A non-robustness in the order structure of the equilibrium set in lattice games

Andrew J. Monaco  
Department of Economics  
University of Kansas  
Lawrence KS, 66045, USA  
monacoa@ku.edu

Tarun Sabarwal  
Department of Economics  
University of Kansas  
Lawrence KS, 66045, USA  
sabarwal@ku.edu

Abstract

The order and lattice structure of the equilibrium set in games with strategic complements are not robust to a minimal introduction of strategic substitutes. In a lattice game in which all-but-one players exhibit strategic complements (with one player exhibiting strict strategic complements), and the remaining player exhibits strict strategic substitutes, no two equilibria are comparable. More generally, in a lattice game, if either (1) just one player has strict strategic complements and another player has strict strategic substitutes, or (2) just one player has strict strategic substitutes and has singleton-valued best-responses, then without any restrictions on the strategic interaction among the other players, no two equilibria are comparable. In such cases, the equilibrium set is a non-empty, complete lattice, if, and only if, there is a unique equilibrium. Moreover, in such cases, with linearly ordered strategy spaces, the game has at most one symmetric equilibrium.

JEL Numbers: C70, C72
Keywords: Lattice games, strategic complements, strategic substitutes, equilibrium set

First Draft: April 2010
This Version: September 16, 2010

*For helpful comments, we are grateful to Sunanda Roy.
1 Introduction

Games with strategic complements (GSC) and games with strategic substitutes (GSS) formalize two basic economic interactions and have widespread applications. In GSC, best-response of each player is weakly increasing (or non-decreasing) in actions of the other players, and in GSS, best-response of each player is weakly decreasing (or non-increasing) in the actions of the other players.

As is well-known, in GSC, the equilibrium set has nice order and structure properties: there always exists a smallest and a largest equilibrium and more generally, the equilibrium set is a non-empty, complete lattice. These properties have proved useful in several ways; for example, they help to provide simple and intuitive algorithms to compute equilibria, and they help to show monotone comparative statics of equilibria in such games.

In contrast, in GSS, the equilibrium set is completely unordered – no two equilibria are comparable (in the standard product order). Consequently, in such games, with multiple equilibria, techniques based on the complete lattice structure of the equilibrium set, or the existence of a smallest or largest equilibrium are invalid. Typically, different techniques are required to analyze such games.

The existing results show that when we move from a setting where all players exhibit strategic complements to a setting where all players exhibit strategic substitutes, the order structure of the equilibrium set is destroyed completely. A central motivation for the present analysis is to inquire when

---

1Such games are defined in Bulow, Geanakoplos, and Klemperer (1985), and as they show, models of strategic investment, entry deterrence, technological innovation, dumping in international trade, natural resource extraction, business portfolio selection, and others can be viewed in a more unifying framework according as the variables under consideration are strategic complements or strategic substitutes. Earlier developments are provided in Topkis (1978) and Topkis (1979).

2Versions of such games arise in diverse economic environments, including competitive strategy, public goods, industrial organization, natural resource utilization, manufacturing analysis, team management, tournaments, resource allocation, business portfolio development, principal-agent modeling, multi-lateral contracting, auctions, technological innovation, behavioral economics, and others.


4See Zhou (1994).

and by how much the order structure of the equilibrium set is affected as we move player-by-player from a setting of all players with strategic complements to a setting of all players with strategic substitutes.

The new results here show that the nice order and structure properties of GSC do not survive a minimal introduction of strategic substitutes, in the following sense. Consider a lattice game in which all-but-one players exhibit strategic complements (with one player exhibiting strict strategic complements), and the remaining player exhibits strict strategic substitutes. In this case, no two equilibria in the game are comparable (in the product order).

The results shown here are stronger, and apply to lattice games with more general strategic interaction among the players. In particular, in any lattice game, if there is reason to believe that either (1) just one player has strict strategic complements and another player has strict strategic substitutes, or (2) just one player has strict strategic substitutes and has singleton-valued best-responses, then without any restrictions on the strategic interaction among the other players, we may conclude that no two equilibria are comparable. In such cases, with multiple equilibria, techniques based on the complete lattice structure of the equilibrium set, or the existence of a smallest or largest equilibrium are invalid.

