Quote Competition in Limit Order Markets *

Wataru Ohta
Osaka University

Abstract

For a dynamic limit order market, we present a Markov perfect equilibrium with Edgeworth cycles. In equilibrium, when sellers enter the market consecutively, the best quote decreases tick by tick, then jumps more than one tick, creating a hole in the book. The next quote rebounds to the less aggressive level, and the same cycle starts over again. Holes can relate to the high kurtosis of transaction prices. If the tick size is large and the trader arrival rate is high, there is another equilibrium under which traders queue at the same quotes.

JEL Classification: G19, G29

Keywords: Limit Order Markets, A Price-Time Precedence Rule, Markov Perfect Equilibrium.

1 Introduction

Securities exchanges in Paris, Stockholm, Toronto, Tokyo, and so forth adopt limit order markets as their execution systems. In these markets, traders can submit both limit orders that are contingent on price and market orders that are not. Limit orders are stored in the book first, and matched with market orders for transactions according to the trading rule such as a price-time precedence rule. The question is what is the optimal order submission strategy under a certain market condition. As a consequence, how do traders compete in quote of limit orders and how do transactions take place? These questions are important because limit order markets are prevalent in financial markets. However,

*We gratefully acknowledge the helpful comments and suggestions by Takao Kobayashi, Makoto Saito, and seminar participants at the Nippon Finance Workshop, University of Tokyo, Nagoya University, Osaka University, and Kyoto University. We are responsible for any remaining errors. This research was supported by a Grant-in-Aid for Scientific Research from the Japanese Government (Grant No. 19530273).
a theoretical analysis of dynamic limit order markets is difficult because the set of states of the book can be complex.

To investigate dynamic limit order markets, we consider a situation in which sellers and buyers arrive randomly in each period and submit an order to the exchange only once. There is no asymmetry of information among traders.\textsuperscript{1} We further assume that limit orders automatically expire in pre-specified periods after their submission. For example, when we assume that a limit order expires in two periods, every limit order automatically expires after two periods but before three periods have elapsed if it is not executed within two periods after its submission. Automatic expiration of limit orders restricts the number of limit orders in the book, which makes the states of the book relatively simple. We offer a pure-strategy Markov perfect equilibrium under the two-period expiration of limit orders. In addition, we demonstrate that characteristics of equilibrium under the two-period expiration are also valid under the three and four-period expiration of limit orders by numerically solved equilibria.

Assuming the short period expiration of limit orders, we can investigate the behavior of fleeting orders. At least traditionally, limit orders are considered to be submitted by patient traders who rarely modify or cancel their orders. However, some empirical studies such as those of Hasbrouck and Saar (2002), and Lo, MacKinlay, and Zhang (2002) report that parts of limit orders are rapidly canceled after their submission if they are not executed.\textsuperscript{2} Furthermore, Hasbrouck and Saar (2008) indicate that over one-third of limit orders are cancelled within two seconds of submission in INET, which is a limit order market. They term limit orders that are canceled very quickly fleeting orders, and show that fleeting orders are prevalent recently. Our model can be considered to study quote competition of fleeting orders.

\textsuperscript{1}For simplicity, we set aside the effect of asymmetric information. Chordia \textit{et al}. (2005) report that information is very quickly incorporated in prices for frequently traded stocks on the New York Stock Exchange, which suggests considerable amounts of transactions take place with less asymmetric information. In addition, Admati and Pfleiderer (1988) argue that orders of liquidity traders affect order submission strategies of informed traders. Thus, investigating the behavior of liquidity traders under symmetric information seems to be a reasonable step.

\textsuperscript{2}Hasbrouck and Saar (2002) indicate that about 25\% (40\%) of limit orders have been canceled within two (ten) seconds after their submission on the Island ECN. Lo, MacKinlay, and Zhang (2002) report that the average time-to-expiration or cancellation of non-executed orders is 34.15 (46.92) minutes for limit sell (buy) orders on the New York Stock Exchange.
Our model is similar to the model in Foucault (1999). He considers a case where limit orders expire in one period after their submission and where the asset value fluctuates over time. We simplify his model by assuming that the asset value does not change, and extend it by stipulating that limit orders survive for longer periods in order to analyze quote competition. Under the one-period expiration of limit orders, the book has at most one limit order at a time, and traders do not compete in price in submitting limit orders. In contrast, the two-period expiration of limit orders is the simplest assumption for limit orders to compete directly with each other. As a consequence, two models share some properties. One of the important properties is that limit order submitters set quotes to induce future traders on the opposite side of the market to submit market orders. Thus, in a typical equilibrium for our model, limit sell orders and limit buy orders are not stored in the book at the same time.

A pure-strategy Markov perfect equilibrium for our model has Edgeworth cycles. When sellers enter the market consecutively, they undercut the best quote by the tick size, which is the smallest amount of price increment stipulated by the exchange. When the best quote drops to a certain aggressive level after one-tick quote-cutting, a seller undercut the best quote by more than one tick to deter further quote-cutting. The next seller sets a less aggressive quote, and the same cycle starts over again. This cycling continues until a buyer arrives at the market, at which a transaction takes place.

Quotes jump under the narrow bid-ask spread during Edgeworth cycles. Cordella and Foucault (1999) show similar quote jumps can occur in dealer markets. Foucault et al. (2005) show that quotes jump under the narrow bid-ask spread in limit order markets if the share of impatient traders is large. They assume that traders must undercut the best quote in submitting limit orders and that a seller and a buyer arrive at the market alternatively with certainty while we assume that traders can set any price in submitting limit orders and that traders arrive at the market randomly. Our results are consistent with theirs because we can consider that the order expiration in short intervals corresponds to the impatience of traders.

case studied by Maskin and Tirole (1988). Eckert (2003) among others empirically observes such cycles in retail gasoline markets while Kruse et al. (1994) report Edgeworth cycles in experimental markets. These articles investigate markets where long-lived sellers set prices and buyers accept or reject prices. We demonstrate an equilibrium with Edgeworth cycles for double auction markets. We call the equilibrium an Edgeworth cycle equilibrium as do Maskin and Tirole (1988).

In the early stage of quote-cutting in an Edgeworth cycle, traders undercut the best quote by only one tick. Consistently, Hasbrouck and Saar (2008) report that fleeting orders are priced more aggressively than limit orders that get cancelled less quickly, and that many fleeting orders undercut the best quote by a small amount. They argue that traders submit fleeting orders whose prices improve the best quote by one tick in order to search for hidden orders. In contrast, our model suggests that even if the book holds no hidden limit order, the quote competition itself induces traders to undercut the best quote by one tick especially when the tick size is small. Harris (1996) shows that traders use hidden limit orders more frequently when the tick size is smaller in the Paris Bourse and the Toronto Stock Exchange. Thus, the smaller tick size can induce both aggressive fleeting limit orders and hidden limit orders. An empirical study on fleeting orders in exchanges which do not receive hidden orders is necessary to examine the intention of fleeting orders.

During cycles, a limit order posting behind the best quote can be optimal when the best quote has been very aggressive after quote-cutting. For duopoly with capacity constraints, Edgeworth (1925) discussed that when one seller had set a very low price, the other seller set a high price because the latter monopolized the residual demand. For the same reason, when the very aggressive limit sell orders have already been posted, a seller sets a high price in serving buyers arriving after these aggressive orders are cleared from the book. In the actual situation, traders who do not observe the book may submit limit orders behind the best quotes. However, such orders can also be submitted by rational traders who observe the book. Empirical studies report that limit orders behind the best quotes are not insignificant. For example, Griffiths et al. (2000) report that the ratio of the number of limit orders placed outside the best quotes relative to all orders is 13.16% (11.06%) for sell (buy) orders on the Toronto Stock Exchange. Hasbrouck and Saar (2002) report that the ratio is 30.5% in the Island ECN. It is necessary to investigate whether
these orders relate with cyclic quote dynamics.

When quotes jump or rebound, “holes” emerge in the book. At a hole there is no limit order even though limit orders currently exist at higher and lower prices. Numerical examples suggest that some transactions take place at the edge of holes, which can make widening bid-ask spreads fast and the distribution of transaction prices fat-tailed. In addition, we show that the size of a hole is greater the more frequently traders arrive and the more balanced the order flow is. Holes in the book have been observed by Biais \textit{et al.} (1995) on the Paris Bourse, by Irvine \textit{et al.} (2000) on the Toronto Stock Exchange, and by Sandås (2001) on the Stockholm Stock Exchange. These empirical studies found holes in the averaged book. Holes are not necessarily static. We leave the empirical investigation of how holes emerge in the book to future studies.

