality in increased congestion:** For each pair \( (l, c) \) and \( (l', c') \in \mathbb{R}^N_+ \times \mathcal{C} \) such that \( l' = l \) and for each \( \pi \in EP(l, c) \) such that there is \( \pi' \in EP(l', c') \) for which \( \pi' = \pi \), if there is \( i \in N \) such that

1. \( c'(s_i, 1) = c(s_i, 1) \),

2. there is \( k \in \{2, \ldots, n\} \) such that \( c'(s_i, k) \geq c(s_i, k) \),

3. for each \( j \in N \setminus \{i\} \) and each \( k \in \{1, \ldots, n\} \), \( c'(s_j, k) = c(s_j, k) \), and

4. there is \( i, j \in N \) such that \( \varphi_i(l, c, \pi) \neq \varphi_i(l', c', \pi') \) and \( \varphi_j(l, c, \pi) \neq \varphi_j(l', c', \pi') \),
then,

\[ \varphi_i(l', c', \pi') - \varphi_i(l, c, \pi) = \varphi_j(l', c', \pi') - \varphi_j(l, c, \pi). \]

The last axiom pertains to situations in which the incremental cost function weakly increases for each segment; and moreover, the structure of these additional incremental costs is "similar" to those in the original problem. This axiom says that each person should pay the same amount when the problem is reassessed to include both incremental costs, or when he pays the total for both problems.

Two problems \((l, c) \in \mathbb{R}_+^N \times \mathcal{C}\) and \((l', c') \in \mathbb{R}_+^N \times \mathcal{C}\) are similar if \(l = l'\) and there is \(\pi \in EP(l, c) \cap EP(l', c')\). Note that if \(\pi \in EP(l, c) \cap EP(l', c')\), then \(\pi \in EP(l, c + c')\).

**Conditional cost additivity:** For each pair \((l, c)\) and \((l', c')\) \(\in \mathbb{R}_+^N \times \mathcal{C}\) that is similar, each \(\pi \in EP(l, c) \cap EP(l', c')\), and each \(i \in N\),

\[ \varphi_i(l, c + c', \pi) = \varphi_i(l', c', \pi) + \varphi_i(l, c, \pi). \]

Without congestion, two problems are similar if the order of costs among people is the same between them. With congestion, the definition of similar problems also requires that they have at least one common efficient partition. To see the relevance of this hypothesis, consider the following example. Let \(N = \{1, 2, 3\}\), \(l = l', c(s_1, 1) = c(s_1, 2) = c(s_2, 1) = c(s_2, 2) = c(s_3, 1) = 10\), \(c(s_1, 3) = 50\), \(c'(s_1, 1) = c'(s_1, 2) = c'(s_2, 2) = c'(s_3, 1) = 10\), and \(c'(s_1, 3) = c(s_1, 3) = 15\). Then, \(EP(l, c) = \{\{1\}, \{1, 2\}\}\), \(EP(l', c') = \{\{1, 2, 3\}\}\), and \(EP(l, c + c') = \{\{1\}, \{2, 3\}\}\). Note that \(mc(l, c) = 40\), \(mc(l', c') = 35\), but \(mc(l, c + c') = 85 > 75 = mc(l, c) + mc(l', c')\).

## 4 Rules

In this section we define some rules for the irrigation problem with congestion. All of them follow the same idea as a rule proposed for the problem without congestion.\(^7\)

Before defining the rules, we need additional notation. Let \((l, c)\) and \(\pi \equiv \{\pi_k\}_{k=1}^N \in EP(l, c)\). For each \(k \in \{1, ..., K\}\), let \(n_k \equiv |\pi_k|\). For simplicity, for each \(k \in \{1, ..., K\}\), let us rename the people in \(\pi_k\) so they are ordered in terms of their needs, that is, \(l_1 \leq l_2 \leq ... \leq l_{n_k}\). For each \(k \in \{1, ..., K\}\) and each \(i, i - 1 \in \pi_k\), \(s_i^{\pi_k} \equiv (l_{i-1}, l_i) \in \mathbb{R}_+^2\) represents the part of the canal that has to be built for \(i\) after all the people with smaller needs have been accommodated.

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\(^6\)An example is provided in Thomson (2007).

\(^7\)This rule is introduced by Baker and Associates (1965) and Littlechild and Owen (1973).
The first rule divides equally the cost of each segment among its users in each canal. It is an extension of the **sequential equal contributions** rule for problems without congestion.

**Sequential equal contributions, SEC:** For each \((l, c) \in \mathbb{R}^N_+ \times \mathcal{C}\), each \(\pi = \{\pi_k\}_{k=1}^K \in EP(l, c)\), each \(k \in \{1, \ldots, K\}\), and each \(i \in \pi_k\), let
\[
SEC_i(l, c, \pi) = \sum_{j \leq i} \frac{c(s_j^\pi_k, n_k - j + 1)}{n_k - j + 1}.
\]

For the next family of rules, for each canal and each segment the cost of serving one person is divided equally among all its users; moreover the cost of serving two or more people is divided among the users whose need is not served when this segment is built. The **weighting functions** decide how to do this.

A **weight function**, \(\sigma : \mathbb{R}^2_+ \times \mathcal{C} \rightarrow \mathbb{R}^N\) such that \(\sum_{i \in N} \sigma(s_i, c) \neq 0\), assigns for each person and each segment a weight. That is, for each \(s_i \in \mathbb{R}^2_+\) and each \(c \in \mathcal{C}\), \(\sigma(s_i, c) \in \mathbb{R}^N\). We define another priority that is used only when all the people in a subset of \(N\) have zero weight. Let \(\sigma^L : \mathbb{R}^2_+ \times \mathcal{C} \rightarrow \mathbb{R}^N_+\) be such that for each \(s_j \in \mathbb{R}^2_+\), each \(c \in \mathcal{C}\), and each \(i \in N\), if \(\sigma_i(s_j, c) = 0\), then \(\sigma_i^L(s_j, c) > 0\). Let \(\Sigma\) be the set of all such weight functions.

**Sequential contributions rule associated with \(\sigma, \sigma^L \in \Sigma\), \(SC^{\sigma, \sigma^L}\):** For each \((l, c) \in \mathbb{R}^N_+ \times \mathcal{C}\), each \(\pi = \{\pi_k\}_{k=1}^K \in EP(l, c)\), each \(k \in \{1, \ldots, K\}\), and each \(i \in \pi_k\), let
\[
SC^\sigma_{i}(l, c, \pi) = \sum_{j \leq i} \frac{c(s_j^\pi_k, 1)}{n_k - j + 1} + \begin{cases} 
\sum_{j < i} \frac{\sigma_i(s_j^\pi_k, c)}{\sum_{m \in \pi_k} \sigma_m(s_j^\pi_k, c)} [c(s_j^\pi_k, n - j + 1) - c(s_j^\pi_k, 1)] & \text{if } \sum_{m \in \pi_k} \sigma_m(s_j^\pi_k, c) \neq 0, \\
\sum_{j < i} \frac{\sigma_i^L(s_j^\pi_k, c)}{\sum_{m \in \pi_k} \sigma_m^L(s_j^\pi_k, c)} [c(s_j^\pi_k, n - j + 1) - c(s_j^\pi_k, 1)] & \text{otherwise}.
\end{cases}
\]

Next, we define an important subfamily of the **sequential contributions** family. Each member of this family selects for each segment who should pay for the cost of serving two or more people. Conditional on being selected to pay, each person pays the same proportion of this cost. That is, for each \(c \in \mathcal{C}\), each \(i \in N\), and each \(j \leq i\), \(\sigma_i(s_j, c) \in \{0, 1\}\); and for each \(m < i\), if \(s_i = s_j\), then \(\sigma_i(s_m, c) = \sigma_j(s_m, c)\). In this case, for each \(s_j \in \mathbb{R}_+\), each \(c \in \mathcal{C}\), and each \(i \in N\), \(\sigma_i^L(s_j, c) = 1\). Let \(\Sigma'\) be the set of all such weight functions. Note that only one person might be paying for the congestion.
Sequential conditional contributions rule associated with $\sigma, \sigma^L \in \Sigma'$, $SCC_{\sigma,\sigma^L}$: For each $(l, c) \in \mathbb{R}_+^N \times \mathcal{C}$, each $\pi = \{\pi_k\}_{k=1}^K \in EP(l, c)$, each $k \in \{1, \ldots, K\}$, and each $i \in \pi_k$, let

$$SCC_{\sigma,\sigma^L}(l, c, \pi) = \sum_{j \leq i} \frac{c(s_{j}^{\pi_k}, 1)}{n_k - j + 1} + \begin{cases} \sum_{j<i} \frac{\sigma_i(s_{j}^{\pi_k}, c)}{\sigma_m(s_{j}^{\pi_k}, c)} [c(s_{j}^{\pi_k}, n-j+1) - c(s_{j}^{\pi_k}, 1)] & \text{if } \sum_{m \in \pi_k} \sigma_m(s_{j}^{\pi_k}, c) \neq 0, \\ \sum_{j<i} \frac{1}{\sigma_m(s_{j}^{\pi_k}, c)} [c(s_{j}^{\pi_k}, n-j+1) - c(s_{j}^{\pi_k}, 1)] & \text{otherwise.} \end{cases}$$

An important member of this family is rule defined next. This rule divides equally, for each segment, the cost of serving two or more people among those who use it to accommodate larger needs. Formally, for each $c, \in \mathcal{C}$, each $i \in N$,

$$\sigma_i(s_j, c) = \begin{cases} 1 & i < j \\ 0 & \text{otherwise.} \end{cases}$$

Sequential contributions with equality in congestion, $SCEC$; For each $(l, c) \in \mathbb{R}_+^N \times \mathcal{C}$, each $\pi \equiv \{\pi_k\}_{k=1}^K \in EP(l, c)$, each $k \in \{1, \ldots, K\}$, and each $i \in \pi_k$, let

$$SCEC_i(l, c, \pi) = \sum_{j \leq i} \frac{c(s_{j}^{\pi_k}, 1)}{n_k - j + 1} + \sum_{j<i} \frac{c(s_{j}^{\pi_k}, n-k-j) - c(s_{j}^{\pi_k}, 1)}{n_k - j}.$$ 