In such cases, the equilibrium set is a non-empty, complete lattice, if, and only if, the equilibrium is unique. In addition, if player strategy spaces are linearly ordered, then the set of symmetric equilibria is non-empty, if, and only if, there is a unique symmetric equilibrium.

Games that have both strategic complements and strategic substitutes arise naturally in many applications; from the simple textbook game of matching pennies, to several examples from competitive strategy and industrial organization in Bulow, Geanakoplos, and Klemperer (1985), to games with contests in Dixit (1987), among others. Therefore, the results here have implications for a diversity of applications.

The proofs here are simple. Some of the simplicity arises naturally in many analyses of questions related to strategic complements and strategic substitutes. Some of it arises from the recent resurgence of work in strategic

\footnote{Intuitively, a lattice game is a strategic game in which every player’s strategy set is a complete lattice, and every player’s payoff function is continuous in own variable. No restriction is placed on strategic interaction across other players. The formal definition is given in the next section.}

\footnote{Intuitively, best response is strictly increasing in other player strategies.}

\footnote{Intuitively, best response is strictly decreasing in other player strategies.}
substitutes that has provided valuable insights regarding similarities and differences between GSC and GSS. And some of it arises from a new insight into fundamental relations underlying the order structure of the equilibrium set in the presence of strategic substitutes.

The results here point out that an important component (the order structure of the equilibrium set) underlying the justifiably celebrated theory of games with strategic complements is not robust to a minimal extension of the theory to include other realistic cases. Therefore, there is a need to develop new techniques to study additional cases of interest.

The results here extend work by Roy and Sabarwal (2008), by covering cases of interest that cannot be covered in that paper. In particular, a central case in this paper, where all-but-one player has strategic complements, is not covered by their assumptions. Moreover, the proofs here are different – they are simpler and rely more directly on economic intuition.

We use the standard product order on the product of the player strategy spaces. This is a natural and intuitive order to consider in lattice games, and is used widely in GSC and in GSS. Recall that in the special case of a two-player GSS, reversing the order on the strategy space of one player transforms that game into a GSC, and results for a GSC apply to this special case. More generally, there may be no such transformation that leaves the equilibrium set invariant. For example, the textbook example of a two-player matching pennies game (a game with both complements and substitutes) has no pure-strategy Nash equilibrium, and therefore, cannot be viewed as a GSC, because a GSC always has a pure-strategy Nash equilibrium. Similarly, Roy and Sabarwal (2010a) provide an example of a three-player, two-action GSS that has no pure-strategy Nash equilibrium, and therefore, cannot be viewed as a GSC.

The next section sets up the model, derives the main results, and provides examples.

---

2 Lattice Games

Let $I$ be a non-empty set of players, and for each player $i$, a strategy space that is a partially ordered set, denoted $(X^i, \preceq^i)$, and a real-valued payoff function, denoted $f^i(x_i, x_{-i})$. As usual, the domain of each $f^i$ is the product of the strategy spaces, $(X, \preceq)$, endowed with the product order. The strategic game $\Gamma = \{I, (X^i, \preceq^i, f^i)_{i \in I}\}$ is a lattice game if for every player $i$,

1. $X^i$ is a non-empty, complete lattice, and
2. For every $x_{-i}$, $f^i$ is order continuous in $x_i$.

The definition of a lattice game here is very general. In particular, no restriction is placed on whether players have strategic complements or strategic substitutes. Consequently, this definition allows for general games with strategic complements, general games with strategic substitutes, and mixtures of the two.

This definition of a lattice game yields well-defined best-responses, as follows. For each player $i$, the best response of player $i$ to $x_{-i}$ is denoted $BR^i(x_{-i})$, and is given by $\arg \max_{x_i \in X_i} f^i(x_i, x_{-i})$. As the payoff function is continuous, and the strategy space is compact in the order interval topology, for every $i$, and for every $x_{-i}$, $BR^i(x_{-i})$ is non-empty. Let $BR : X \to X$, given by $BR(x) = (BR^i(x_{-i}))_{i \in I}$, denote the joint best-response correspondence. As usual, a (pure strategy) Nash equilibrium of the game is a profile of player actions $x$ such that $x \in BR(x)$. The equilibrium set of the game is given by $E = \{x \in X | x \in BR(x)\}$.