When the the tick size is large, there can be multiple equilibria. The tick size is the minimum cost in price for price priority, and affects how traders compete on price. We show that an Edgeworth cycle equilibrium exists if the tick size is small. Under the large tick size, the high cost for price priority inhibits quote-cutting, leading to the existence of an equilibrium where both sellers and buyers do not compete in price but queue at the same quotes. We call it queuing, and show that an equilibrium with queuing on both sides of the market is more likely to exist if the tick size is larger and the trader arrival rate is higher. Though these results are suggested by Foucault \textit{et al.} (2005), we also show the following relation between queuing and order imbalance.

Under the large tick size, there can be another type of equilibrium where only sellers queue at the same ask or only buyers queue at the same bid. We show that sellers (buyers) are more likely to queue at the same quote if sellers (buyers) outnumber buyers (sellers). Order imbalance to sell (buy) orders makes limit sell (buy) orders more aggressive; there remains little room of quote-cutting for sellers (buyers); sellers (buyers) queue at the same quote. In such a case, the depth at the ask (bid) is greater than the depth at the bid (ask). Order imbalance caused by information is considered not to increase the depth because traders use market orders in order to trade quickly. On the other hand, our model predicts that order imbalance caused by liquidity needs increase the depth on the big side of the market.

The bid-ask spared can be very small when traders queue at the same quote. This is because the first trader who arrives at the market submits a limit order at a very aggressive
price to prevent future traders from undercutting his price. Moreover, sellers and buyers can submit limit orders at the same price. In such a case, executions of market sell orders and executions of market buy orders take place at the same price even though the observed bid-ask spread is positive. In dealer markets, transaction prices bounce between the best ask and the best bid, which is the source of the profits of dealers who supply liquidity to the market. In contrast, bouncing of transaction prices is not necessarily observed in limit order markets because liquidity can be supplied by public traders. Transactions at a single price during continuous auction is not empirically investigated. However, Figure 3b presented in Biais et al. (1995) for an example of transactions in the Paris Bourse suggests such a possibility.

We obtain the above results when limit orders expire in two periods after their submission. For robustness, we present some numerical examples of equilibria under the three and four-period expiration of limit orders. One of the most important effects of longer expiration periods is to make limit orders more aggressive. Similar to Foucault (1999), limit order submitters in our model set quotes to which future traders on the opposite side of the market submit market orders. When limit orders survive longer periods, the execution probabilities of limit orders are higher, and the expected payoffs from limit orders are greater. Because limit orders are more attractive, traders submit more aggressive limit orders to bring out market orders from the opposite side of the market under the longer expiration periods. We conjecture that these aggressive limit orders reduce the size of holes.

In addition to the above-mentioned Foucault (1999) and Foucault et al. (2005), several studies have investigated the dynamic limit order markets. Parlour (1998) presents a model for a limit order market where the bid-ask spread is the same as the tick size. By contrast, we consider a situation in which the bid-ask spread is so wide relative to the tick size that traders compete on price. Goettler et al. (2005) solve for equilibrium numerically. Rosu (2008a) studies a continuous-time model with the zero tick size. Our analysis differs from existing studies in investigating the effect of the tick size and order imbalance. While these models including ours assume that information is symmetric among traders, Goettler et al. (2008) and Rosu (2008b) study the effect of information in limit order markets.

This article is organized as follows. Section 2 provides the model, and Section 3
presents an equilibrium under the one-period expiration of limit orders for the base case. In Section 4, we demonstrate quote dynamics in an Edgeworth cycle equilibrium under the two-period expiration of limit orders. Section 5 shows the possibility of another equilibrium under which traders queue at the same quotes when the tick size is large. In Section 6, we present numerical examples to show that equilibria exhibit Edgeworth cycles even if limit orders survive more than two periods. Section 7 is devoted to some concluding remarks. Theorem 2 in the Appendix formally presents a Markov perfect equilibrium with Edgeworth cycle. We put it in the Appendix because it is long and complex. In addition, all proofs can be found in the Appendix.

2 The Model

This section provides the model. We explain the types of traders, orders traders can choose, the state of the book, the trading rule, and equilibrium concept.

2.1 Types of traders

There are discrete and infinite periods, which are represented by \( \tau \in \{0, +1, \ldots, +\infty\} \). In each period, one potential trader arrives at the exchange and submits an order. Traders are risk neutral with the discount factor of one, and choose an order to maximize their expected payoffs. A trader can submit an order only when he arrives at the market. We assume that a trader himself cannot cancel or modify his order once he submits it.\(^3\)

A trader is either a seller, \( s \), or a buyer, \( b \). We denote the set of trader types by \( \Theta = \{s, b\} \). A seller holds one share of the asset and evaluates it as \( v_L \geq 0 \). A buyer holds no shares and evaluates a share as \( v_H \). We assume \( \Delta = v_H - v_L > 0 \). A payoff for a seller is \( P - v_L \) if he sells a share at a price \( P \) while a payoff for a buyer is \( v_H - P \) if he buys a share at a price \( P \). A trader receives zero payoff if he does not trade.

The trader arrival is stochastic in the following way. Let \( \alpha \in (0, 1) \) be the probability of a trader arriving, and \( \beta \in (0, 1) \) be the probability of a seller conditional on a trader arriving. That is, in each period, the trader is a seller with the probability \( \pi_s = \alpha \beta \in \)

\(^3\)In our model, a trader faces a static problem in choosing an order, which circumvents the complexity of a dynamic problem. Both Goettler et al. (2005) and Foucault et al. (2005) exclude, as we do, the possibility of resubmission of orders. An exception at present is Rosu (2008a), who assumes that traders can cancel and change orders at will.
(0, 1), and a buyer with the probability \( \pi_b = \alpha(1 - \beta) \in (0, 1) \). No trader arrives with the probability \( \pi_n = 1 - \pi_s - \pi_b \in [0, 1) \). Let \( \pi = (\pi_s, \pi_b) \) and \( \Pi = \{ \pi : 0 < \pi_s < 1, 0 < \pi_b < 1, \pi_s + \pi_b \leq 1 \} \). We assume that \( \pi \) is exogenous and constant.

### 2.2 The orders

Similar to typical limit order markets, an exchange allows a trader to submit a market sell order, a market buy order, a limit sell order, and/or a limit buy order. A trader specifies a price, or a quote, in submitting a limit order. The quote of a limit sell order is an ask, say \( A \), and the quote of a limit buy order is a bid, say \( B \). The exchange designates the tick size \( k > 0 \), which is the minimum price variation, and a trader must choose a quote from the pricing grid \( N_k = \{0, k, 2k, \ldots\} \). For simplicity, we assume that \( v_L \) and \( v_H \) are on the pricing grid, \( v_L \in N_k \) and \( v_H \in N_k \). We restrict the volume of each order to one share. In summary, the set of available orders is \( \bar{X} = \{\text{no order, a market sell order, a market buy order, a limit sell order at } A, \text{ a limit buy order at } B : A \in N_k, B \in N_k\} \). We say a limit buy order at \( B \) is symmetric to a limit sell order at \( A \) if \( A - v_L = v_H - B \), that is, if the limit orders yield the same payoff once executed.

We assume that limit orders automatically expire in two periods after their submission in Section 4. The analysis of limit order markets is complicated because of the large number of possible states of the book. The two-period expiration makes the set of books simple while it maintains direct quote competition of limit orders. For the base case, we present an equilibrium when limit orders automatically expire in one period in Section 3, which is investigated by Foucault (1999). In Section 6, we will discuss the case where limit orders expire in three or four periods by numerically solving equilibria.

If limit orders survive more than one period and if the tick size is zero, the optimal strategy may not exist because the maximum ask undercutting other asks and the minimum bid overbidding other bids do not exist due to the openness problem. To ensure the existence of optimal strategies, we need to assume the positive tick size \( k > 0 \) as in real exchanges. However, the discreteness in price can cause a multiplicity of equilibria. For example, Maskin and Tirole (1988) assume a positive tick size, yielding multiple equilibria. We will discuss the multiplicity of equilibria in Section 3.3.


2.3 The state of the book

When a limit order is submitted to an exchange, it is stored in the book until its execution or expiration. The book holds information regarding a price of a limit order and when the limit order is submitted. Let \( \omega \) be a state of the book, or simply a book, and let \( \Omega \) be the set of all states of the book. Because we assume the positive tick size and automatic expiration of limit orders, the number of the states of the book is finite. See Appendix A.1 for the concrete description of \( \Omega \).

A lower (higher) price for an ask (bid) is called a more aggressive price. In a book, the best ask (bid) is the most aggressive ask (bid). Let \( A^*(\omega) \) be the best ask and \( B^*(\omega) \) be the best bid of the book \( \omega \).

The bid-ask spread is the difference between the best ask and the best bid. In order to calculate the bid-ask spread for the book without a limit sell order or a limit buy order, we assume, similar to Seppi (1997) and Rosu (2008a), that a trading crowd implicitly provides limit sell orders at \( v_H \) and limit buy orders at \( v_L \). In other words, if the book does not have any limit sell (buy) order, the best ask (bid) is assumed to be \( v_H \) (\( v_L \)). The purpose of this assumption is to calculate the bid-ask spread for any book, and is irrelevant to an equilibrium.