Example 1. Allocations recommended by different rules. Let $(l, c) \in \mathbb{R}_+^N \times \mathcal{C}$ be as in the next figure. Since $N \in EP(l, c)$, let $\pi = \{N\}$. Note that $C(l, c, N) = c(s_1, 4) + c(s_2, 3) + c(s_3, 2) + c(s_4, 1) = 50 + 10 + 10 + 10 = 80.$

In this case,
\[
SEC(l, c, N) = \left(\frac{50}{4}, \frac{50}{4} + \frac{10}{3}, \frac{50}{4} + \frac{10}{3}, \frac{10}{2}, \frac{50}{4} + \frac{10}{3} + \frac{10}{2} + 10\right) \\
= \left(\frac{150}{12}, \frac{190}{12}, \frac{250}{12}, \frac{270}{12}, \frac{290}{12} + \frac{40}{3}, \frac{290}{12} + \frac{40}{3} + \frac{10}{2} + 10\right) \\
= \left(\frac{75}{6}, \frac{95}{6}, \frac{125}{6}, \frac{185}{6}\right).
\]

\[
SCEC(l, c, N) = \left(\frac{10}{4}, \frac{10}{3} + \frac{5}{2} + \frac{10}{3}, \frac{10}{4} + \frac{5}{3} + \frac{10}{2} + \frac{10}{3} + \frac{10}{2} - 10\right), \\
= \left(\frac{30}{12}, \frac{210}{12}, \frac{300}{12}, \frac{320}{12}\right) = \left(\frac{15}{6}, \frac{155}{6}, \frac{210}{6}\right).
\]

Let \(\sigma(s_1, c) = (0, 4, 1, 1)\) and \(\sigma(s_2, c) = (0, 0, 1, 0)\). In this case,

\[
SC^\sigma \sigma^L (l, c, N) = \left(\frac{1}{4} \cdot 10, \frac{1}{4} \cdot 10 + \frac{4}{6} \cdot [50 - 10], \frac{1}{4} \cdot 10 + \frac{4}{6} \cdot [50 - 10] + \frac{1}{3} \cdot [50 - 10] + \frac{1}{3} \cdot \frac{1}{2} \cdot [10 - 5] + \frac{1}{2}, \\
= \left(\frac{30}{12}, \frac{370}{12}, \frac{300}{12}, \frac{320}{12}\right) = \left(\frac{15}{6}, \frac{155}{6}, \frac{210}{6}\right).
\]

Let \(\sigma'(s_1, c) = (0, 1, 1, 1)\) and \(\sigma'(s_2, c) = (0, 0, 1, 0)\). In this case,

\[
SCC^\sigma \sigma'(l, c, N) = \left(\frac{1}{4} \cdot 10, \frac{1}{4} \cdot 10 + \frac{1}{3} \cdot [50 - 10], \frac{1}{3} \cdot \frac{1}{2} \cdot [10 - 5] + \frac{1}{2}, \\
= \left(\frac{30}{12}, \frac{210}{12}, \frac{330}{12}, \frac{390}{12}\right) = \left(\frac{15}{6}, \frac{105}{6}, \frac{165}{6}, \frac{195}{6}\right).
\]

5 Results

In the absence of congestion, the sequential equal contributions rule is characterized by equal treatment of equals and independence of at-least-as-large costs.

---

*In this example, we do not need to specify \(\sigma^L\).*
(Moulin and Shenker 1992). When there is congestion, this is not the case anymore. We show in Theorem 1 that although the extension of the sequential equal contributions rule without congestion satisfies equal treatment of equals per canal and independence of at-least-as-large lengths, there is an extensive family of rules satisfying these axioms.

Similarly, without congestion, equal treatment of equals and conditional cost additivity characterize the sequential equal contributions rule (Dubey 1982).\textsuperscript{9} When there is congestion, this is not the case anymore. Moreover, in the absence of congestion, when equal treatment of equals is imposed, independence of at-least-as-large costs and conditional cost additivity imply the sequential equal contributions rule. This is not the case with congestion anymore. We show in Table 1 that some members of the sequential contributions family violate conditional cost additivity.

Our first result is that no-subsidy across canals and equal treatment of equals are incompatible. Since no-subsidy implies no-subsidy across canals, then no-subsidy and equal treatment of equals are incompatible. This is in contrast with the no-congestion case, since there, almost all rules satisfy these axioms.

**Proposition 1.** No rule satisfies no-subsidy across canals and equal treatments of equals.

**Proof.** Let \((l, c) \in \mathbb{R}^N_+ \times C\) be such that for each \(i \in N\), \(l_i = \bar{l}\), \(s_i = \bar{s} \equiv (0, \bar{l})\), \(c(\bar{s}, 2) = c(\bar{s}, 3) = \ldots = c(\bar{s}, n - 1) = (n - \frac{3}{2})c(\bar{s}, 1)\), and \(c(\bar{s}, n) = (n + 1)c(\bar{s}, 1)\). Let \(\varphi\) be a rule satisfying no-subsidy across canals and equal treatment of equals. For notational simplicity, for each \(N' \subset N\), we use \(C(N')\) instead of \(C(l, c, N')\).

Note that \(C(N) = (n + 1)c(l, 1) > n * c(l, 1) = C(\{1\}) + C(\{2\}) + \ldots + C(\{n\})\) and for each \(i \in N\), \(n * c(\bar{s}, 1) = C(\{1\}) + C(\{2\}) + \ldots + C(\{n\}) > (n - \frac{1}{2}) = C(\{\bar{s}\})\). Then, \(mc(l, c)\) is obtained when all people but one are served by the same canal. That is, \(mc(l, c) = (n - \frac{3}{2} + 1)c(\bar{s}, 1) = (n - \frac{1}{2})c(\bar{s}, 1)\). In addition, for each \(i \in N\), \(\pi_i = \{\{i\}, N \setminus \{i\}\} \in EP(l, c)\). Without loss of generality, let \(\pi_i = \{\{1\}, \{2, \ldots, n-1\}\}\) be the partition selected. By equal treatment of equals, for each \(i \in N\), \(\varphi_i(l, c, \pi_i) = \frac{mc(l, c)}{n} = (n - \frac{1}{2})\frac{c(\bar{s}, 1)}{n}\). Then, \(\varphi_i(l, c, \pi_1) < c(\bar{s}, 1) = C(\{1\})\). This violates no-subsidy across canals. \(\square\)

To state the next result, we need additional notation. Let \((l, c) \in \mathbb{R}^N_+ \times C\) and \(\pi \equiv \{\pi_k\}_{k=1}^K \in \Pi^N\). For each \(i \in N\), let \(n_k^i\) be the number of people in \(\pi_k\) who use segment \(s_i\). That is, \(n_k^i \equiv |\{j \in \pi_k \mid l_j \geq l_i\}|\). Then, \(c(s_i, n_k^i)\) is the incremental cost of constructing \(s_i\) when \(n_k^i\) people are using this segment.

\textsuperscript{9}When the definition of a rule includes that no person should pay more than his cost, these two axioms are enough to the characterization. When this assumption is not considered, the characterization needs an additional axiom. This axiom says that if a person has no need, then he pays nothing.
For each $i \in N$, let the savings from building a unique canal for segment $s_i$ be $S_i(c) \equiv c(s_i, n - i + 1) - \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_i, n_k) \right\}$. 

Lemma 1 gives sufficient conditions on a problem to guarantee that it is efficient to construct a unique canal.

**Lemma 1.** For each $(l, c) \in \mathbb{R}_+^N \times C$, if $\sum_{i=1}^{n-1} S_i(c) \leq 0$, then $N \in EP(l, c)$.

**Proof.** We prove the contrapositive. Let $(l, c) \in \mathbb{R}_+^N \times C$ and suppose $N \notin EP(l, c)$. Then, there is $\pi = \{\pi_k\}_{k=1}^K \in \Pi^N$ such that $C(l, c, N) > \sum_{k=1}^{K(\pi)} C(l, c, \pi_k)$. Then,

\[
c(s_1, n) + \ldots + c(s_i, n - i + 1) + \ldots + c(s_n, 1) > c(s_1, n_1^1) + c(s_1, n_2^1) + \ldots + c(s_1, n_K^1) + \ldots + c(s_i, n_i^1) + \ldots + c(s_i, n_K^i) + \ldots + c(s_n, n_{n_K}^n).
\]

Then,

\[
c(s_1, n) - (c(s_1, n_1^1) + c(s_1, n_2^1) + \ldots + c(s_1, n_K^1)) + \ldots + c(s_i, n - i + 1) - (c(s_i, n_1^i) + \ldots + c(s_i, n_K^i)) + \ldots + c(s_n, 1) - (c(s_n, n_1^n) + \ldots + c(s_n, n_{n_K}^n)) > 0.
\]

Since for each $i \in N$, $\sum_{k=1}^{K(\pi)} n_k^i = n - i + 1$, then $c(s_n, 1) - (c(s_n, n_1^n) + \ldots + c(s_n, n_{n_K}^n)) = c(s_n, 1) - c(s_n, 1) = 0$. In addition, for each $i \in N$, $\sum_{k=1}^{K(\pi)} c(s_i, n_k^i) \geq \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_i, n_k^i) \right\}$. Then,

\[
c(s_1, n) - \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_i, n_k^i) \right\} + \ldots + c(s_i, n - i + 1) - \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_i, n_k^i) \right\} + \ldots + c(s_{n-1}, 2) - \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_{n-1}, n_k^i) \right\} > 0.
\]

Then, $\sum_{i=1}^{n-1} S_i(c) > 0$. \hfill \Box

Before proving the main theorem we define an operator on cost functions that lowers the costs of serving more than one person for a segment. For each $(l, c) \in \mathbb{R}_+^N \times C$, each $i \in N$, and each $p \in \{1, \ldots, n - i + 1\}$, let the lowered incremental cost at $s_i = (l_{i-1}, l_i)$ for $p$ people, $w_p^{s_i}(c)$, be defined as follows:
Lemma 2. For each $(l, c) \in \mathbb{R}_+^N \times C$, if $N \in EP(l, c)$, then for each $i \in N$, each $p \in \{1, ..., n\}$, and each $(l, w_p^i(c))$, $N \in EP(l, w_p^i(c))$.