Of particular interest to us are cases where the best-response of a player is either increasing (the case of strategic complements) or decreasing (the case of...
strategic substitutes) with respect to the strategies of the other players. Here, increasing or decreasing are with respect to an appropriately defined set order.

Recall that in a lattice game, if the payoff function of each player \( i \) is quasi-supermodular in \( x_i \) and satisfies the single-crossing property in \((x_i; x_{-i})\)\(^{14}\) then the best-response correspondence of each player is nondecreasing\(^{16}\) in the standard induced set order\(^{17}\). Such a game is termed a **lattice game with strategic complements**, or GSC, for short.

In a GSC, the equilibrium set is a non-empty, complete lattice (see Zhou (1994)), and there exists a smallest equilibrium and a largest equilibrium (various versions of this result can be seen in Topkis (1978), Topkis (1979), Lippman, Mamer, and McCardle (1987), Sobel (1988), Milgrom and Roberts (various versions of this result can be seen in Topkis (1978), Topkis (1979), Milgrom and Roberts (1990), Vives (1990), Milgrom and Shannon (1994), among others).

Similarly, in a lattice game, if the payoff function of each player \( i \) is quasi-supermodular in \( x_i \), and satisfies the decreasing single-crossing property in \((x_i; x_{-i})\)\(^{19}\) then the best-response correspondence of each player is nonincreasing\(^{19}\) in the standard induced set order. Such a game is termed a **lattice game with strategic substitutes**, or GSS, for short.

Notice that the case where a player’s best response is a constant function can be viewed as either strategic complements or strategic substitutes. Therefore, in a lattice game with strategic complements, strategic substitutes may be introduced trivially by having some players with constant best response functions. Of course, such games remain lattice games with strategic comple-

\(^{14}\)As in Milgrom and Shannon (1994), a function \( f : X \rightarrow \mathbb{R} \) (where \( X \) is a lattice) is **quasi-supermodular** if (1) \( f(x) \geq f(x \land y) \iff f(x \lor y) \geq f(y) \), and (2) \( f(x) > f(x \land y) \implies f(x \lor y) > f(y) \).

\(^{15}\)A function \( f : X \times T \rightarrow \mathbb{R} \) (where \( X \) is a lattice and \( T \) is a partially ordered set) satisfies **single-crossing property** in \((x; t)\) if for every \( x' < x'' \) and \( t' < t'' \), (1) \( f(x', t') \leq f(x'', t'') \implies f(x', t') \leq f(x'', t'') \), and (2) \( f(x', t') < f(x'', t'') \implies f(x', t') < f(x'', t'') \).

\(^{16}\)A function \( f : X \times T \rightarrow \mathbb{R} \) (where \( X \) is a lattice and \( T \) is a partially ordered set) satisfies **decreasing single-crossing property** in \((x; t)\) if for every \( x' < x'' \) and \( t' < t'' \), (1) \( f(x', t') \leq f(x'', t'') \implies f(x', t') \leq f(x'', t'') \), and (2) \( f(x', t') < f(x'', t'') \implies f(x', t') < f(x'', t'') \).

\(^{17}\)The standard induced set order is defined as follows: for non-empty subsets \( A, B \) of a lattice \( X \), \( A \) is weakly lower than \( B \), if for every \( a \in A \), and for every \( b \in B \), \( a \land b \in A \), and \( a \lor b \in B \). It is sometimes also termed the strong set order.

\(^{18}\)A function \( f : X \times T \rightarrow \mathbb{R} \) (where \( X \) is a lattice and \( T \) is a partially ordered set) satisfies **decreasing single-crossing property** in \((x; t)\) if for every \( x' < x'' \) and \( t' < t'' \), (1) \( f(x', t') \leq f(x'', t'') \implies f(x', t') \leq f(x'', t'') \), and (2) \( f(x', t') < f(x'', t'') \implies f(x', t') < f(x'', t'') \).