From the perspective of Hasbrouck and Saar (2008), our model can be explained in the following way.\(^4\) Patient sellers submit hidden limit sell orders at \( v_H \), patient buyers submit hidden limit buy orders at \( v_L \), and impatient traders submit visible limit orders or market orders. Limit orders of impatient traders are fleeting because their cost of immediate execution drops in two periods after their submission, and they switch limit orders to market orders yielding the zero payoff.\(^5\)

\(^{4}\)Hasbrouck and Saar (2008) argue that patient investors use hidden orders to supply liquidity while impatient traders use limit orders priced inside the bid-ask spread. For a motive of cancellation of limit orders, they offer three hypotheses, and one of them is the cost-of-immediacy hypothesis, mentioning that limit orders are canceled and switched to market orders in response to a drop in the cost of immediate execution.

\(^{5}\)In an equilibrium of the model of Rosu (2008a), patient traders submit fleeting limit orders when the book is full of limit orders, and that order is executed immediately by a trader on the other side of the market. Thus, submission of fleeting limit orders is endogenized, one fleeting limit order is submitted at a time, and there is no quote competition between fleeting limit orders. In contrast, we consider the situation where multiple fleeting limit orders are exogenously submitted to the market.
2.4 The trading rule

We consider a transparent market with the pure price-time precedence rule and the discriminatory pricing rule. The specific rule used here is typical and is as follows: (1) the market is transparent in the sense that traders can observe the book when submitting an order. (2) The exchange treats a market sell order as the limit sell order at $v_L$, and a market buy order as the limit buy order at $v_H$. (3) A limit sell order at or below the best bid and a limit buy order at or above the best ask are called marketable. (4) If an incoming order is not marketable, it is stored in the book. If it is marketable, it is matched with an unfilled limit order on the opposite side of the book. (5) The priority among limit orders is assigned by price, and by time for limit orders at the same price due to a price-time precedence rule. (6) The transaction price is the price of the limit order waiting in the book due to the discriminatory pricing rule.

Under this trading rule, marketable orders are a market sell order, a limit sell order at or below the best bid, a market buy order, and a limit buy order at or above the best ask. In what follows, we call marketable orders market orders and the other orders limit orders. Market orders are executed at the best price in the book immediately after their submission while limit orders are stored in the book and await future market orders.

From the set of available orders, $\bar{X}$, a seller never chooses no order, any buy order, a limit sell order at or below $v_L$, or a limit sell order at or above $v_H$ in an equilibrium.\footnote{Because a buyer submits a market buy order to the book with a very aggressive limit sell order, there is always a limit sell order yielding the positive expected payoff for a seller. Thus, sellers never submit orders yielding zero or negative expected payoff, which are no order, any buy order, a limit sell order at or below $v_L$, or a market sell order to the book with a limit buy order at or below $v_L$. The orders which buyers never submit are analogous. Because buyers never submit a market buy order to the book with a limit sell order at or above $v_H$, a seller never submits a limit sell order at or above $v_H$. Symmetrically, a buyer never submits a limit buy order at or below $v_L$.} He chooses orders from a market sell order and limit sell orders whose price is above the best bid. Thus, we restrict the set of orders for a seller facing the book $\omega$ as $X(s, \omega) = \{a \text{ market sell order, a limit sell order at } A: A \in (B^*(\omega), v_H) \cap N_k\} \subseteq \bar{X}$. Similarly, the set of orders for a buyer facing the book $\omega$ is restricted to $X(b, \omega) = \{a \text{ market buy order, a limit buy order at } B: B \in (v_L, A^*(\omega)) \cap N_k\} \subseteq \bar{X}$.
2.5 Equilibrium concept

The strategy of a trader specifies an order he submits. We consider a Markov strategy which depends only on the type of trader and on a current book. In addition, we focus only on a pure strategy. We denote a pure Markov strategy of a trader $i \in \Theta$ facing the book $\omega \in \Omega$ as $x(i, \omega) \in X(i, \omega)$, and a profile of strategies as $x = \{x(i, \omega) : i \in \Theta, \omega \in \Omega\}$.

The expected payoff of a seller in submitting an order is as follows. Submission of a limit sell order at $A$, along with the expiration of the limit orders submitted two periods ago, if any, changes the book. After the transition of the book, the next trader arrives at the market according to the trader arrival $\pi$, and submits a new order according to the profile of strategies $x$. Thus, the execution probability of a limit sell order at $A$ depends on $\omega$, $\pi$, and $x$, and we denote it as $\Phi(s, A, \omega, \pi, x)$. The payoff of a market sell order is $B^*(\omega) - v_L$. Consequently, the expected payoff of an order $x \in X(s, \omega)$ for a seller is

$$V(s, x, \omega, \pi, x) = \begin{cases} 
\Phi(s, A, \omega, \pi, x)(A - v_L) & \text{if } x \text{ is a limit sell order at } A \\
B^*(\omega) - v_L & \text{if } x \text{ is a market sell order.}
\end{cases}$$

Symmetrically, the expected payoff of an order $x \in X(b, \omega)$ for a buyer is

$$V(b, x, \omega, \pi, x) = \begin{cases} 
\Phi(b, B, \omega, \pi, x)(v_H - B) & \text{if } x \text{ is a limit buy order at } B \\
v_H - A^*(\omega) & \text{if } x \text{ is a market buy order.}
\end{cases}$$

where $\Phi(b, B, \omega, \pi, x)$ represents the execution probability of a limit buy order at $B$ submitted to the book $\omega$ under $\pi$ and $x$.

As for equilibrium, we consider a pure-strategy Markov perfect equilibrium. A profile of pure Markov strategies $x^* = \{x^*(i, \omega) : i \in \Theta, \omega \in \Omega\}$ consists of an equilibrium if

$$x^*(i, \omega) \in \arg \max_{x \in X(i, \omega)} V(i, x, \omega, \pi, x^*) \quad \text{for } \forall i \in \Theta, \forall \omega \in \Omega.$$ 

Though a Markov perfect equilibrium allowing mixed strategies exists for which both actions and states are finite, a pure-strategy Markov perfect equilibrium does not necessarily exist. However, Theorem 2 in the Appendix demonstrates that a pure-strategy Markov perfect equilibrium indeed exists if the tick size is sufficiently small.

3 Equilibrium under one-period expiration

Before presenting an equilibrium under the two-period expiration of limit orders, let us revisit an equilibrium under the one-period expiration. Most of the implications in this
section have been discussed in Foucault (1999), and are valid for equilibrium under longer expiration periods of limit orders. Readers not interested in the detailed explanation can skip to Section 4.

Section 3.1 presents an equilibrium when traders can choose prices from real numbers, and Section 3.2 explains levels of quotes in the equilibrium. Section 3.3 presents numerical examples of pure-strategy Markov perfect equilibria under the positive tick size, and we discuss multiplicity of equilibria.

### 3.1 Equilibrium

The next theorem provides a unique equilibrium under the one-period expiration of limit orders. It is a special case of Proposition 3 in Foucault (1999) in the sense that the asset value does not change over time.

**Theorem 1** Suppose that limit orders expire in one period after their submission. If a limit order can be contingent on a price of a real number, the following profile of strategies is a unique equilibrium. Let $A_r$ and $B_r$ be

$$
A_r = \frac{(1 - \pi_s)v_H + \pi_s(1 - \pi_b)v_L}{1 - \pi_s \pi_b}, \quad B_r = \frac{\pi_b(1 - \pi_s)v_H + (1 - \pi_b)v_L}{1 - \pi_s \pi_b}.
$$

The strategy for a seller is to submit a market sell order if the book has a limit buy order at $B$ such as $B_r \leq B$, but otherwise to submit a limit sell order at $A_r$. The strategy for a buyer is to submit a market buy order if the book has a limit sell order at $A$ such as $A \leq A_r$, but otherwise to submit a limit buy order at $B_r$.

The proof is straightforward, since $A_r$ and $B_r$ are the solutions to the simultaneous equations

$$v_H - A_r = \pi_s(v_H - B_r), \quad (1)$$

$$B_r - v_L = \pi_b(A_r - v_L). \quad (2)$$

The limit prices $A_r$ and $B_r$ are constructed for a buyer to submit a market buy order to a limit sell order at $A_r$ if a seller submits a market sell order to a limit buy order at $B_r$, and vice versa.
3.2 The level of quotes under one-period expiration

As the following proposition states, the ask is higher than the bid due to the discontinuity of the execution probability in price

**Proposition 1** Under the equilibrium in Theorem 1, the ask is higher than the bid \( (A_r > B_r) \).

If the book has a limit buy order at \( B_r \), the execution probability of a limit sell order at \( A \in (B_r, A_r] \) is \( \pi_b \). The execution probability of a sell order is discontinuous in price at \( B_r \) because a seller can trade at \( B_r \) by a market sell order whose execution probability is unity. For a limit sell order at an ask slightly above \( B_r \), its gain in price relative to a market sell order cannot compensate for its loss in execution probability due to discontinuity, which inhibits a seller from submitting a limit sell order at \( A \in (B_r, A_r) \). Cohen et al. (1981) investigate an effect of this discontinuity on the bid-ask spread, referring to it as a “gravitational pull.” As we will see, the discontinuity of execution probability in price causes holes in the book under the two-period expiration.