Proof. Let $(l, c) \in \mathbb{R}_+^N \times C$ be such that $N \in EP(l, c)$. Since $N \in EP(l, c)$, for notational simplicity for each $i \in N$, we denote $s_i^N$ by $s_i$. Let $i \in N$ and $p \in \{1, ..., n\}$. By Lemma 1 $\sum_{i=1}^{n-1} S_i(c) \leq 0$. Note that for each $j \neq i$, $S_j(c) = S_j(w_p^i(c))$. Then, it is enough to show that $S_i(w_p^i(c)) \leq 0$. By definition,

$$S_i(w_p^i(c)) = w_p^i(c)(s_i, n - i + 1) - \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} w_p^i(c)(s_i, n_k^i) \right\}.$$  

Note that $\min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} w_p^i(c)(s_i, n_k^i) \right\} \leq \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} c(s_i, n_k^i) \right\}.$

Let $\pi' \in \arg \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} c(s_i, n_k^i) \right\}$ and $\pi' = \{\pi_k^{(\pi')}\}_{k=1}^{K}$ for each $k \in \{1, ..., K\}$, let $n_k^i \equiv |\pi_k^{(\pi')}|$. Let $\pi_w \in \arg \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} w_p^i(c)(s_i, n_k^i) \right\}$ and $\pi_w = \{\pi_k^{(\pi_w')}\}_{k=1}^{K}$ for each $k \in \{1, ..., K\}$, let $n_k^{w,i} \equiv |\pi_k^{(\pi_w')}|$.  

There are three cases:

CASE 1: There is no $\pi' \in \arg \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} c(s_i, n_k^i) \right\}$ such that $\pi' \in \arg \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} w_p^i(c)(s_i, n_k^i) \right\}$. Let $\pi' \in \arg \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} c(s_i, n_k^i) \right\}$ and $\pi' \in \arg \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} w_p^i(c)(s_i, n_k^i) \right\}$.  

Note that there is $k \in \{1, ..., K(\pi''')\}$ such that $n_k^{w,i} > p$ or there is $k \in \{1, ..., K(\pi'')\}$ such that $n_k > p$. Otherwise, $\sum_{k=1}^{K} c(s_i, n_k^i) = \sum_{k=1}^{K} w_p^i(c)(s_i, n_k^{w,i})$ and $\pi' \in \arg \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} c(s_i, n_k^i) \right\}$.  

If each $k \in \{1, ..., K(\pi''')\}$, $n_k^{w,i} \leq p$ and there is $k \in \{1, ..., K(\pi'')\}$ such that $n_k > p$, then $\sum_{k=1}^{K} w_p^i(c)(s_i, n_k^{w,i}) = \sum_{k=1}^{K} c(s_i, n_k^{w,i}) < \sum_{k=1}^{K} c(s_i, n_k^i)$, which is a contradiction to $\pi' \in \arg \min_{\pi \in (\prod_{k=1}^{K}\{s_i, n_k^i\})} \left\{ \sum_{k=1}^{K} w_p^i(c)(s_i, n_k^i) \right\}$. Then, there is $k \in \{1, ..., K(\pi''')\}$ such that $n_k^{w,i} > p$. Let $m \equiv \{\pi_k | n_k^{w,i} > p\}$. Note that $m \geq 1$. We introduce an indicator function to distinguish the incremental cost of components whose cardinality is smaller than $p$. That is, $I(n_k^{w,i} \leq p) \equiv \sum_{k=1}^{K} w_p^i(c)(s_i, n_k^{w,i}) = \sum_{k=1}^{K} c(s_i, n_k^i)$.
\[
\begin{cases}
1 & \text{if } n_k^{w,i} \leq p \\
0 & \text{otherwise.}
\end{cases}
\]

Then,

\[
K(\pi^w) \sum_{i=1}^{K(\pi^w)} w_p^{s_i}(c)(s_i, n_k^{w,i}) = m \ast w_p^{s_i}(c)(s_i, p) + \sum_{k=1}^{K(\pi^w)} I(n_k^{w,i} \leq p)w_p^{s_i}(c)(s_i, n_k^{w,i}).
\]

Then, the savings from building a unique canal in the new problem is given by

\[
S_i(w_p^{s_i}(c)) = w_p^{s_i}(c)(s_i, n - i + 1) - m \ast w_p^{s_i}(c)(s_i, p) - \sum_{k=1}^{K(\pi^w)} I(n_k^{w,i} \leq p)w_p^{s_i}(c)(s_i, n_k^{w,i}) \]

\[
= (1 - m)c(s_i, p) - \sum_{k=1}^{K(\pi^w)} I(n_k^{w,i} \leq p)w_p^{s_i}(c)(s_i, n_k^{w,i}) \leq 0.
\]

**CASE 2:** For each \(\pi' \in \arg \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_i, n_k^i) \right\} \), there is \(\pi^w \in \arg \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} w_p^{s_i}(c)(s_i, n_k^{w,i}) \right\} \) such that \(\pi = \pi^w\) and for each \(k \in \{1, \ldots, K\}, n_k^i \leq p\).

In this case, \(\sum_{k=1}^{K(\pi^w)} c(s_i, n_k^i) = \sum_{k=1}^{K(\pi^w)} w_p^{s_i}(c)(s_i, n_k^{w,i})\). Then,

\[
S_i(w_p^{s_i}(c)) = w_p^{s_i}(c)(s_i, n - i + 1) - \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_i, n_k^i) \right\}
\]

\[
= c(s_i, r) - \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_i, n_k^i) \right\}
\]

\[
\leq c(s_i, n - i + 1) - \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_i, n_k^i) \right\} = S_i(c) \leq 0.
\]

**CASE 3:** For each \(\pi' \in \arg \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} c(s_i, n_k^i) \right\} \), there is \(\pi^w \in \arg \min_{\pi \in (\Pi^N \setminus \{N\})} \left\{ \sum_{k=1}^{K(\pi)} w_p^{s_i}(c)(s_i, n_k^{w,i}) \right\} \) such that \(\pi = \pi^w\) and there is \(k \in \{1, \ldots, K(\pi)\}\) such that \(n_k^i > p\).

Let \(m \equiv |\{\pi_k^i \mid n_k^i > p\}|\). Note that \(m \geq 1\). We introduce an indicator function to distinguish the incremental cost of components whose cardinality is smaller than \(p\). That is, \(I(n_k^i \leq p) \equiv \begin{cases} 1 & \text{if } n_k^i \leq p \\
0 & \text{otherwise.} \end{cases} \) Then,

\[
\sum_{i=1}^{K(\pi^w)} w_p^{s_i}(c)(s_i, n_k^{w,i}) = m \ast w_p^{s_i}(c)(s_i, p) + \sum_{k=1}^{K(\pi^w)} I(n_k^i \leq p)w_p^{s_i}(c)(s_i, n_k^{w,i}).
\]
Then, the savings from building a unique canal in the new problem is given by

\[ S_i(w^s_p(c)) = w^s_p(c)(s_i, n - i + 1) - m \ast w^s_p(c)(s_i, r) - \sum_{k=1}^{K(p^w)} I(n^i_k \leq p)w^s_p(c)(s_i, n^w,i) \]

\[ = (1 - m) \ast c(s_i, p) - \sum_{k=1}^{K(p^w)} I(n^i_k \leq p)w^s_p(c)(s_i, n^w,i) \leq 0. \]

Before proving the main theorem we define an operator on the cost function that reduces the incremental costs of serving more than one person for some segments. This operator is similar to the one introduced before Lemma 2. For each \((l, c) \in \mathbb{R}^+_N \times \mathcal{C}\), each \(i \in N\), and each \(p \in \{1, ..., n - i + 1\}\), let the **reduced incremental cost until** \(s_i = (l_{i-1}, l_1)\) for \(p\) people, \(r^s_p(c)\), be defined as follows:

1. For each \(k \in \{1, ..., p\}\), \(r^s_p(c)(s_i, k) \equiv c(s_i, k)\).
2. For each \(k \in \{p, ..., n - i + 1\}\), \(r^s_p(c)(s_i, k) \equiv c(s_i, p)\).
3. For each \(j < i\) and each \(k \in \{1, ..., n\}\), \(r^s_p(c)(s_j, k) \equiv c(s_j, 1)\).
4. For each \(j > i\) and each \(k \in \{1, ..., n\}\), \(r^s_p(c)(s_j, k) \equiv c(s_j, k)\).

**Theorem 1.** A rule satisfies no-subsidy across canals, equal treatment of equals per canal, independence of at-least-as-large segments, and strong congestion monotonicity, if and only if there is \(\sigma, \sigma^L \in \Sigma\) such that the rule is the sequential contributions rule associated with \(\sigma, \sigma^L\).

**Proof.** It is easy to show that each member of the sequential contributions family satisfies the axioms in the theorem. Conversely, let \(\varphi\) be a rule satisfying the axioms.

Let \((l, c) \in \mathbb{R}^+_N \times \mathcal{C}\) and \(\pi = \{\pi_k\}_{k=1}^K \in EP(l, c)\). Since \(\varphi\) satisfies no-subsidy across canals, it is enough to show that there is \(\sigma \in \Sigma\) such that for each \(k \in \{1, ..., K\}\) and each \(i \in \pi_k\), \(\varphi_i(l, c, \pi) = SC^\sigma_{\pi^L}(l, c, \pi)\). Without loss of generality, we consider the case in which \(N \in EP(l, c)\) and \(\pi = \{N\}\). When \(\pi = \{N\}\), we write \(\varphi(l, c)\) instead of \(\varphi(l, c, \pi)\), and for each \(i \in N\), we write \(s_i\) instead of \(s^N_i\).