\(^{19}\)A function \( f : X \times T \rightarrow \mathbb{R} \) (where \( X \) is a lattice and \( T \) is a partially ordered set) satisfies **decreasing single-crossing property** in \((x; t)\) if for every \( x' < x'' \) and \( t' < t'' \), (1) \( f(x', t') \leq f(x'', t'') \implies f(x', t') \leq f(x'', t'') \), and (2) \( f(x', t') < f(x'', t'') \implies f(x', t') < f(x'', t'') \).
ments, and the equilibrium set in such games remains a non-empty, complete lattice.

For non-trivial results involving strategic substitutes, it is useful to consider players with best responses that are “strictly” decreasing with respect to the strategies of the other players. When player \( i \)’s best response is a function, this is the standard definition of a strictly decreasing function. When player \( i \)’s best response is a correspondence, the analogous definition is stated as follows: for every \( x'_{-i} < x''_{-i} \), and for every \( x'_i \in BR_i(x'_{-i}) \) and \( x''_i \in BR_i(x''_{-i}) \), \( x''_i < x'_i \); that is every best-response to \( x''_{-i} \) is strictly lower than every best-response to \( x'_{-i} \).

Incorporating such a definition in a more general definition, Roy and Sabarwal (2008) show that in a lattice game with strict strategic substitutes, the equilibrium set is completely unordered; that is, no two equilibria are comparable.

In effect, moving from all strategic complements to all strategic substitutes destroys the order structure of the equilibrium set.

A motivating feature of the present analysis is to inquire how far we can move from strategic complements toward strategic substitutes while maintaining (or before destroying) the order structure of the equilibrium set.

To formalize strict strategic substitutes, consider the following set order. Let \( X \) be a lattice. For non-empty subsets \( A, B \) of \( X \), \( A \) is strictly lower than \( B \), if for every \( a \in A \), and for every \( b \in B \), \( a < b \). This definition is a slight strengthening of the following set order defined in Shannon (1995): \( A \) is completely lower than \( B \), if for every \( a \in A \), and for every \( b \in B \), \( a \leq b \). Notice that when \( A \) and \( B \) are non-empty, complete sub-lattices of \( X \), \( A \) is strictly lower than \( B \), if, and only if, \( \sup A < \inf B \); and similarly, \( A \) is completely lower than \( B \), if, and only if, \( \sup A \leq \inf B \).

Strictly decreasing correspondences are defined in a natural manner using the strictly lower than set order. Let \( X \) be a lattice and \( T \) be a partially ordered set. A correspondence \( \phi : T \rightarrow X \) is strictly decreasing, if for every \( t' < t'' \), \( \phi(t'') \) is strictly lower than \( \phi(t') \). In particular, strictly decreasing correspondences that are singleton-valued are equivalent to the standard definition of a strictly decreasing function. Similarly, we shall also find it useful to define strictly increasing correspondences. A correspondence \( \phi : T \rightarrow X \) is strictly increasing, if for every \( t' < t'' \), \( \phi(t'') \) is strictly lower than \( \phi(t') \).

\( 20 \) \( x'_i < x''_i \Rightarrow BR_i(x''_i) < BR_i(x'_{-i}) \).

\( 21 \) \( t' < t'' \Rightarrow \phi(t'') < \phi(t') \).
correspondence is a function, this is equivalent to the standard definition of a strictly increasing function.

With these ideas in place, player $i$ has **strict strategic substitutes**, if $BR^i$ is strictly decreasing, and player $i$ has **strict strategic complements**, if $BR^i$ is strictly increasing.

Here are the main results in this paper. Theorem 1 shows that in any lattice game, one player with strict strategic substitutes and one player with strict strategic complements are sufficient to destroy the order structure of the equilibrium set, as follows.

**Theorem 1.** Let $\Gamma$ be a lattice game in which one player has strict strategic substitutes and another player has strict strategic complements. If $x^*$ and $\hat{x}$ are distinct equilibria, then $x^*$ and $\hat{x}$ are not comparable.

**Proof.** Suppose, without loss of generality, that player 1 has strict strategic substitutes, player 2 has strict strategic complements, and suppose the distinct equilibria are comparable, with $\hat{x} \prec x^*$.