Basically, the following relations between the trader arrival and the level of quotes under the one-period expiration are preserved even under the longer-period expiration.

**Proposition 2** Under the equilibrium in Theorem 1, (1) the ask \( A_r \) decreases in \( \pi_s \) given \( \pi_b \). The bid side is symmetric. (2) The ask \( A_r \) increases in \( \pi_b \) given \( \pi_s \). The bid side is symmetric. (3) For \( \beta > (3 - \sqrt{5})/2 \simeq 0.38 \), the ask \( A_r \) decreases in \( \alpha \) given \( \beta \). The bid side is symmetric. (4) The ask \( A_r \) decreases in \( \beta \) given \( \alpha \). The bid side is symmetric. (5) The expected bid-ask spread decreases in \( \alpha \) given \( \beta \). (6) Given \( \alpha \), the expected bid-ask spread increases in \( \beta \) if \( \beta < 1/2 \), decreases in \( \beta \) if \( \beta > 1/2 \), and is maximized at \( \beta = 1/2 \).

The ask \( A_r \) decreases in \( \pi_s \) as Proposition 2(1) states. In submitting a limit order, a trader has a monopoly power over future traders. However, he needs to satisfy participation constraints of future traders to induce them to submit market orders. Participation constraints of a future trader require that the expected payoff from a market order is equal to or higher than the expected payoff from a limit order. Because the higher execution probability \( \pi_s \) of a limit buy order raises the expected payoff of a buyer in submitting a limit buy order, a more aggressive ask is required to induce a future buyer to submit a market buy order.
Symmetrically, the ask $A_r$ increases in $\pi_b$ as Proposition 2(2) states. The higher execution probability $\pi_b$ of a limit sell order increases the expected payoff of a seller; a buyer submits a more aggressive limit buy order, which reduces his expected payoff; a less aggressive limit sell order is sufficient to extract a market buy order.

Proposition 2(3) suggests that comparative statics regarding $\alpha$ is not simple. Equations (1) and (2) suggest that not $\alpha$ and $\beta$ but $\pi_s$ and $\pi_b$ directly determine the level of quotes. Proposition 2(4) shows that when sellers (buyers) arrive at the market more frequently, both sellers and buyers post lower (higher) quotes.

Proposition 2(5) shows that the bid-ask spread is narrower under the higher trading activity. This relation is supported by many articles such as Lehmann and Modest (1994). On the other hand, Proposition 2(6) shows that the bid-ask spread is narrow when the order flow is not balanced. This relation is supported empirically by Handa et al. (2003) and Huang and Chou (2007).

3.3 Multiplicity of equilibria under the positive tick size

Theorem 1 demonstrates that an equilibrium is unique if limit orders expire in one period and if the tick size is zero. On the other hand, if limit orders survive more than one period, we need to assume the positive tick size $k > 0$ as in real exchanges to ensure the existence of optimal strategies. However, the discreteness in price can cause multiplicity of equilibria as we will show in this subsection.

Consider parameter values of $v_H = 3$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$ under one-period expiration of limit orders. The price of a limit order is restricted to one or two, and there are five states of the book, each of which is represented by an order in the book. Table 1 presents the states of the book and two equilibria, Equilibrium A and Equilibrium B, under these parameter values. For example, in Equilibrium A, a seller submits a limit sell order at 2 to the book with no order.

The discrete pricing grid can cause multiple equilibria because optimal strategies can be multiple. There are three equilibria under these parameter values, Equilibria A, Equilibrium B, and an equilibrium whose strategies are symmetric to those in Equilibrium B. Equilibrium A is a discretized version of a unique equilibrium in Theorem 1 in which $A_r = 2$ and $B_r = 1$. Under Equilibrium A, the optimal order for a seller to the book with a limit buy order at one is either a market sell order or a limit sell order at 2 because
both orders yield one as the expected payoff. Equilibrium A designates a seller to submit a market sell order while Equilibrium B designates a seller to submit a limit sell order at 2. As this example suggests, discreteness in price results in multiplicity of optimal strategies, which can lead to multiple equilibria.

Another source of multiple equilibria is that unconstrained prices may fail on the pricing grid. In this numerical example, if \( k \neq 1/n \) for some integer \( n \), \( A_r \) and \( B_r \) in Theorem 1 are not on the pricing grid. In such a case, there are multiple substitutes for unconstrained prices \( A_r \) and \( B_r \), which can lead to multiple equilibria. To avoid this kind of multiplicity of equilibria under the two-period expiration of limit orders, Theorem 2 in the appendix assumes that the critical prices belong to the pricing grid.\(^7\)

4 Edgeworth cycle equilibrium

This section is devoted to explaining an Edgeworth cycle equilibrium under the two-period expiration of limit orders. First, a corollary in Section 4.1 presents the equilibrium quote dynamics, and a numerical example in Section 4.2 demonstrates Edgeworth cycles. Next, we explain how equilibrium quotes are constructed. Though the numerical example in Section 4.2 is an equilibrium in our theorem, we have found multiple equilibria numerically. In Section 4.4, we document that numerically solved equilibria are close to an equilibrium in the theorem. Then, we explain the quote-cutting process, the level of quotes, and the size of holes emerging during an Edgeworth cycle.

There are three types of equilibria according to the trader arrival \( \pi \). Since the equilibrium quote dynamics are essentially the same for all types, this section explains mainly the case where sellers and buyers arrive proportionally, that is, \( \beta = 1/2 \) and \( \pi_s = \pi_b \), although the propositions in this section are not restricted to these parameter values. Refer to the Appendix for a complete description of an Edgeworth cycle equilibrium.

\(^7\)A positive tick size generates multiple equilibria in some models. For example, the subgame perfect equilibrium of the bargaining game in Rubinstein (1982) is unique when the set of alternatives is a continuum. In contrast, Van Damme et al. (1990) show that the bargaining game has multiple equilibria when the set of alternatives is finite due to a positive tick size. Another example is the model of oligopolistic markets in Maskin and Tirole (1988). They assume a positive tick size, yielding multiple equilibria.


4.1 Equilibrium quote dynamics

In an Edgeworth cycle equilibrium, there are four critical asks, $A_m$, $A_u$, $A_f$, and $A_l$, defined by $\pi_s$, $\pi_l$, $v_H$, and $v_L$. They satisfy $v_L < A_m < A_u < A_f < A_l < v_H$. As we will see, $A_f$ is the first ask submitted to an empty book; $A_u$ is the end of the range of one-tick quote-cutting; $A_m$ is the most aggressive ask; and $A_l$ is the least aggressive ask submitted on the equilibrium path. We suppose that the tick size $k$ is so fine that $A_m$, $A_f$, and $A_l$ are on the pricing grid. Since the critical ask $A_u$ may not be on the pricing grid, let $A_u^* \in [A_u, A_u + k) \cap N_k$, i.e., $A_u^*$ is the last ask of one-tick quote-cutting.

The next corollary stems from Theorem 2 in the Appendix, and presents the quote dynamics of an Edgeworth cycle equilibrium.

Corollary 1: In an Edgeworth cycle equilibrium in Theorem 2, if sellers arrive consecutively, the asks submitted on the equilibrium path are follows. The first ask posted in an empty book is $A_f$. After that, the ask declines from $A_f - k$ to $A_u^*$ tick by tick, jumps to $A_m$, rebounds to $A_l$, then falls to $A_f$. After the return to $A_f$, the same cycle repeats itself until a buyer arrives. A buyer submits a market buy order if the book has limit sell orders on the equilibrium path. The bid side is symmetric.

On the equilibrium path, the book does not have both limit sell orders and limit buy orders at the same time because a seller (buyer) submits a limit order to which future buyers (sellers) will submit market orders.

4.2 A numerical example

To present a numerical example, let $v_H = 21$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. The tick size $k = 1$ satisfies the conditions for the existence of an Edgeworth cycle equilibrium. Under these parameter values, $A_m = 5$, $A_u = 15/2$, $A_u^* = 8$, $A_f = 12$, and $A_l = 15$. Figure 1 illustrates an example of the equilibrium quote dynamics. The solid lines indicate the asks and the broken lines the bids. The two horizontal lines in one period mean that the book has two limit orders. For example, the book has a limit sell order at 12 in period 0, and limit sell orders at 11 and 12 in period 1. The point indicates a transaction. For example, a transaction takes place at price 10 in period 10.