For each \(\{j, i\} \subset N\),
• If there is \( k \in \{2, \ldots, n - i + 1\} \) such that \( c(s_i, k) > c(s_i, k - 1) \), then
\[
\sigma_j(s_i, c) = \begin{cases} 
\sum_{m \in N} \sigma_m(s_i, c) \frac{\varphi_j(l, x_{m+1}^i(c)) - \varphi_j(l, x_{m}^i(c))}{c(s_i, k) - c(s_i, k - 1)} & \text{if } k = n - i + 1, \\
\sum_{m \in N} \sigma_m(s_i, c) \frac{\varphi_j(l, x_{m+1}^i(c)) - \varphi_j(l, x_{m}^i(c))}{c(s_i, k) - c(s_i, k - 1)} & \text{otherwise.}
\end{cases}
\]

• If for each \( k \in \{2, \ldots, n - i + 1\} \), \( c(s_i, k) = c(s_i, k - 1) \), then \( \sigma_j(s_i, c) = 0 \).

Let \( x \equiv SC^\sigma\sigma^L(l, c) \).

**CASE 1:** Without congestion costs (Figure 3.)

To prove this case we solve the sequence of problems \( \{(l^i, c^i)\}_{i=1}^n \) defined as follows. For each \( i \in \{1, \ldots, n\} \),

1. \( l^i_j = \left\{ \begin{array}{ll}
  l_j & j \leq i, \\
  l_i & j > i,
\end{array} \right. \) and

2. for each \( j \in N, c^i(s_j^i, k) = c(s_j, 1) \).

Note that \( l^n = l \).

**Step 1:** Solving \( (l^1, c^1) \in \mathbb{R}_+^N \times \mathcal{C} \).

Note that \( N \in EP(l, c^1) \). Let \( \pi = \{N\} \) and let \( x^1 \equiv SC^\sigma\sigma^L(s^1, c^1) \). By equal treatment of equals per canal, for each \( i \in N \), \( \varphi_1(l^1, c^1) = \frac{c^1(s^1, n)}{n} = \frac{c(s^1, 1)}{n} \).

Note that for each \( i \in N \), \( x^1_i = \frac{c^1(s^1, n)}{n} \). Furthermore, \( x^1_1 = x_1 = \frac{c(s^1, 1)}{n} \). Then, \( \varphi_1(l^1, c^1) = x_1 \).

**Step 2:** Solving \( (l^2, c^2) \in \mathbb{R}_+^N \times \mathcal{C} \).

Note that \( N \in EP(l, c^2) \). Let \( \pi = \{N\} \) and let \( x^2 \equiv SC^\sigma\sigma^L(p^2, c^2) \). By independence of at-least-as-large segments, \( \varphi_1(l^2, c^2) = \varphi_1(l^1, c^1) = \frac{c(s^1, 1)}{n} = x_1 \).

By equal treatment of equals per canal, for each \( i \in N \setminus \{1\} \),
\[
\varphi_1(l^2, c^2) = \frac{c^2(s^1, n)}{n} + \frac{c^2(s^2, n - 1)}{n - 1} = \frac{c(s^1, 1)}{n} + \frac{c(s^2, 1)}{n - 1}.
\]

Note that \( x^2_i = x_1 = \frac{c(s^1, 1)}{n} \) and for each \( i \in N \setminus \{1\} \), \( x^2_i = \frac{c(s^1, 1)}{n} + \frac{c(s^2, 1)}{n - 1} \).

:**Step n:** Solving \( (l^n, c^n) \in \mathbb{R}_+^N \times \mathcal{C} \).

We follow the same argument as in Step 2. Note that \( l^n = l \). Then, for each \( i \in N \),
\[
\varphi_1(l, c^n) = \frac{c(s, 1)}{n} = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1}.
\]
Figure 3: Without congestion costs. (a) Original problem $(l, c) \in \mathbb{R}^N_+ \times \mathcal{C}$. (b) Problem $(l^1, c^1)$: People need the same length $l_1$ and the cost of connecting one or more people is the same. (c) Problem $(l^2, c^2)$: Person 1 needs $l_1$, but all others need the same length $l_2$ and the cost of connecting one or more people is the same. (d) Problem $(l, c^n)$: Each person needs its original length and the cost of connecting one or more people is the same.

**CASE 2:** With congestion costs (Figure 4.)

**Step 1:** Solving $(l, r_1^{n-2}(c)) \in \mathbb{R}^N_+ \times \mathcal{C}$.

By applying Lemma 2 successively, $N \in EP(l, r_1^{n-2}(c))$. Let $\pi = \{N\}$ and let $x_{i}^{n-2} = SC^{\pi}(l, r_1^{n-2}(c))$. By strong congestion monotonicity, for each $i \in N \setminus \{n\}$, $\varphi_i(l, r_1^{n-2}(c)) = \varphi_i(l, c^n) = \sum_{j \leq i} \frac{c(s_j, 1)}{n-j+1}$. Then, $\varphi_n(l, r_1^{n-2}(c)) = \sum_{j \leq n} \frac{c(s_j, 1)}{n-j+1} + [c(s_{n-1}, 2) - c(s_{n-1}, 1)]$. Note that by definition of $SC^{\pi}$, for each $i \in N \setminus \{n\}$, $x_{i}^{n-2} = x_i^n = \sum_{j \leq i} \frac{c(s_j, 1)}{n-j+1}$. In addition, since for each $i \in N \setminus \{n\}$,
\[ \sigma_i(s_{n-1}, 2) = 0, \text{ then} \]
\[ x_{i}^{n-2} = \sum_{j \leq n} \frac{c(s_j, 1)}{n - j + 1} + [c(s_{n-1}, 2) - c(s_{n-1}, 1)]. \]

**Step 2:** Solving \((l, r_2^{s_{n-2}}(c)) \in \mathbb{R}_+^N \times \mathcal{C}.

By applying Lemma 2 successively, \(N \in EP(l, r_2^{s_{n-2}}(c))\). Let \(\pi = \{N\}\) and let \(x^{n-2} \equiv SC^{\sigma, \sigma^2}(l, r_2^{s_{n-2}}(c))\). By strong congestion monotonicity,

1. For each \(i \in N \setminus \{n - 1, n\}\), \(\varphi_i(l, r_2^{s_{n-2}}(c)) = \varphi_i(l, r_1^{s_{n-2}}(c)) = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1}. \)
2. For each \(i \in \{n - 1, n\}\), \(\varphi_i(l, r_2^{s_{n-2}}(c)) \geq \varphi_i(l, r_1^{s_{n-2}}(c)). \)

Then, there is \(\alpha^{n-2} \in \Delta(\{n - 1, n\})\) such that,
\[
\varphi_i(l, r_2^{s_{n-2}}(c)) = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1} + \begin{cases} 
\alpha^{n-2}_{i-1} [c(s_{n-2}, 2) - c(s_{n-2}, 1)] & i = n - 1 \\
\alpha^{n-2}_i [c(s_{n-1}, 2) - c(s_{n-1}, 1)] + \alpha^{n-2}_n [c(s_{n-2}, 2) - c(s_{n-2}, 1)] & i = n. 
\end{cases}
\]

By definition of \(SC^{\sigma, \sigma^L}\), for each \(i \in \{n, n - 1\}\), \(\frac{\sigma_i(s_{n-2}, c)}{\sigma_{n-1}(s_{n-2}, c) + \sigma_n(s_{n-2}, c)} = \alpha^{n-2}_i\)
and
\[
x^{n-2}_i = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1} + \begin{cases} 
0 & i \in N \setminus \{n - 1, n\} \\
\frac{\sigma_{n-1}(s_{n-2}, c)}{\sigma_{n-2}(c) + \sigma_n(s_{n-2}, c)} [c(s_{n-2}, 2) - c(s_{n-2}, 1)] & i = n - 1 \\
\frac{c(s_{n-1}, 2) - c(s_{n-1}, 1)}{\sigma_{n-1}(s_{n-2}, c) + \sigma_n(s_{n-2}, c)} [c(s_{n-2}, 2) - c(s_{n-2}, 1)] & i = n. 
\end{cases}
\]

**Step 3:** Solving \((l, r_1^{s_{n-3}}(c)) \in \mathbb{R}_+^N \times \mathcal{C}.

By applying Lemma 2 successively, \(N \in EP(l, r_1^{s_{n-3}}(c))\). Let \(\pi = \{N\}\) and let \(x^{n-3} \equiv SC^{\sigma, \sigma^L}(l, r_1^{s_{n-3}}(c))\). By strong congestion monotonicity,

1. For each \(i \in N \setminus \{n - 1, n\}\), \(\varphi_i(l, r_1^{s_{n-3}}(c)) = \varphi_i(l, r_2^{s_{n-2}}(c)) = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1}. \)
2. For each \(i \in \{n - 1, n\}\), \(\varphi_i(l, r_1^{s_{n-3}}(c)) \geq \varphi_i(l, r_2^{s_{n-2}}(c)). \)

Then, there is \(\tilde{\alpha}^{n-2} \in \Delta(\{n - 1, n\})\) such that
We claim that $\alpha_n^{-2} = \bar{\alpha}_n^{-2}$. Suppose not. Then without loss of generality, assume $\alpha_n^{-2} < \bar{\alpha}_n^{-2}$. Let $(l, c') \in \mathbb{R}_+^N \times C$ be defined by,

1. For each $i \in N \setminus \{n-2, n-1, n\}$ and each $k \in \{1, ..., n\}$, $c'(s, k) = r_{s-3}^n(c)(s, k)$.
2. For each $i \in \{n-2, n-1, n\}$ and each $k \in \{1, 3, 4, ..., n\}$, $c'(s, k) = r_{s-3}^n(c)(l, k)$.
3. $c'(s_2, 2) = r_{s-3}^n(c)(l_2, 3)$.