As case 1, suppose $\hat{x}_{-1} \prec x^*_{-1}$. Then $\hat{x}_1 = BR^1(\hat{x}_{-1})$ and $x^*_1 = BR^1(x^*_{-1})$, and by strict strategic substitutes, $x^*_1 \prec \hat{x}_1$, contradicting $\hat{x} \prec x^*$.

As case 2, suppose $\hat{x}_1 < x^*_1$. Then $\hat{x}_{-2} < x^*_{-2}$. Then $\hat{x}_2 \in BR^2(\hat{x}_{-2})$ and $x^*_2 \in BR^2(x^*_{-2})$, and by strict strategic complements, $\hat{x}_2 < x^*_2$, whence $\hat{x}_{-1} < x^*_{-1}$, and we are in case 1. Thus $x^*$ and $\hat{x}$ are not comparable.

Notice the simple economic intuition in this proof. In case 1 in the proof, if opponents of player 1 play higher strategies in the $x^*$ equilibrium than in the $\hat{x}$ equilibrium, then player 1 (with strict strategic substitutes) must be playing a strictly lower strategy in the $x^*$ equilibrium than in the $\hat{x}$ equilibrium, and therefore, the equilibria are non-comparable. Case 2 essentially says that with $\hat{x} \prec x^*$, player 1 cannot be playing a higher strategy in the $x^*$ equilibrium. For if he did, then player 2 (with strict strategic complements) is playing a higher strategy in the $x^*$ equilibrium, and therefore, the opponents of player 1 are playing higher strategies in the $x^*$ equilibrium, whence player 1 is playing a strictly lower strategy in the $x^*$ equilibrium, which is a contradiction.

---

22Recall that Shannon (1995) provides conditions on payoff functions that guarantee a comparison in the completely lower than set order. Moreover, in finite-dimensional Euclidean spaces, Edlin and Shannon (1998) provide an additional intuitive and easy-to-use differentiable condition regarding strictly increasing marginal returns to derive a comparison in the strictly lower than set order. Both these conditions can be adapted easily for strategic substitutes.
Taking this intuition one step further, when the best-response of the player with strict strategic substitutes is singleton-valued, the requirement of a player with strict strategic complements can be dropped, as follows.

**Theorem 2.** Let \(\Gamma\) be a lattice game in which one player has strict strategic substitutes, and this player’s best-response is singleton-valued. If \(x^*\) and \(\hat{x}\) are distinct equilibria, then \(x^*\) and \(\hat{x}\) are not comparable.

**Proof.** Suppose, without loss of generality, that player 1 has strict strategic substitutes, and suppose the distinct equilibria are comparable, with \(\hat{x} \prec x^*\).

Case 1 remains the same as above. Suppose \(\hat{x} \prec x^*_1\). Then \(\hat{x}_1 \in BR^1(\hat{x}_1 - 1)\) and \(x^*_1 \in BR^1(x^*_1 - 1)\), and by strict strategic substitutes, \(x^*_1 \prec \hat{x}_1\), contradicting \(\hat{x} \prec x^*\).

For case 2, suppose \(\hat{x} = x^*_1\) and \(\hat{x}_1 \prec x^*_1\). Then \(\hat{x}_1 = BR^1(\hat{x}_1) = BR^2(x^*_2) = x^*_2\), contradicting \(\hat{x}_1 \prec x^*_1\). Thus \(x^*\) and \(\hat{x}\) are not comparable.

Intuitively, in theorem 2, if \(\hat{x} \prec x^*\), then we need only consider the case when the opponents of player 1 play higher strategies; that is, \(\hat{x} - 1 \prec x^*_1 - 1\). For if \(\hat{x} - 1 = x^*_1\), then by singleton-valued best responses, the best response of player 1 to \(\hat{x}_1\) is the same as her best response to \(x^*_1\), and thus both equilibria are the same, which is a contradiction.

Theorem 2 formalizes the intuition that adding one player with strict strategic substitutes completely destroys the order structure of the equilibrium set. The above results yield the following corollary immediately.