In Figure 1, the first ten traders are assumed to be all sellers. After the first ask posted in an empty book at $A_f = 12$, the best ask decreases tick by tick from $A_f = 12$ to $A_u^* = 8$, 16
and then jumps from $A_u^* = 8$ to $A_m = 5$ by three ticks. When the book has a limit sell order at $A_m = 5$ as period 6, the next seller submits a limit sell order at $A_l = 15$, and the best ask remains at 5. Then, the same cycle of submission of limit sell orders starts over again until a buyer arrives, i.e., a limit sell order at $A_f = 12$ is submitted, and a limit sell order at $A_l = 5$ expires, which makes the best ask rebound seven ticks from 5 to 12. After that, the best ask walks down the pricing grid tick-by-tick again. In period 10, a buyer appears, a transaction takes place, and the book becomes empty. Quote dynamics of the bid side are symmetric as shown from period 11 to 20.

One caveat for an Edgeworth cycle equilibrium is that the probability of observing an actual Edgeworth cycle is small. For example, an Edgeworth cycle from the first $A_f$ to the next $A_f$ is observed if eight sellers arrive at the market consecutively, and the probability of a trader arrival in such a sequence is $\pi_8^s = 1/2^8$. In other words, order submissions are not necessarily inconsistent with an Edgeworth cycle equilibrium even though Edgeworth cycles are rarely observed.

During quote competition, quotes jump and holes emerge, causing rapid quote changes. In period 28 in Figure 1, the book has a large hole where an old limit sell order at $A_m = 5$ is posted along with a new limit sell order at $A_l = 15$. If a buyer arrives at this book, he submits a market buy order as shown in period 28, the limit sell order at $A_l = 5$ is executed, and the best ask falls back ten ticks from $A_m = 5$ to $A_l = 15$. If sellers arrive after such a large quote jump, the best ask returns to $A_f = 12$ as shown in period 29. On the other hand, if two buyers arrive consecutively when the asks in the book are $A_m = 5$ and $A_l = 15$, as in periods 36 and 37, a transaction at 5 is immediately followed by a transaction at 15. That is, due to holes in the book, widening the bid-ask spread is faster than narrowing the bid-ask spread, and transaction prices can be fat-tailed.

### 4.3 How to construct equilibrium quotes

This subsection explains the conditions for the critical quotes of an Edgeworth cycle equilibrium. To simplify the notation, let $A_u$ be the second most aggressive ask submitted on the equilibrium path, which is denoted by $A_u^*$ in the previous subsection. Analogous with the critical asks, let $B_f$ be the first bid submitted to an empty book, $B_u$ the second most aggressive bid, $B_m$ the most aggressive bid, and $B_l$ the least aggressive bid submitted on the equilibrium path. The following eight conditions from (3) to (10) specify the eight
critical quotes, \(A_m, A_u, A_f, A_l, B_m, B_u, B_f, \) and \(B_l\). There can be multiple solutions for the conditions from (3) to (10), and we will discuss multiplicity in the next subsection. Figure 2 depicts the relation between the four critical asks and the expected payoffs in the following when \(v_H = 21, v_L = 0, k = 1, \alpha = 1, \) and \(\beta = 1/2\). The parameter values are the same as those used for a numerical example presented in the previous section.

First, we explain how to construct the first ask \(A_f\) and the first bid \(B_f\) submitted to an empty book. A buyer facing an empty book chooses a bid to attract a market sell order from the next seller, and the next seller submits a market sell order if it yields a higher expected payoff than any limit order. Let \(v_m^s (v_m^b)\) be the expected payoff a seller (buyer) can receive by submitting not a market order but the optimal limit order to the book with a limit buy (sell) order at \(B_f (A_f)\) submitted in the previous period. Given \(v_m^s (v_m^b),\) a buyer (seller) facing an empty book chooses a limit order whose price is the lowest (highest) among prices to which the next seller (buyer) submits a market order. Thus, \(A_f\) and \(B_f\) satisfy

\[
B_f - v_L \geq v_m^s \geq B_f - v_L - k, \\
v_H - A_f \geq v_m^b \geq v_H - A_f - k.
\]

(3) (4)

The expected payoff \(v_m^s\) is determined as follows. In an equilibrium, if a seller submits a limit sell order at \(A_f\) to the book with a limit buy order at \(B_f\) submitted in the previous period, the limit sell order in the book crowds out the next seller, leading to submit a market sell order. As a result, the seller posting \(A_f\) can trade with the buyer after the next seller. Because he can also trade with the next buyer and the buyer after no trader, \(v_m^s = \pi_b(1 + \pi_n + \pi_s)(A_f - v_L).\) Analogously, \(v_m^b = \pi_s(1 + \pi_n + \pi_b)(v_H - B_f).\) Conditions (3) and (4) along with \(v_m^s\) and \(v_m^b\) offer the candidate of \(A_f\) and \(B_f.\)

Next, the least aggressive ask submitted on the equilibrium path, \(A_l\), satisfies the following condition. A seller does not compete with any seller in trading with the buyer after no trader and the buyer after the next buyer. Let \(v_f^b\) be the expected payoff these buyers can obtain. In order to receive the greatest payoff on the condition of trading with them, the seller submits a limit sell order at \(A_l\) such as

\[
v_H - A_l \geq v_f^b \geq v_H - A_l - k.
\]

(5)

Let the expected payoff of a limit sell order at \(A_l\) be \(v_l^s.\) By construction of \(A_l, v_l^s = \pi_b(\pi_b + \pi_n)(A_l - v_L).\) To any book, a seller can obtain \(v_l^s\) by submitting a limit sell order
at $A_l$. That is, $v^s_i$ is the reservation payoff for sellers, and no seller submits a limit sell order yielding the expected payoff less than $v^s_i$.

The most aggressive ask submitted on the equilibrium path, $A_m$, is determined as follows. By submitting a limit sell order at $A_m$, the seller can get the expected payoff of $\pi_b(1 + \pi_n + \pi_s)(A_m - v_L)$ because the next seller does not undercut $A_m$ by definition and because the seller can trade with the buyer after the next seller, the buyer after no trader, and the next buyer. In addition, no seller undercuts $A_m$ when further quote-cutting yields the expected payoff less than the reservation payoff $v^s_s$. Thus, $A_m$ satisfies

$$\pi_b(1 + \pi_n + \pi_s)(A_m - v_L) \geq v^s_i \geq \pi_b(1 + \pi_n + \pi_s)(A_m - v_L - k).$$  \hspace{1cm} (6)$$

In other words, if the best ask is larger than $A_m$, a seller has an incentive to undercut it. A limit sell order at $A$ such as $A > A_m$ is undercut by the future seller, and yields the expected payoff $\pi_b(1 + \pi_n)(A - v_L)$. Let $v^s_i$ be the expected payoff for a seller facing an empty book. Because he can submit a limit sell order at $A_f$, $v^b_f = \pi_b(1 + \pi_n)(A_f - v_L)$. Analogously, $v^b_f = \pi_s(1 + \pi_n)(v_H - B_f)$. Up to this point, Condition (5) along with $v^b_f$ determines $A_l$, $v^s_i$, and $A_m$.

A limit sell order at the second most aggressive ask $A_u$ must yield the equal or greater expected payoff than the expected payoff of a limit sell order at $A_m$. Otherwise, the seller submits a limit sell order not at $A_u$ but at $A_m$. Thus, $A_u$ satisfies

$$\pi_b(1 + \pi_n)(A_u - v_L) \geq v^s_i \geq \pi_b(1 + \pi_n + \pi_s)(A_u - v_L) \geq \pi_b(1 + \pi_n + \pi_s)(A_u - k - v_L).$$  \hspace{1cm} (7)$$

because $A_u$ is the lowest among asks above $A_m$. Thus far, we have the conditions for $A_f$, $B_f$, $A_l$, $A_m$, and $A_u$.

The bids, $B_m$, $B_l$, and $B_u$ are constructed analogously. Let $v^b_i$ be the reservation payoff for buyers, and $v^b_i = \pi_s(\pi_s + \pi_n)(v_H - B_l)$. Similar to Conditions (5), (6), and (7), $B_m$, $B_l$, and $B_u$ satisfy the following inequalities:

$$B_l - v_L \geq v^b_i \geq B_l - v_L - k,$$  \hspace{1cm} (8)$$

$$\pi_s(1 + \pi_n + \pi_b)(v_H - B_m) \geq v^b_i \geq \pi_s(1 + \pi_n + \pi_b)(v_H - B_m - k),$$  \hspace{1cm} (9)$$

$$\pi_s(1 + \pi_n)(v_H - B_u) \geq \pi_s(1 + \pi_n + \pi_b)(v_H - B_m) \geq \pi_s(1 + \pi_n)(v_H - B_u - k).$$  \hspace{1cm} (10)$$

As a whole, the eight critical quotes satisfy Conditions from (3) to (10).
4.4 The effect of tick size on critical quotes

If \( k = 0 \), the eight conditions from (3) to (10) uniquely determine the eight critical quotes, which are used in the definition of an Edgeworth cycle equilibrium in Theorem 2 in the Appendix. On the other hand, there can be multiple solutions satisfying the eight conditions if the tick size is positive.