In this new problem only the cost of serving more than one person increases for $s_{n-2}$. By Lemma \[2], $N \in EP(l, r_{s-3}^n(c))$. Let $\pi = \{N\}$. Using the same argument as for the problem $(l, r_{s-3}^n(c))$, we know that

$$\varphi_{n-1}(l, c') = \sum_{j \leq i} \frac{c(s_j, 1)}{n-j+1} + \alpha_{n-1}^{-2}[c(s_{n-2}, 2) - c(s_{n-2}, 1)] + \alpha_{n-1}^{-2}[c(s_{n-2}, 3) - c(s_{n-2}, 2)]$$

$$< \varphi_{n-1}(l, r_{s-3}^n(c)).$$

This violates strong cost monotonicity. If $\alpha_{n-1}^{-2} > \bar{\alpha}_n^{-2}$, we can reach the same conclusion for person $n$. Then, $\alpha_n^{-2} = \bar{\alpha}_n^{-2}$. Then,

$$\varphi_i(l, r_{s-3}^n(c)) = \sum_{j \leq i} \frac{c(s_j, 1)}{n-j+1} + \left\{ \begin{array}{ll}
\alpha_{n-1}^{-2}[c(s_{n-2}, 3) - c(s_{n-2}, 1)] & i = n-1 \\
[\alpha_{n-1}^{-2}[c(s_{n-1}, 2) - c(s_{n-1}, 1)] + \alpha_{n-2}^{-2}[c(s_{n-2}, 3) - c(s_{n-2}, 1)] & i = n. \end{array} \right.$$
Step 4: Solving \((l, r_2^{s_{n-3}}(c)) \in \mathbb{R}_+^N \times C\).

By applying Lemma 2 successively, \(N \in \text{EP}(l, r_2^{s_{n-3}}(c))\). Let \(\pi = \{N\}\) and let \(x_i^{s_{n-3}} \equiv SC^{\pi,\pi^2}(l, r_2^{s_{n-3}}(c))\). By strong congestion monotonicity,

1. For each \(i \in N \setminus \{n-2, n-1, n\}\), \(\varphi_i(l, r_2^{s_{n-3}}(c)) = \varphi_i(l, r_1^{s_{n-3}}(c)) = \sum_{j \leq i} c(l_j)\).

2. For each \(i \in \{n-2, n-1, n\}\), \(\varphi_i(l, r_2^{s_{n-3}}(c)) \geq \varphi_i(l, r_1^{s_{n-3}}(c))\).

Then, there is \(\alpha^{-3} \in \Delta(\{n-2, n-1, n\})\) such that

\[
\varphi_i(l, r_2^{s_{n-3}}(c)) = \sum_{j \leq i} c(s_j, 1) n - j + 1 + \begin{cases} 
\alpha_{n-2}^{i-3}[c(s_{n-3}, 2) - c(s_{n-3}, 1)] & i = n - 2 \\
\alpha_{n-1}^{i-3}[c(s_{n-3}, 3) - c(s_{n-3}, 1)] + \alpha_{n-3}^{i-3}[c(s_{n-3}, 2) - c(s_{n-3}, 1)] & i = n - 1 \\
[c(s_{n-1}, 2) - c(s_{n-1}, 1)] + \alpha_{n}^{i-2}[c(s_{n-2}, 3) - c(s_{n-2}, 1)] + \alpha_{n}^{i-3}[c(s_{n-3}, 2) - c(s_{n-3}, 1)] & i = n.
\end{cases}
\]

Note that by definition of \(SC^{\pi,\pi^L}\), for each \(i \in \{n-2, n-1, n\}\), \(\sum_{j=n-2}^{n} \sigma_i(s_j) = \alpha_i^{n-3} \) and

\[
x_i^{s_{n-3}} = \sum_{j \leq i} c(s_j, 1) n - j + 1 + \begin{cases} 
0 & i \in N \setminus \{n-2, n-1, n\} \\
\sum_{j=n-2}^{n} \sigma_j(s_{n-3}, c) [c(s_{n-3}, 2) - c(s_{n-3}, 1)] & i = n - 2 \\
\sum_{j=n-1}^{n} \sigma_j(s_{n-3}, c) [c(s_{n-3}, 3) - c(s_{n-3}, 1)] + \sum_{j=n-2}^{n} \sigma_j(s_{n-3}, c) [c(s_{n-3}, 2) - c(s_{n-3}, 1)] & i = n - 1 \\
[c(s_{n-1}, 2) - c(s_{n-1}, 1)] + \sum_{j=n-1}^{n} \sigma_j(s_{n-2}, c) [c(s_{n-2}, 3) - c(s_{n-2}, 1)] & i = n.
\end{cases}
\]

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Step $\frac{n(n-1)}{2}$: Solving $(l, c) \in \mathbb{R}_+^N \times C$.

We follow the same argument of the previous step. Recall that $N \in EP(l, c)$. Let $\pi = N$. Then, for each $i \in N$,

$$\varphi_i(l, c) = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1} + \sum_{j < i} \sum_{m \in N} \sigma_m(s_j, c) [c(s_j, n - j + 1) - c(s_j, 1)] = x_i.$$ 

The independence of the axioms of Theorem 1 is discussed with the independence of the axioms of Theorem 2, after the proof of the latter. Theorem 2 characterizes a subfamily of the sequential contributions family, the sequential conditional contributions family when equality in increased congestion is imposed.

**Theorem 2.** A rule satisfies no-subsidy across canals, equal treatment of equals per canal, independence of at-least-as-large segments, strong congestion monotonicity, and equality in increased congestion if and only if there is $\sigma, \sigma^L \in \Sigma'$ such that the rule is the sequential conditional contributions rule associated with $\sigma, \sigma^L$.

**Proof.** It is easy to show that each member of the sequential contributions family satisfies the axioms in the theorem. Conversely, let $\varphi$ be a rule satisfying the axioms. Let $(l, c) \in \mathbb{R}_+^N \times C$ and $\pi \equiv \{\pi_k\}_{k=1}^K \in EP(l, c)$. Since $\varphi$ satisfies no-subsidy across canals, it is enough to show that for each $\pi_k \in \pi$ and each $i \in \pi_k$, there is $\sigma \in \Sigma'$ such that $\varphi_i(l, c) = SCC_i^{\sigma, \sigma^L}(s, c, \pi)$. Then, without loss of generality, we consider the case in which $N \in EP(l, c)$ and $\pi = N$. When $\pi = N$, we write $\varphi(l, c)$ instead of $\varphi(l, c, \pi)$.

For each $\{j, i\} \in N$,

- If there is $k \in \{2, \ldots, n - i + 1\}$ such that $c(s_i, k) > c(s_i, k - 1)$, then
  $$\sigma_j(s_i, c) = \begin{cases} 
  \sum_{m \in N} \sigma_m(s_i, c) \frac{\varphi_j(l, c, s_i, k - 1)}{c(s_i, k) - c(s_i, k - 1)} & \text{if } k = n - i + 1, \\
  \sum_{m \in N} \sigma_m(s_i, c) \frac{\varphi_j(l, c, s_i, k - 1)}{c(s_i, k) - c(s_i, k - 1)} & \text{otherwise.}
  \end{cases}$$

- If for each $k \in \{2, \ldots, n - i + 1\}$, $c(s_i, k) = c(l_i, k - 1)$, then $\sigma_j(s_i, c) = 0$.

Let $x \equiv SCC^{\sigma, \sigma^L}(l, c)$.

**CASE 1:** Without congestion costs (Figure 3.)
The argument is the same as in Case 1 of the proof of Theorem 1. Then, for each \( i \in N \), \( \varphi_i(l, c^n) = SCC_i^{\sigma_i}(l, c^n) = \sum_{j \leq n} c(j, 1)\frac{n-j+1}{n} \).

**CASE 2:** With congestion costs (Figure 4.)

We follow the same argument as in Case 2 of the proof of Theorem 1.

**Step 1:** Solving \((l, r_1^{n-2}(c)) \in \mathbb{R}_+^N \times C\).

By Step 2 of Case 2 of the proof of Theorem 1, \( \varphi(l, r_1^{n-2}(c)) = x_1^{n-2} \).

**Step 2:** Solving \((l, r_2^{n-2}(c)) \in \mathbb{R}_+^N \times C\).

By Step 2 of Case 2 of the proof of Theorem 1, for each \( i \in N \setminus \{n-1, n\} \), \( \varphi_i(l, r_2^{n-2}(c)) = \sum_{j \leq n} c(j, 1)\frac{n-j+1}{n} \), and there is \( \alpha_i^{n-2} \in \Delta(\{n-1, n\}) \) such that

\[
\varphi_i(l, r_2^{n-2}(c)) = \sum_{j \leq i} c(j, 1)\frac{n-j+1}{n} + \begin{cases} \alpha_i^{n-2}[c(s_{n-2}, 2) - c(s_{n-2}, 1)] & i = n - 1 \\ \lfloor c(s_{n-1}, 2) - c(s_{n-1}, 1) \rfloor + \alpha_i^{n-2}[c(s_{n-2}, 2) - c(s_{n-2}, 1)] & i = n. \end{cases}
\]

There are two cases:

**Case 1:** \( \varphi_{n-1}(l, r_2^{n-2}(c)) > 0 \) and \( \varphi_n(l, r_2^{n-2}(c)) > 0 \).

By equality in increased congestion, \( \varphi_{n-1}(l, r_2^{n-2}(c)) - \varphi_{n-1}(l, r_1^{n-2}(c)) = \varphi_n(l, r_2^{n-2}(c)) - \varphi_n(l, r_1^{n-2}(c)) \). Then,

\[
\varphi_{n-1}(l, r_2^{n-2}(c)) = \sum_{j \leq i} c(j, 1)\frac{n-j+1}{n} + \frac{1}{2} \lfloor c(s_{n-2}, 2) - c(s_{n-2}, 1) \rfloor + \begin{cases} 0 & i = n - 1 \\ \lfloor c(s_{n-1}, 2) - c(s_{n-1}, 1) \rfloor & i = n. \end{cases}
\]

Then, \( \alpha_{n-1}^{n-2} = \frac{\sigma_{n-1}(s_{n-2}, c)}{\sigma_{n-1}(s_{n-2}, c) + \sigma_n(s_{n-2}, c)} = \frac{\sigma_n(s_{n-2}, c)}{\sigma_{n-1}(s_{n-2}, c) + \sigma_n(s_{n-2}, c)} = \alpha_n^{n-2} \).