**Corollary 1.** Let \(\Gamma\) satisfy the conditions of either theorem 1 or theorem 2. The following are equivalent.

1. \(E\) is a non-empty lattice,
2. \(E\) is a singleton, and
3. \(E\) is a non-empty, complete lattice.

**Proof.** The only part that needs to checked is that (1) implies (2). Suppose \(E\) is a non-empty lattice. If it has one element only, then (2) is proved. If it has two elements, say \(x^* \neq \hat{x}\), then it contains the join and meet of these two elements, and the join and meet are distinct and comparable, contradicting the theorems above.

Theorems 1 and 2 also imply immediately that when strategy spaces of players are linearly ordered\(^{23}\), the game has at most one symmetric equilib-

\(^{23}\)As usual, linearly ordered means that every pair of strategies is comparable. A linear order is sometimes termed a complete order.
Corollary 2. Let $\Gamma$ satisfy the conditions of either theorem 1 or theorem 2, and suppose the strategy space of each player is linearly ordered. The set of symmetric equilibria is non-empty, if, and only if, there is a unique symmetric equilibrium.

The examples below provide some intuition about these results.

Example 1. Consider a lattice game with two players. Player 1’s strategy space is a standard four-point lattice, $X_1 = \{a_1, b_1, c_1, d_1\}$, with $b_1$ and $c_1$ unordered, and $a_1 = b_1 \land c_1$, and $d_1 = b_1 \lor c_1$, shown graphically in figure 1. Similarly, $X_2 = \{a_2, b_2, c_2, d_2\}$, also shown graphically in figure 1. Suppose player 1’s best response correspondence is given as follows: $BR_1^1(a_2) = \{d_1\}$, $BR_1^1(b_2) = BR_1^1(c_2) = \{b_1, c_1\}$, and $BR_1^1(d_2) = \{a_1\}$, and player 2’s best response correspondence is given as follows: $BR_2^2(a_1) = \{a_2\}$, $BR_2^2(b_1) = BR_2^2(c_1) = \{b_2, c_2\}$, and $BR_2^2(d_1) = \{d_2\}$. Both are depicted in figure 1. It is easy to check that this example satisfies the conditions of theorem 1: player 1 has strict strategic substitutes, player 2 has strict strategic complements. Consequently, the four Nash equilibria $(b_1, b_2)$, $(b_1, c_2)$, $(c_1, b_2)$, and $(c_1, c_2)$ are all non-comparable.

Figure 1: An application of theorem 1

Theorem 2 shows that when the best-response of the player with strict strategic substitutes is singleton-valued, the condition in theorem 1 regarding one player with strict strategic complements can be dropped. Example 2 below shows that when the best-response of the player with strict strategic substitutes is not singleton-valued, this condition in theorem 1 cannot be dropped, in general.

---

24 An equilibrium is symmetric, if every player plays the same strategy in equilibrium.
Example 2. Consider the lattice game with two players given in figure 2, where for player 1, $L \prec M \prec H$, and for player 2, $L \prec M$. In this case, player 1 has strict strategic substitutes, with non-singleton-valued best-response; $BR^1(L) = \{M, H\}$, and $BR^1(M) = \{L\}$. Player 2 has “weak” strategic complements; in fact, player 2’s best-response function is constant, $BR^2(L) = BR^2(M) = BR^2(H) = \{L\}$. This game has two Nash equilibria, $(M, L)$ and $(H, L)$, and these equilibria are comparable, with $(M, L) \prec (H, L)$.

\[
\begin{array}{c|cc}
\text{Player 1} & L & M \\
\hline
L & 0, 5 & 5, 0 \\
M & 5, 5 & 0, 0 \\
H & 5, 5 & 0, 0 \\
\end{array}
\]

Figure 2: Comparing theorems 1 and 2

Theorems 1 and 2 above highlight a particular non-robustness in the order structure of the equilibrium set in lattice games.

If we consider a lattice game in which all players have strategic complements, then the equilibrium set is a non-empty, complete lattice. In particular, every pair of equilibria has a smallest larger equilibrium, and a largest smaller equilibrium.

If we modify this game slightly to require that one player has strict strategic complements, and another has strict strategic substitutes, then we destroy the order structure of the equilibrium set completely. That is, no two equilibria are comparable.