Correspondingly, when we search for equilibrium numerically, we have found multiple equilibria for most parameter values. However, critical quotes of these equilibria satisfy the eight conditions if quotes in equilibria satisfy \( A_f > A_m \) and \( B_f < B_m \), if sellers (buyers) do not queue at \( A_f \) nor \( A_m \) (\( B_f \) nor \( B_m \)) on the equilibrium path, and if \( k/(v_H - v_L) \) is smaller than \( 1/7 \). In this sense, numerically solved equilibria with cyclic quote dynamics we have found are close to the Edgeworth cycle equilibrium described in Theorem 2 when the tick size is small relative to the difference of valuations \( v_H - v_L \). On the other hand, equilibria with queuing can exist if the tick size is large as we will show in Section 5.

4.5 Process of quote-cutting

The quote-cutting process proceeds as follows. Posting at \( A_f \) to an empty book is followed by quote-cutting until the best ask reaches the most aggressive ask, \( A_m \). A seller allows the future seller to undercut their ask because the cost of deterring quote-cutting is significant enough. By how much do sellers undercut the best ask? When the best extant ask \( A_{-1} \) is \( A_f \geq A_{-1} > A_m \), a seller obtains \( \pi_b(1+\pi_n)(A_{-1} - k - v_L) \) by one-tick quote-cutting while he obtains the expected payoff close to the reservation payoff \( v_s^a \) by submitting a limit sell order at the most aggressive level, \( A_m \). If the best ask is at or higher than \( A_u + k \), a seller prefers \( A_{-1} - k \) to \( A_m \) by definition of \( A_u \), and undercuts the best ask by only one tick so as to minimize the cost in price to assure a higher priority against the existing best ask.

When the best ask reaches \( A_u \), a seller submits a limit sell order at \( A_m \). Such an aggressive order deters further quote-cutting, and attains a high execution probability, which compensates for the large cost in price. If the best ask is \( A_m \), the next seller gives up undercutting it further, and submits a limit sell order at the least aggressive ask, \( A_l \), to receive the reservation payoff \( v_s^a \). After the submission of a limit sell order at \( A_l \), the next seller submits a limit sell order at \( A_f \), and the same cycle repeats itself. This is because the optimality of \( A_f \) for a seller facing an empty book implies that \( A_f \) is also the
optimal for a seller facing the book whose best ask is higher than $A_f$.

### 4.6 The level of quotes in Edgeworth cycle equilibrium

The conditions of the first quotes submitted to an empty book under the two-period expiration require that traders on the other side of the market submit market orders. When $k = 0$, Conditions (3) and (4) are

$$B_f - v_L = \pi_b(1 + \pi_n + \pi_s)(A_f - v_L),$$

(11)

$$v_H - A_f = \pi_s(1 + \pi_n + \pi_b)(v_H - B_f).$$

(12)

These conditions are similar to conditions under the one-period expiration, Equations (1) and (2). Thus, essentially the same conditions are applied, and we expect that some properties of quotes are similar even under the longer expiration periods of limit orders.\(^8\)

The next proposition shows that in an Edgeworth cycle equilibrium the ask submitted to an empty book is higher than the bid submitted to an empty book, which are similar to Proposition 1 under the one-period expiration. However, more aggressive quotes are also posted due to quote competition under the two-period expiration.

**Proposition 3** Under an Edgeworth cycle equilibrium, (1) the first ask submitted to an empty book, $A_f$, is higher than the first bid submitted to an empty book ($A_f - B_f > 0$). (2) The most aggressive ask submitted on the equilibrium path is lower than the most aggressive bid submitted on the equilibrium path ($B_m > B_u > A_u > A_m$).

Proposition 3(2) suggests that an outside dealer can make a profit by market orders if he buys an asset when an ask is low and sells it when a bid is high. This profitable opportunity would attract dealers to limit order markets.\(^9\)

---

\(^8\)If we assume that traders are heterogeneous in patience, impatient traders would submit market orders to less aggressive limit orders, and patient traders could trade by less aggressive limit orders. Thus, conditions such as Equations (11) and (12) would become less restrictive. Consequently, in the equilibrium in Foucault et al. (2005) and Rosu (2008a), who assume that traders are heterogeneous in patience, the first patient trader submits the least aggressive limit order to an empty book, and the following patient traders undercut the best quote. On the other hand, in this model and the model in Foucault (1999), traders are homogeneous in patience, and limit order submitters facing an empty book set prices for future traders on the opposite side to submit market orders.

\(^9\)Bloomfield et al. (2005) experimentally show an endogenous liquidity provision in limit order markets.

---
The relations between the trader arrival and the critical quotes are similar to those under the one-period expiration as the next proposition shows.

**Proposition 4** Under an Edgeworth cycle equilibrium, (1) the critical asks, $A_m$, $A_u$, $A_f$, and $A_l$ decrease in $\pi_s$ given $\pi_b$. The bid side is symmetric. (2) The critical asks, $A_m$, $A_u$, $A_f$, and $A_l$ increase in $\pi_b$ given $\pi_s$. The bid side is symmetric. (3) Suppose $\beta = 1/2$ and $\pi_s = \pi_b = \alpha/2$. The asks $A_f$ and $A_m$ decrease in $\alpha$. Let $a_1 = 2(\sqrt{2} - 1) \approx 0.83$. The asks $A_l$ and $A_u$ decrease in $\alpha$ for $\alpha < a_1$ but increase in $\alpha$ for $\alpha > a_1$. The bid side is symmetric. (4) The critical asks, $A_m$, $A_u$, $A_f$, and $A_l$ decrease in $\beta$ given $\alpha$. The bid side is symmetric.

Propositions 4(1) to (4) correspond to Propositions 2(1) to (4).

Limit orders are priced more aggressively under the longer expiration period because limit orders earn the greater execution probability and traders have to submit more aggressive orders to attract market orders. The following proposition formally states this property between one-period expiration and two-period expiration.

**Proposition 5** The difference between the first ask and the first bid submitted to an empty book, $A_f - B_f$, under the two-period expiration is smaller than the difference between the ask $A_r$ and the bid $B_r$ under the one-period expiration ($A_r - B_r > A_f - B_f$).

While Propositions 2(5) and (6) state that the expected bid-ask spread is larger if the trader arrival rate is lower and the order flow is more balanced under the one-period expiration, it is not easy to calculate the expected bid-ask spread under the longer expiration period because it depends on the number of steps of one-tick quote-cutting, which depends on the tick size. However, we conjecture that the similar properties hold under the two-period expiration because $A_f$ is the most frequently observed ask and it decreases in the trader arrival rate $\alpha$ and the order imbalance $\beta$ as Propositions 4(3) and (4) state. In addition, Proposition 5 implies that the expected bid-ask spread is narrower if limit orders expire in longer periods. We will present numerical examples consistent with these conjecture in Section 6.

4.7 The size of holes

Some empirical studies observe holes in the book. An Edgeworth cycle equilibrium offers an explanation for the existence of holes, which emerge in the book by quote-jumping and
quote-rebounding. There are three types of holes on the ask side in an Edgeworth cycle equilibrium: (1) a hole with the size of $A_u - A_m$ created by quote-jumping as in period 5, Figure 1; (2) a hole with the size of $A_l - A_m$ created by quote-rebounding as in period 6, Figure 1; and (3) a hole with the size of $A_l - A_f$ occurring when cyclical quote dynamics resume again from $A_f$ as in period 7, Figure 1. The first hole is observed more frequently than the others.

If $A_u - A_m > k$, a hole emerges in the book when the most aggressive ask $A_m$ is submitted. This hole is created by the discontinuity of the execution probability. In equilibrium, the execution probability of a limit sell order at $A$ is $\pi_b(1 + \pi_n + \pi_s)$ if $A \leq A_m$, whereas it is equal to or less than $\pi_b(1 + \pi_n)$ if $A > A_m$. That is, the execution probability is discontinuous in price at $A_m$ because of the price priority rule. For a limit sell order at an ask slightly above $A_m$, the gain in price does not compensate for the loss in execution probability because of the discontinuity. $A_u$ is the minimum price offsetting the loss in execution probability, and a seller never submits a limit sell order at $A \in (A_m, A_u)$, thereby creating a hole. A hole with the size of $A_l - A_f$ is also created by the discontinuity of the execution probability.