**Case 2:** There is \( i \in \{n-1, n\} \) such that \( \varphi_i(l, r_2^{n-2}(c)) = 0 \).

In this case, \( \alpha_i^{n-2} = \frac{\sigma_i(s_{n-2}, c)}{\sigma_{n-1}(s_{n-2}, c) + \sigma_n(s_{n-2}, c)} = 0 \) and for \( j = \{n-1, n\} \setminus \{i\} \),

\[
\alpha_j^{n-2} = \frac{\sigma_j(s_{n-2}, c)}{\sigma_{n-1}(s_{n-2}, c) + \sigma_n(s_{n-2}, c)} = 1.
\]

For both cases, \( \varphi(l, r_2^{n-2}(c)) = x_2^{n-2} \).

**Step 3:** Solving \((l, r_1^{n-3}(c)) \in \mathbb{R}_+^N \times C\).

By Step 3 of Case 2 of the proof of Theorem 1, \( \varphi(l, r_1^{n-3}(c)) = x_1^{n-3} \).

**Step 4:** Solving \((l, r_2^{n-3}(c)) \in \mathbb{R}_+^N \times C\).

By Step 4 of Case 2 of the proof of Theorem 1, for each \( i \in N \setminus \{n-2, n-1, n\} \), \( \varphi_i(l, r_2^{n-3}(c)) = \sum_{j \leq n} c(j, 1)\frac{n-j+1}{n} \), and there is \( \alpha_i^{n-3} \in \Delta(\{n-2, n-1, n\}) \) such that

\[
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\]
Case 2: There is \( \phi \) such that 
\[
\phi \in (l, r_{l,n} - 1) \quad \text{and} \quad \phi \in (l, r_{l,n} - 1) \quad \text{for each} \quad n \geq 2.
\]

There are three cases:

**Case 1:** \( \phi_{n-2}(l, r_{l,n} - 3) > 0 \), \( \phi_{n-1}(l, r_{l,n} - 3) > 0 \), and \( \phi_{n}(l, r_{l,n} - 3) > 0 \).

By equality in increased congestion, \( \phi_{n-2}(l, r_{l,n} - 3) - \phi_{n-2}(l, r_{l,n} - 3) = \phi_{n-1}(l, r_{l,n} - 3) - \phi_{n-1}(l, r_{l,n} - 3) = \phi_{n}(l, r_{l,n} - 3) - \phi_{n}(l, r_{l,n} - 3) \). Then,

\[
\varphi_i(l, r_{l,n} - 3(c)) = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1} + \begin{cases} 
\frac{1}{3}[c(s_{n-2}, 2) - c(s_{n-3}, 1)] & i = n - 2 \\
\alpha_{n-1}[c(s_{n-2}, 3) - c(s_{n-2}, 1)] & i = n - 1 \\
\alpha_{n-2}[c(s_{n-2}, 3) - c(s_{n-2}, 1)] & i = n.
\end{cases}
\]

Then, for each \( i \in \{n - 2, n - 1, n\} \), \( \alpha_i = \frac{\sigma_i(s_{n-3}, c)}{\sum_{j \in \{n - 2, n - 1, n\}} \sigma_j(s_{n-3}, c)} = \frac{1}{3} \).

**Case 2:** There is \( i \in \{n - 2, n - 1, n\} \) such that \( \varphi_i(l, r_{l,n} - 3(c)) = 0 \) and each \( j \in \{n - 2, n - 1, n\} \setminus \{i\}, \varphi_j(l, r_{l,n} - 3(c)) > 0 \).

Without loss of generality, assume \( n - 2 = i \). By equality in increased congestion, \( \varphi_{n-1}(l, r_{l,n} - 3(c)) - \varphi_{n-1}(l, r_{l,n} - 3(c)) = \varphi_{n}(l, r_{l,n} - 3(c)) - \varphi_{n}(l, r_{l,n} - 3(c)) \). Then,
\[
\varphi_i(l, r_2^{s_{n-3}}(c)) = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1} + \begin{cases} 
0 & i = n - 2 \\
\alpha_{n-1}^{n-2}[c(s_{n-2}, 3) - c(s_{n-2}, 1)] \\
\frac{1}{2}[c(s_{n-3}, 2) - c(s_{n-3}, 1)] & i = n - 1 \\
[c(s_{n-1}, 2) - c(s_{n-1}, 1)] \\
+ \alpha_{n-1}^{n-2}[c(s_{n-2}, 3) - c(s_{n-2}, 1)] & i = n. \\
+ \frac{1}{2}[c(s_{n-3}, 2) - c(s_{n-3}, 1)]
\end{cases}
\]

Then, \( \alpha_{n-2}^{n-3} = \frac{\sigma_{n-2(s_{n-3}, c)}}{\sum_{j \in \{s_{n-2}, n-1, n\}} \sigma_j(s_{n-3}, c)} = 0 \) and for each \( i \in \{n - 1, n\} \), \( \alpha_i^{n-3} = \frac{\sigma_{i(s_{n-3}, c)}}{\sum_{j \in \{s_{n-2}, n-1, n\}} \sigma_j(s_{n-3}, c)} = \frac{1}{2} \).

Case 3: There is \( i \in \{n - 2, n - 1, n\} \) such that \( \varphi_i(l, r_2^{s_{n-3}}(c)) > 0 \) and each \( j \in \{n - 2, n - 1, n\} \setminus \{i\} \), \( \varphi_j(l, r_2^{s_{n-3}}(c)) = 0. \)

Without loss of generality, assume \( n - 2 = i \), then

\[
\varphi_i(l, r_2^{s_{n-3}}(c)) = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1} + \begin{cases} 
\frac{1}{2}[c(s_{n-3}, 2) - c(s_{n-3}, 1)] & i = n - 2 \\
\alpha_{n-1}^{n-2}[c(s_{n-2}, 3) - c(s_{n-2}, 1)] & i = n - 1 \\
[c(s_{n-1}, 2) - c(s_{n-1}, 1)] \\
+ \alpha_{n-1}^{n-2}[c(s_{n-2}, 3) - c(s_{n-2}, 1)] & i = n. \\
+ \frac{1}{2}[c(s_{n-3}, 2) - c(s_{n-3}, 1)]
\end{cases}
\]

Then, \( \alpha_{n-2}^{n-3} = \frac{\sigma_{n-2(l_{n-3}, c)}}{\sum_{j \in \{n - 2, n-1, n\}} \sigma_j(l_{n-3}, c)} = 1 \) and for each \( i \in \{n - 1, n\} \), \( \alpha_i^{n-3} = \frac{\sigma_i(l_{n-3}, c)}{\sum_{j \in \{n - 2, n-1, n\}} \sigma_j(l_{n-3}, c)} = 0. \)

Then, for each previous case, \( \varphi(l, r_2^{s_{n-3}}(c)) = x_2^{n-3}. \)

Step \( \frac{n(n-1)}{2} \): Solving \( (l, c) \in \mathbb{R}^N_+ \times C \).

We follow the same argument of the previous step. Recall that \( N \in EP(l, c). \)

Let \( \pi = N. \) Then, for each \( i \in N, \)

\[
\varphi_i(l, c) = \sum_{j \leq i} \frac{c(s_j, 1)}{n - j + 1} + \sum_{j < i} \frac{\sigma_i(s_j, c)}{\sum_{m \in \pi_k} \sigma_j(s_j, c)}[c(s_j, n - j + 1) - c(s_j, 1)] = x_i.
\]
Remark 1. Independence of axioms for Theorem 1 and Theorem 2

1. It is easy to see that the sequential equal contributions rule satisfies no-subsidy across canals, equal treatment of equals per canal, independence of at-least-as-large segments, and equality in increased congestion. However, it violates strong congestion monotonicity. To see this consider the following example:

Let $N \equiv \{1, 2, 3\}$ and $(l, c) \in \mathbb{R}_+^N \times \mathcal{C}$ be such that for each $i \in N$, $c(s_i, 1) = 10$, $c(s_1, 2) = c(s_1, 3) = 30$, and $c(s_2, 2) = 10$. Note that $N \in \mathcal{E}(l, c)$.

Let $\pi = \{N\}$. Then $\text{SEC}(l, c, N) = (10, 15, 25)$. Let $(l, c') \in \mathbb{R}_+^N \times \mathcal{C}$ be such that for each $i \in N$, $c'(s_i, 1) = c(s_i, 1)$, $c'(s_2, 2) = c'(s_2, 3) = c(s_1, 2) = c(s_1, 3)$, and $c'(s_2, 2) = 20$. Note that $N \in \mathcal{E}(l', c')$. Let $\pi = \{N\}$. Then $\text{SEC}(l', c', N) = (10, 20, 30)$. Note that $c'(s_2, 2) > c(s_2, 2)$ and $\text{SEC}_2(l', c', N) > \text{SEC}_2(l, c, N)$. This violates strong cost monotonicity.

2. It is easy to see that each member of the following family satisfies no-subsidy across canals, independence of at-least-as-large segments, strong congestion monotonicity, and equality in increased congestion. However, there is at least one rule in the family that violates equal treatment of equals per canal.

Let $\prec: N \to N$ be an order over $N$.

For each $(l, c) \in \mathbb{R}_+^N \times \mathcal{C}$, each $\pi \equiv \{\pi_k\}_{k=1}^K \in \mathcal{E}(l, c)$, each $k \in \{1, \ldots, K\}$, and each $i \in \pi_k$, let

$$\phi^\pi_{i}(l, c, \pi) = \sum_{j<i} c(s^\pi_k, n-j+1) - c(s^\pi_k, 1) \frac{n-j}{n} + \begin{cases} 0 & \text{if } \exists j \in \pi_k \text{ s.t. } i \prec |i, j \text{ and } l_i = l_j, \\ c(s^\pi_k, 1) & \text{otherwise.} \end{cases}$$

To see that at least one member of the family violates equal treatment of equals per canal consider the following example:

Let $N \equiv \{1, 2, 3, 4\}$ and $(l, c) \in \mathbb{R}_+^N \times \mathcal{C}$ be such that $l_1 = l_2 < l_3 < l_4$, $c(s_1, 1) = 5$, $c(s_1, 2) = 10$, $c(s_1, 3) = c(s_1, 4) = 15$, $c(s_2, 1) = 2$, $c(s_3, 2) = 5$, $c(s_4, 1) = 4$. Note that $N \in \mathcal{E}(l, c)$. Let $\pi = \{N\}$ and $\prec$ be $1 \prec 2 \prec 3 \prec 4$. Then $\phi^\pi_{i}(l, c, N) = (0, 5, 2+5, 4+5+3) = (0, 5, 7, 12)$. Note that $l_1 = l_2$, but $\phi^\pi_{i}(l, c, N) \neq \phi^\pi_{i}(l, c, N)$. This violates equal treatment of equals per canal.