Similarly, if we modify this game to require that one player has strict strategic substitutes, and that player’s best-response is singleton-valued (perhaps because that payoff function is strictly quasi-concave), then again the order structure of the equilibrium set is destroyed completely.

Of course, the results here are stronger, and apply to general lattice games, not just to lattice games with strategic complements. In particular, in any lattice game, if there is reason to believe that either (1) one player has strict strategic complements and another player has strict strategic substitutes, or
(2) just one player has strict strategic substitutes and has singleton-valued best-responses, then without any restrictions on the strategic interaction among the other players, we may conclude that no two equilibria are comparable.

In all such cases, with multiple equilibria, techniques based on the complete lattice structure of the equilibrium set, or the existence of a smallest or largest equilibrium are invalid. Consequently, these results point to the need to develop alternative techniques to study such cases.

The results here extend work by Roy and Sabarwal (2008), who assume that the best-response correspondence satisfies a never-increasing property, defined as follows. Let $X$ be a lattice and $T$ be a partially ordered set. A correspondence $\phi : T \to X$ is not increasing, if for every $t' < t''$, for every $x' \in \phi(t')$, and for every $x'' \in \phi(t'')$, $x' \not\leq x''$. This property is satisfied in lattice games in which every player has strict strategic substitutes, but it is not satisfied in a central case in this paper: all-but-one players have strategic complements, and the remaining player has strategic substitutes. To see this, consider the following.

Let $X$ be a lattice and $T$ be a partially ordered set. A correspondence $\phi : T \to X$ is weakly completely increasing, if for every $t' < t''$, $\phi(t')$ is completely lower than $\phi(t'')$. Player $i$ has weak strategic complements, if $BR^i$ is weakly completely increasing. We have the following proposition.

**Proposition 1.** Let $\Gamma$ be a lattice game in which all-but-one players exhibit weak strategic complements, and the remaining player has at least two actions. The best response correspondence in such a game does not satisfy the never-increasing property.

**Proof.** Suppose, without loss of generality, that all-but-player-1 have weak strategic complements. Consider $x'^1_1 < x'^1_2$ in $X_1$, and $x'_{-1} \in X_{-1}$. Then $(x'^1_1, x'_{-1}) \prec (x'^1_2, x'_{-1})$. Let $y'^1_1 \in BR^1(x'_{-1})$. For each $i \neq 1$, let $x'^i_{-i} = (x'^i_1, x'_{-(1,i)})$ and $x'^i_{i} = (x'^i_1, x'_{-(1,i)})$. Then for each $i \neq 1$, $x'^i_{-i} < x'^i_{i}$. For each such $i$, fix $y'^i_1 \in BR^i(x'^i_{-i})$ and $y'^i_{-i} \in BR^i(x'^i_{-i})$ arbitrarily. By weak strategic complements, $y'^i_1 \leq y'^i_{-i}$. Thus, $(x'^1_1, x'^i_{-i}) \prec (x'^1_2, x'^i_{-i})$, $(y'^1_1, y'^i_{-i}) \in BR(x'^1_1, x'^i_{-i})$, $(y'^1_1, y'^i_{-i}) \in BR(x'^1_2, x'^i_{-i})$, and $(y'^1_1, y'^i_{-i}) \not\leq (y'^1_2, y'^i_{-i})$, contradicting the never-increasing property. Q.E.D.

---

25When best-responses are functions, this coincides with the definition of a not-increasing function, $t' < t'' \Rightarrow \phi(t') \not\leq \phi(t'')$, and in linearly ordered $X$, this is equivalent to a strictly decreasing function.

26When the correspondence is a function, this is equivalent to the standard definition of a weakly increasing function.

27Notice that strict strategic complements implies weak strategic complements.
Consequently, the case where all-but-one players exhibit weak strategic complements, and the remaining player has strategic substitutes is not covered by Roy and Sabarwal (2008). It is precisely this case that starts the analysis in this paper: consider a movement away from the case of all strategic complements by introducing one player with strategic substitutes. In particular, example 1 above (in which one player has strategic complements and one player has strategic substitutes) is not covered by Roy and Sabarwal (2008), but is admissible here.
References