The size of holes depends on the trader arrival if the tick size is fine enough. The next proposition states that the size of holes is greater under the higher trader arrival rate because the gain in the execution probability is higher under the higher trader arrival rate.\footnote{A hole created by the most aggressive quote in this model is similar to the price impact at the most aggressive quote in Rosu (2008a). By assuming that sellers submit only limit sell orders and that buyers submit only market buy orders, his Proposition 4 suggests that the average price impact increases as the arrival rate of market order submitters, which is consistent with our results.}

\textbf{Proposition 6} Suppose $\beta = 1/2$ and $\pi_s = \pi_b = \alpha/2$. The sizes of the holes $A_u - A_m$ and $A_l - A_f$ increase in $\alpha$. Let $a_2 = \sqrt{6} - \sqrt{4\sqrt{6} - 6} \approx 0.50$. The size of the hole $A_l - A_m$ increases in $\alpha$ for $\alpha > a_2$ and decreases in $\alpha$ for $\alpha < a_2$. The sizes of holes on the bid side...
have the same relation with $\alpha$.

The next proposition relates with the size of holes and the order imbalance.

**Proposition 7** Suppose $\alpha = 1$. Thus, $\pi_s = \beta$ and $\pi_b = 1 - \beta$. Let $b_1$, $b_2$, and $b_3$ be solutions between zero and one of the equations presented in the proof in the Appendix, and $b_1 \simeq 0.3326$, $b_2 \simeq 0.6389$, and $b_3 \simeq 0.4274$. The size of the hole $A_u - A_m$ increases in $\beta$ for $\beta < b_1$ and decreases in $\beta$ for $\beta > b_1$. The size of the hole $A_l - A_f$ increases in $\beta$ for $\beta < b_2$ and decreases in $\beta$ for $\beta > b_2$. The size of the hole $A_f - A_u$ increases in $\beta$ for $\beta < b_3$ and decreases in $\beta$ for $\beta > b_3$. The sizes of holes on the bid side have similar relation with $\beta$.

When sellers (buyers) outnumber buyers (sellers), sellers (buyers) submit aggressive limit orders as Proposition 4 states, which collapse the holes. As a whole, the size of holes is smaller under the less balanced order flow.

As for the effect of the expiration periods on the size of holes, we conjecture that the size of the hole is smaller under the longer expiration periods. This is because traders submit more aggressive orders as Proposition 5 states, which leaves no room for further quote-cutting.\(^{12}\)

The size of holes relates with the execution probability, which may make it possible to infer the execution probability from the size of holes. By the construction of quotes on the edge of holes, limit orders at $A_u$, $A_m$, and $A_l$ yield the reservation payoff. More specifically, Equations (6) and (7) with $k = 0$ lead to the condition

$$
\pi_b(1 + \pi_n + \pi_s)(A_m - v_L) = \pi_b(\pi_n + \pi_s)(A_l - v_L) = \pi_b(1 + \pi_n)(A_u - v_L). \tag{13}
$$

Because the valuation of sellers and buyers, $v_L$ and $v_L$, can be inferred from the price range in the long run, we can infer the ex-ante trader arrival $\pi$ from the observed critical quotes. Thus, we can examine if the ex-ante trader arrival implied in the size of the hole is consistent with the ex-post trader arrival estimated by the actual order flow.\(^{13}\)

\(^{12}\)Foucault et al. (2005) show that quote-jumping is less likely under the narrow bid-ask spreads when the share of patient traders is large. Our conjecture is consistent with the results of Foucault et al. (2005) because the longer expiration periods of limit orders can correspond to more patient traders.

\(^{13}\)In contrast, every limit order in the book yields the same expected payoff in an equilibrium of Rosu (2008a) because he assumes that traders can freely modify their limit orders. In our model, short-term commitment of limit orders yields the monopoly profit to traders who arrive in the early stage of quote-cutting, and traders on the edge of holes obtain the same expected payoff.
5 Equilibrium with queuing

This section demonstrates that traders can queue at the same quotes if the pricing grid is coarse. Section 5.1 presents the condition under which sellers or buyers queue at the same quote. In Section 5.2, we discuss queuing on both sides of the market.

We define queuing in Definition 1 in the Appendix. Let $A_{q}$ be the ask and let $B_{q}$ be the bid. We call equilibrium strategies queuing at the ask if a seller submits a limit sell order at $A_{q}$ to the book with the two limit sell orders at $A_{q}$, if a buyer submits a market buy order to the book with a limit sell order at $A_{q}$ submitted two periods ago, and if some additional conditions are satisfied. We define queuing at the bid and queuing on both sides analogously. Our queuing does not restrict where traders queue. That is, traders may queue at the first quote submitted to an empty book, or they may queue at the best quote after the quote-cutting.

5.1 Queuing on one side of the market

In equilibrium, queuing on one side of the market does not occur when the tick size is small enough as the next theorem shows.

**Theorem 3** Let $k_{q1}^a$ and $k_{q1}^b$ be defined as

\[ k_{q1}^a = (1 - \pi_s)(1 - \pi_b)(1 - \pi_s(1 + \pi_n))/\{2 - \pi_b + (1 - \pi_s)(1 - \pi_b)\}, \]

\[ k_{q1}^b = (1 - \pi_s)(1 - \pi_b)(1 - \pi_b(1 + \pi_n))\{2 - \pi_s + (1 - \pi_s)(1 - \pi_b)\}. \]

If the tick size $k$ is equal to or smaller than $k_{q1}^a \Delta (k_{q1}^b \Delta)$, queuing at the ask (queuing at the bid) defined by Definition 1 in the Appendix can not occur in equilibrium.

The conditions for traders neither to overbid the extant quote nor to undercut the extant quote lead to the conditions in the above theorem. As a corollary, an equilibrium can exhibit queuing under the coarse pricing grid because the large tick size hinders quote-cutting by raising the cost in price to obtain price priority.

The next proposition relates queuing and the critical quotes in an Edgeworth cycle equilibrium.

**Proposition 8** Let $A_f$, $A_m$, $B_f$, and $B_f$ be quotes submitted in an Edgeworth cycle equilibrium in Theorem 2. $k_{q1}^a \Delta < A_f - A_m$ and $k_{q1}^b \Delta < B_m - B_f$ if $\alpha = 1$ or $\beta = 1/2$. 

25
That is, when the tick size is small enough for an equilibrium with queuing at the ask (bid) not to exist, the tick size is smaller than the difference between the first quote submitted to an empty book and the most aggressive quote submitted on the equilibrium path of an Edgeworth cycle equilibrium.

Proposition 8 implies that if \( k > A_f - A_m \), that is, if there is no room between \( A_f \) and \( A_m \) to submit limit sell orders at the multiple prices, sellers can queue at the same ask.

The higher the arrival rate of sellers (buyers) is, the more likely it is that sellers (buyers) will queue at the same quote as the next proposition shows.

**Proposition 9** (1) An equilibrium with queuing at the ask (queuing at the bid) less likely exists if \( \pi_s (\pi_b) \) is smaller given \( k \) and \( \pi_b (\pi_s) \). (2) Suppose \( \beta = 1/2 \). An equilibrium with queuing at the ask (queuing at the bid) less likely exists if \( \alpha \) is smaller given \( k \).

Thus, the depth increases on the thicker side of the market, that is, the depth at the ask (bid) increases because of queuing when sellers (buyers) outnumber buyers (sellers).

When a larger number of sellers arrive at the market, sellers submit more aggressive limit sell orders to induce buyers to submit market buy orders. Such aggressive limit sell orders reduce the range of prices that a trader can choose from under the large tick size, and sellers are more likely to queue at the same ask. The large depth at the ask can indicate that sellers want to trade by the liquidity need. In contrast, if bad news drives sellers to trade, they will not wait but use market orders to trade quickly.

### 5.2 Queuing on both sides of the market

Next, we consider the queuing on both sides of the market. Similar to queuing on one side, queuing on both sides can not occur if the tick size is small relative to the difference in valuations between sellers and buyers as the next theorem shows.

**Theorem 4** Let \( k_{q2}^s \) and \( k_{q2}^b \) be defined as

\[
k_{q2}^s = (1 - \pi_s)(1 - \pi_b)/\{1 + (1 - \pi_b)(2 - \pi_s)(1 + 2\pi_s - \pi_s^2)\},
\]

\[
k_{q2}^b = (1 - \pi_s)(1 - \pi_b)/\{1 + (1 - \pi_s)(2 - \pi_b)(1 + 2\pi_b - \pi_b^2)\}.
\]

If the tick size \( k \) is equal to or smaller than \( k_{q2}^s \Delta \) or \( k_{q2}^b \Delta \), an equilibrium with queuing on both sides defined by Definition 1 in the Appendix can not exist.
The conditions in the above theorem are derived from the conditions for traders neither
to overbid the extant quote nor to undercut the extant quote. For a numerical example,
if \( \pi_s = \pi_b = 1/2 \), \( k^s_{q2} = k^b_{q2} \) and \( 9k^s_{q2} < 1 < 10k^s_{q2} \), suggesting that an equilibrium with
queuing on both sides can exist when the difference in valuations between sellers and
buyers is equal to or smaller than nine ticks.