3. It is easy to see that the following rule satisfies no-subsidy across canals, equal treatment of equals per canal, independence of at-least-as-large segments, and strong congestion monotonicity, but not equality in increased congestion.

Sequential contributions with proportion over congestion, SCPC,

For each $(l, c) \in \mathbb{R}_+^N \times \mathcal{C}$, each $\pi \equiv \{\pi_k\}_{k=1}^K \in \mathcal{E}(l, c)$, each $k \in \{1, \ldots, K\}$, and each $i \in \pi_k$, let
SCPC\(_i(l, c, \pi) = \sum_{j \leq i} \frac{c(s_j^{\pi}, 1)}{n-j+1} + \sum_{j<i} \frac{l_i - l_{i-1}}{\sum_{j<i} l_j - l_{j-1}} (c(s_j^{\pi}, n-j+1) - c(s_j^{\pi}, 1)).

To see that the sequential contributions with proportion over congestion rule violates equality in increased congestion consider the following example:

Let \( N = \{1, 2, 3\} \) and \((l, c) \in \mathbb{R}_+^N \times \mathcal{C}\) be defined by \(l = (1, 2, 3)\), for each \(i \in N\), \(c(s_i, 1) = 30\), \(c(s_1, 2) = c(s_1, 3) = 40\), and \(c(s_2, 2) = 30\). Note that \(N \in \text{EP}(l, c)\). Let \(\pi = \{N\}\). Then \(\text{SCPC}(l, c, N) = (10, 29, 61)\). Let \((l, c') \in \mathbb{R}_+^N \times \mathcal{C}\) be such that for each \(i \in N\), \(c'(s_i, 1) = c(s_i, 1)\), \(c'(s_1, 2) = c(s_1, 2)\), \(c'(s_1, 3) = 45\), and \(c'(s_2, 2) = c(s_2, 2)\). Note that \(N \in \text{EP}(l', c')\). Let \(\pi = \{N\}\). Then \(\text{SCPC}(l, c', N) = (10, 31, 64)\). Note that \(c'(s_2, 2) > c(s_2, 2)\) and \(\text{SCPC}_2(l, c', N) - \text{SCPC}_3(l, c, N) = 2 \neq 3 = \text{SCPC}_3(l, c', N) - \text{SCPC}_3(l, c, N)\). This violates equality in increased congestion.

4. It is easy to see that the following rule satisfies no-subsidy across canals, equal treatment of equals per canal, strong congestion monotonicity, and equality in increased congestion. However, it violates independence of at-least-as-large lengths.

For each \((l, c) \in \mathbb{R}_+^N \times \mathcal{C}\), each \(\pi = \{\pi_k\}_{k=1}^K \in \text{EP}(l, c)\), each \(k \in \{1, ..., K\}\), and each \(i \in \pi_k\), let

\[
\mu_i(l, c, \pi) = \sum_{j \in \pi} \frac{c(s_j^{\pi}, 1)}{n-j+1} + \sum_{j<i} \frac{c(s_j^{\pi}, n-j+1) - c(s_j^{\pi}, 1)}{n-j}.
\]

To see that it violates independence of at-least-as-large lengths consider the following example:

Let \(N = \{1, 2\}\) and \((l, c) \in \mathbb{R}_+^N \times \mathcal{C}\) be defined by \(l = (1, 2)\), \(c(s_1, 1) = 10\), \(c(s_1, 2) = 12\), and \(c(s_2, 1) = 10\). Note that \(N \in \text{EP}(l, c)\). Let \(\pi = \{N\}\). Then \(\mu(l, c, N) = (10, 12)\). Let \((l', c') \in \mathbb{R}_+^N \times \mathcal{C}\) be such that \(l' = (1, 3)\), for each \(i \in N\), \(c(s_1, 1) = c(s_1, 1)\), \(c'(s_1, 2) = c(s_1, 2)\), \(c'(s_2, 1) = 16\). Note that \(N \in \text{EP}(l', c')\). Let \(\pi = \{N\}\). Then \(\mu(l, c', N) = (13, 15)\). Since \(l'_2 > l_2\), \(l'_1 = l_1\) and \(\mu_1(l', c', N) \neq \mu_1(l', c', N)\). This violates independence of at-least-as-large segments.

5. It is easy to see that each member of the following family satisfies equal treatment of equals per canal, independence of at-least-as-large segments, strong congestion monotonicity, and equality in increased congestion. However, there is at least one rule in the family that violates no-subsidy across canals.
For each \((l,c) \in \mathbb{R}_+^N \times \mathcal{C}\), each \(\pi \equiv \{\pi_k\}_{k=1}^K \in EP(l,c)\), each \(\delta \leq \frac{\min_{j \in N} c(s_j,1)}{n-j+1}\), and each \(i \in N\), let

\[
\phi^\delta_i(l,c,\pi) = \begin{cases} 
SCEC_i(l,c,\pi) + \delta & \text{if } |\pi| = 2, |\pi_1| < |\pi_2|, i \in \pi_1, \text{ and } i = n_1 \\
SCEC_i(l,c,\pi) - \delta & \text{if } |\pi| = 2, |\pi_1| < |\pi_2|, i \in \pi_2, \text{ and } i = n_2 \\
SCEC_i(l,c,\pi) & \text{otherwise.}
\end{cases}
\]

To see that at least one member of the family violates no-subsidy across canals consider the following example:

Let \(N \equiv \{1, 2, 3\}\) and \((l,c) \in \mathbb{R}_+^N \times \mathcal{C}\) be such that for each \(i \in N\), \(c(s_i,1) = 10\), \(c(s_1,2) = 15\), \(c(s_1,3) = 40\), and \(c(s_2,2) = 30\). Note that \(N \not\in EP(l,c)\) and \(\pi = \{\{1\}, \{2,3\}\} \in EP(l,c)\). Let \(\delta = 1\). Then, \(\phi^\delta(l,c,\pi) = (10 + 1, 12.5, 22.5 - 1) = (11, 12.5, 21.5)\). Note that \(\phi^\delta(l,c,\pi) > 10\). This violates no-subsidy across canals.

### 6 Variable-population model

In this section we introduce a variable-population model to study problems in which some people go away from the original problem. We extend in a natural way the sequential equal contributions rule, the sequential contributions family, and the sequential conditional contributions family to allow for changes in population. We introduce two additional axioms, a solidarity property and an invariance property.

Let \(N\) be the finite set of potential people. Let \(N'\) be the set of all possible finite sets of \(N\), and \(N \in N'\) be a finite set of people.

A problem is a finite set of people, \(N \in N'\), a vector of positive lengths, \(l \in \mathbb{R}_+^N\), and a cost function \(c \in \mathcal{C}^N\).

Let \(\Pi^N\) be the set of all partitions over \(N\), and \(\pi = (\pi_k)_{k=1}^K\) be a typical element.

An allocation for \((N,l,c)\) is a vector \(x \in \mathbb{R}^N\) such that for each \(i \in N\), \(x_i \geq 0\), and \(\sum_{i \in N} x_i = mc(N,l,c)\). Let \(X(c)\) be the set of allocations.

Let \(EP(l,c) \subset \Pi^N\) be the set of efficient partitions, that is, the set of partitions over \(N\) for which the total cost is \(mc(N,l,c)\).

A rule, \(\varphi\), recommends for each problem and each efficient partition of it an allocation of it. That is, for each \((N,l,c) \in N' \times \mathbb{R}_+^N \times \mathcal{C}\) and each \(\pi \in EP(l,c)\), \(\varphi(N,l,c,\pi) \in X(c)\).

We introduce the rules.

**Sequential equal contributions, SEC:** For each \((N,l,c) \in N' \times \mathbb{R}_+^N \times \mathcal{C}^N\), each \(\pi = \{\pi_k\}_{k=1}^K \in EP(l,c)\), each \(k \in \{1, ..., K\}\), and each \(i \in \pi_k\), let
\[ SEC_i(N, l, c, \pi) = \sum_{j \leq i} \frac{c(s_j^{\pi_k}, n_k - j + 1)}{n_k - j + 1}. \]

For each \( s_i \in \mathbb{R}_+ \) and each \( c \in C \), let \( \sigma(s_i, c) \in \mathbb{R}^N_+ \), and let \( \sigma^L \in \mathbb{R}^N_+ \) be such that for each \( s_j \in \mathbb{R}_+^2 \), each \( c \in C \), and each \( i \in N \), if \( \sigma_i(s_j, c) = 0 \), then \( \sigma^L_i(s_j, c) > 0 \). Let \( \Sigma \) be the set of all such weight functions.

**Sequential contributions rule associated with \( \sigma, \sigma^L \in \Sigma \), \( SC^\sigma, \sigma^L \):** For each \( (N, l, c) \in N \times \mathbb{R}^N_+ \times C^N \), each \( \pi = \{\pi_k\}_{k=1}^K \in EP(l, c) \), each \( \sigma, \sigma^L \in \Sigma \), each \( k \in \{1, ..., K\} \), and each \( i \in \pi_k \), let

\[
SC^\sigma \sigma^L_i (N, l, c, \pi) = \sum_{j \leq i} \frac{c(s_j^{\pi_k}, 1)}{n_k - j + 1} + \begin{cases} 
\sum_{j \leq i} \frac{\sigma_i(s_j^{\pi_k}, c)}{\sum_{m \in \pi_k} \sigma_m(s_j^{\pi_k}, c)} [c(s_j^{\pi_k}, n - j + 1) - c(s_j^{\pi_k}, 1)] & \text{if } \sum_{m \in \pi_k} \sigma_m(s_j^{\pi_k}, c) \neq 0 \\
\sum_{j \leq i} \frac{\sigma^L_i(s_j^{\pi_k}, c)}{\sum_{m \in \pi_k} \sigma^L_m(s_j^{\pi_k}, c)} [c(s_j^{\pi_k}, n - j + 1) - c(s_j^{\pi_k}, 1)] & \text{otherwise.}
\end{cases}
\]

For each \( i \in N \), and each \( j \leq i \), \( \sigma_i(s_j, c) \in \{0, 1\} \), and for each \( m < i \), if \( s_i = s_j \), then \( \sigma_i(s_m, c) = \sigma_j(s_m, c) \). For each \( s_j \in \mathbb{R}_+ \), each \( c \in C \), and each \( i \in N \), \( \sigma^L_i(s_j, c) = 1 \). Let \( \Sigma' \) be the set of such weight functions.

**Sequential conditional contributions rule associated with \( \sigma, \sigma^L \in \Sigma' \), \( SCC^\sigma, \sigma^L \):** For each \( (N, l, c) \in N \times \mathbb{R}^N_+ \times C^N \), each \( \pi = \{\pi_k\}_{k=1}^K \in EP(l, c) \), each \( \sigma \in \Sigma' \), each \( k \in \{1, ..., K\} \), and each \( i \in \pi_k \), let

\[
SCC^\sigma \sigma^L_i (N, l, c, \pi) = \sum_{j \leq i} \frac{c(s_j^{\pi_k}, 1)}{n_k - j + 1} + \begin{cases} 
\sum_{j \leq i} \frac{\sigma_i(s_j^{\pi_k}, c)}{\sum_{m \in \pi_k} \sigma_m(s_j^{\pi_k}, c)} [c(s_j^{\pi_k}, n - j + 1) - c(s_j, 1)] & \text{if } \sum_{m \in \pi_k} \sigma_m(s_j^{\pi_k}, c) \neq 0 \\
\frac{1}{\sum_{m \in \pi_k} \sigma^L_m(s_j^{\pi_k}, c)} [c(s_j^{\pi_k}, n - j + 1) - c(s_j, 1)] & \text{otherwise.}
\end{cases}
\]

The first axiom says that when a person leaves a problem everybody should be affected in the same direction.\footnote{This axiom is introduced by Thomson (1983a) and Thomson (1983b) for bargaining problems.} To formally state this axiom, we need additional
notation. For each \((N, l, c) \in \mathcal{N} \times \mathbb{R}_+^N \times C\) and each \(i \in N\), a subproblem, \((N', l|_{N'}, c|_{N'})\) is defined as follows:

1. Let \(N' = N \setminus \{i\}\).
2. Let \(l|_{N'}\) be the restriction of the lengths for people in \(N'\). That is, for each \(j \in N', l_j|_{N'} = l_j\).
3. For each \(j < i\), each \(j > i + 1\) and each \(k \in \{1, ..., n - 1\}\), let \(c|_{N'}(s_j, k) \equiv c(s_j, k)\).
4. For each \(k \in \{1, ..., n\}\), let \(c|_{N'}(s_{i+1}, k) \equiv c(s_i, k) + c(s_{i+1}, k)\).

**Population monotonicity:** For each \((N, l, c) \in \mathcal{N} \times \mathbb{R}_+^N \times C\), each \(\pi \in EP(l, c)\), and each subproblem \((N', l|_{N'}, c|_{N'})\) such that there is \(\pi' \in EP(l, c|_{N'})\) for which for each \(k \in \{1, ..., K\}\), \(\pi'_k = \pi_k\) or \(\pi'_k = \pi_k \setminus \{i\}\), and for each \(j \in N'\),

\[
\varphi_j(N, l, c, \pi) \leq \varphi_j(N', l|_{N'}, c|_{N'}, \pi') \quad \text{or} \quad \varphi_j(N, l, c, \pi) \geq \varphi_j(N', l|_{N'}, c|_{N'}, \pi').
\]

Without congestion, this axiom is weak in the sense that all the well-known rules satisfy it (Thomson 2007). However, with congestion, the sequential equal contributions rule and the sequential contributions with equality in congestion rule are not population monotonic. Therefore, some members of the sequential conditional contributions and sequential contributions families are not population monotonic. However, we do not know if some members of these families satisfy this axiom.

**Example 2.** The sequential equal contributions rule and the sequential contributions with equality in congestion rule are not population monotonic.

Let \((N, l, c) \in \mathcal{N} \times \mathbb{R}_+^N \times C\) be such that \(N = \{1, 2, 3, 4\}\), for each \(i \in N\) and \(k \in \{1, ..., 3\}\), \(c(s_i, k) = 20\), and \(c(s_1, 4) = 40\). In this case, \(\{N\} \in EP(l, c)\). Let \(\pi = \{N\}\). Let \(x \equiv SEC(N, l, c, \pi)\) and \(y \equiv SCEC(N, l, c, \pi)\). In this problem, \(x = (10, \frac{50}{3}, \frac{80}{3}, \frac{140}{3})\) and \(y = (5, \frac{55}{3}, \frac{85}{3}, \frac{145}{3})\).

Let \(N' = N \setminus \{4\}\) and \((N', l|_{N'}, c|_{N'})\) be a subproblem. Note that \(\{N'\} \in EP(l, c|_{N'})\). Let \(\pi' = \{N\}\). Let \(x' \equiv SEC((N', l|_{N'}, c|_{N'}, \pi'))\) and \(y' \equiv SCEC(N', l|_{N'}, c|_{N'}, \pi')\). Then, \(x' = y' = (\frac{30}{3}, \frac{50}{3}, \frac{10}{3})\).

Note that \(x_1 > x'_1\) and \(x_3 < x'_3\). Thus, the sequential equal contributions rule is not population monotonic. Moreover, \(y_1 < y'_1\) and \(y_2 > y'_2\). Thus, the sequential contributions with equality in congestion rule is not population monotonic.

We introduce the last axiom. Consider a problem and an allocation recommended for it by a rule. Suppose that the person with the smallest need leaves, but he pays what the rule assigned to him in the original problem. Consider the
new problem in which this person is not there and the cost of the first segment 
decreases by the amount that he paid in the original problem. This axiom says 
that in this new problem, everyone should pay the same amount as in the original 
problem. 

To formally state the axiom, we need additional notation. For each \((N, l, c) \in N \times \mathbb{R}^N_+ \times C^N\) and each \(x \in \mathcal{X}(c)\), the \textbf{first-segment reduction of \((N, l, c)\) with respect to \(N \setminus \{1\}\) at \(x\),} \(f^x_{N \setminus \{1\}}(N, l, c)\), is the problem with set of 
people \(N' \in N\), list of lengths \(s' \in \mathbb{R}^N_+\), and cost functions \(c' \in C\) defined as 
follows:

1. \(N' \equiv N \setminus \{i\}\).
2. For each \(j \in N'\), \(l'_j = l_j\).
3. For \(j = 2\), and each \(k \in \{1, ..., n\}\), \(c'(s_2, k) \equiv \max\{c(s_2, k) + [c(s_1, k) - x_1], 0\}\).
4. For each \(j > 2\), \(c'(s_j, k) \equiv \begin{cases} 
  c(s_j, k) & \text{if } c'(s_{j-1}, n) - x_1 \geq 0, \\
  \max\{c(s_j, k) - x_1, 0\} & \text{otherwise.}
\end{cases}\)

\textbf{First-person consistency:} For each \((N, l, c) \in N \times \mathbb{R}^N_+ \times C\), each \(\pi \in EP(l, c)\), and each \(f^x_{N \setminus \{1\}}(N, l, c)\) such that there is \(\pi' \in EP(l', c')\) such that for each \(k \in \{1, ..., K\}\), \(\pi'_k = \pi_k\) or \(\pi'_k = \pi_k \setminus \{1\}\), and for each \(j \in N'\), 

\[ \varphi_j(N, l, c, \pi) = \varphi_j(f^x_{N \setminus \{1\}}(N, l, c), \pi'). \]

This axiom is satisfied by the \textit{sequential equal contributions rule}, but not by 
the \textit{sequential contributions with equality in congestion rule}. Therefore, there is 
a member of the \textit{sequential conditional contributions family} and the \textit{sequential 
contributions family} that are not \textit{first-person consistent}. However, some members 
of these families do satisfy the axiom. More work is needed to identify the entire 
subfamily of rules that satisfy this property.

The next table summarizes the results in the paper.

\section*{References}

Aadland, David and Van Kolpin (1998) “Shared irrigation costs: An empirical and 

the Association of Local Transport Airlines, Jackson, Miss.

\footnote{Without congestion, Potters and Sudhölter (1999) study of different interpretations of 
this axiom.}
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<th>Seq. equal contr.</th>
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Table 1: Summary of the results. The axioms are divided into three categories. The first axioms are punctual axioms, they only make reference to a particular problem. The second category shows the axioms that are related with changes in costs or lengths. The last category shows the axioms that are related with changes in population. A + means that the rule satisfies the property, and a – means the opposite. The number (1) shows the characterization in Theorem 1 and (2) the characterization in Theorem 2.


Figure 4: With congestion costs. (a) Problem \((l, r_1^{n-2}(c))\): For each segment between \(s_1\) and \(s_{n-2}\) the cost of connecting one or more people is the same. Only for \(s_{n-1}\) the cost of connecting one person is different from the cost of connecting two or more people. (b) Problem \((l, r_2^{n-2}(c))\): Only for \(s_{n-2}\) and \(s_{n-1}\) the cost of connecting one person is different from the cost of connecting two or more people. (c) Problem \((l, r_1^{n-3}(c))\): In this case, \(c(s_{n-2}, 3) \leq c(s_{n-2}, 2)\) (d) The original problem is recovered \((l, c)\).