Similar to queuing on one side, traders are more likely to queue on both sides when
the trader arrival rate, \( \alpha \), is larger, as the following proposition implies.

**Proposition 10** Suppose \( \beta = 1/2 \). An equilibrium with queuing on both sides of the
market less likely exists if \( \alpha \) is smaller given \( k \).

When the trader arrival rate is high, the execution probability of limit orders is high,
and traders submit more aggressive limit orders to induce traders on the other side of the
market to submit market orders. These aggressive orders along with the large tick size
discourage further quote-cutting, and traders queue at the same quotes.

The next proposition shows that traders submit very aggressive limit orders in an
equilibrium with queuing.

**Proposition 11** Under an equilibrium with queuing on both sides, \( A_q - B_q \leq ((2 - \pi_b)/(1 - \pi_b) + (2 - \pi_s)/(1 - \pi_s))k - \Delta \).

For example, under the parameter values \( v_H = 6, v_L = 0, k = 1, \alpha = 1 \), and \( \beta = 1/2 \),
Proposition 11 states \( A_q - B_q \leq 0 \). In fact, there are numerical examples of queuing
equilibria whose quotes are \( (A_q, B_q) = (3, 3), (3, 4), (2, 3), \) and \( (2, 4) \). For an equilibrium
with \( (A_q, B_q) = (3, 3) \), sellers and buyers queue at the same quote. Consequently, though
the positive bid-ask spread can be observed, every transaction takes place at the same
price like a transition in a call auction. This case is similar to that reported in Figure
3b by Biais et al. (1995). Propositions 10 and 11 predict that a sequence of transactions
at the same price are frequently observed for actively traded stocks under the coarse tick
size.

6 Equilibria under longer expiration periods

So far, we have assumed that limit orders expire in two periods after their submission. In
this section, we demonstrate that numerical examples of equilibria under the three and
four-period expiration also exhibit similar characteristics proved under the two-period expiration. First, we summarize the results under the two-period expiration, and present some conjecture. Next, we explain the parameter values of examples. After that, we show properties of numerically solved equilibria. Finally, we discuss equilibrium in which the book holds both limit sell orders and limit buy orders at the same time on the equilibrium path.

6.1 Summary of characteristics under two-period expiration

We show the following characteristics of equilibrium under the two-period expiration of limit orders, and present consistent numerical examples under the longer expiration periods in this section. (1) Cyclic quote dynamics are observed when the tick size is small (Theorem 2); (2) Traders are more likely to queue at the same quote if the proportion of traders on the same side of the market is larger (Proposition 9); (3) Traders submit more aggressive limit orders if the trader arrival rate is higher (Proposition 4), if limit orders survive over longer periods (Proposition 5), and if the proportion of traders on the same side of the market is larger (Proposition 4); (4) The size of the hole is larger if the trader arrival rate is higher (Proposition 6).

Though unproved, we speculate regarding the following properties. (5) The bid-ask spread is narrower when the trader arrival rate is higher, the limit orders have longer expiration periods, or when the order flow is less balanced; (6) The size of holes is larger if limit orders expire in shorter periods; (7) Limit orders are more frequently submitted behind the market, and holes created by quote-rebounding are more frequently observed under the higher trader arrival rate and with the longer expiration periods of limit orders because of the greater possibility that the most aggressive limit orders are submitted; (8) The kurtosis of transaction prices positively relates with the frequency of holes to be observed under the balanced order flow.

The following are additional characteristics of equilibrium proved in this article though we do not present consistent numerical examples under the longer expiration periods of limit orders. (9) Traders are more likely to queue at the same quote if the tick size is larger (Theorems 3 and 4), and the trader arrival rate is higher (Propositions 9 and 10); (10) The size of holes is small when the order flow is far from balanced (Proposition 7).
6.2 Parameter values

We consider the parameter values $v_H = 14$, $v_L = 0$, and $k = 1$. For these values, the number of states of the book is 547 under the two-period expiration, 9,855 under the three-period expiration, and 166,531 under the four-period expiration. Table 2 reports characteristics of equilibria for the trader arrival rate $\alpha$ of $1/3$, $2/3$, and 1, under the respective two, three and four-period expiration of limit orders for the share of a seller $\beta$ of $1/2$. On the other hand, Table 3 reports characteristics of equilibria for the share of a seller $\beta$ of $1/2$, $3/5$, and $2/3$ under the respective two and four-period expiration of limit orders for the trader arrival rate $\alpha$ of 1.\footnote{We do not present numerical examples of equilibria under the three-period expiration of limit orders in Table 3 because we have not found equilibria for $\alpha = 1$ and $\beta \neq 1/2$. One possible reason we have not been able to find equilibrium for these cases is that a pure strategy Markov perfect equilibrium does not exist while a mixed strategy Markov perfect equilibrium does.} For equilibria under $\beta = 1/2$, strategies for sellers and buyers are symmetric. Statistics in the table are calculated from the stationary distribution of the states of the books on the equilibrium path.

6.3 Cyclic quote dynamics and queuing

Tables 2 and 3 show that cyclic quote dynamics are observed even if limit orders expire in more than two periods. The “First eight asks” in the tables report the asks submitted if the first seller arrives at an empty book and the other seven sellers arrive consecutively. Analogously, the “First eight bids” in Table 3 report the bids submitted if the first buyer arrives at an empty book and the other seven buyers arrive consecutively. For example, under the two-period expiration with the trader arrival of $\alpha = 1$ and the share of sellers of $\beta = 1/2$, there is an equilibrium where sellers submit limit sell orders at 8, 7, 6, 4, and 10, consecutively, and the next seller restarts the same cycle by submitting a limit sell order at 8. Under the four-period expiration with the trader arrival of $\alpha = 1$ and the share of sellers of $\beta = 1/2$, there is an equilibrium where sellers submit limit sell orders at 7, 6, 5, 4, and 3, consecutively, the next ask rebounds to 8, and the same cycle starts again. For other cases presented in the tables, equilibrium quote dynamics are also cyclic.

Table 3 indicates that sellers queue at 2 for $\alpha = 1$ and $\beta = 2/3$ under the two-period expiration, and that sellers queue at 1 for $\alpha = 1$ and $\beta = 2/3$ under the four-period expiration. Traders on the big side of the market queue at the same quote even under
the longer expiration periods of limit orders.

6.4 Order aggressiveness and the bid-ask spread

Tables 2 and 3 show that traders submit more aggressive limit orders when the trader arrival rate is higher, when limit orders expire in longer periods after their submission, and when the trader arrival on the same side of the market is larger.

The more aggressive limit orders traders submit, the narrower the expected bid-ask spread is. The expected bid-ask spread for each equilibrium is reported in the “Expected bid-ask spread” in the tables. The tables show that the expected bid-ask spreads become narrower as traders arrive at the market more frequently, as the order flow imbalance between sellers and buyers is greater, or as limit orders survive over longer periods. These examples are consistent with our conjecture.

6.5 Holes and order flow composition

Table 2 exhibits quote-jumping at the most aggressive ask for $\alpha = 2/3$ and $\beta = 1/2$ under the two-period expiration, for $\alpha = 1$ and $\beta = 1/2$ under the two-period expiration, and for $\alpha = 1$ and $\beta = 1/2$ under the three-period expiration. There is no quote-jumping at the most aggressive quotes for equilibria under the four-period expiration. These examples are consistent with our results such as holes by quote-jumping are more frequently observed under the higher trader arrival and under shorter periods of order expiration.

Because quote-jumping and quote-rebounding create holes, the probability of emerging holes relates with the order flow composition. In the “Order flow composition of sell orders” in the tables, we classify market orders as “Market order,” limit orders undercutting the best quote by more than one tick as “Cutting by more than one tick,” limit orders undercutting the best quote by one tick as “Cutting by one tick,” limit orders at the best quote as “At the market,” and limit orders behind the best quote as “Behind the best quote.” We classify the other orders, limit orders submitted to an empty book, as “Empty,” because our traders frequently submit such orders. Table 2 indicates that the share of limit orders submitted behind the best quote increases as the trader arrival rate or the expiration periods increase because of the small variation of Edgeworth cycles.

In Table 2, the “Probability of holes” denotes the probability of emerging holes in the book. As expected from the order flow composition, holes are more likely observed under
a higher trader arrival rate. When limit orders expire in longer periods, holes created by quote-jumping collapse while holes created by quote-rebounding are observed more frequently.

6.6 Distribution of transaction prices

The last row in Tables 2 and 3 report kurtosis of transaction prices for each equilibrium. Table 2 suggests that kurtosis tends to increase as the trader arrival rate increases or the expiration periods of limit orders increase. Under the higher trader arrival rate or the longer expiration periods, traders submit more aggressive limit orders, and quotes far from the average price are more frequently posted. Thus, transactions take place more frequently at the extreme prices, which increases kurtosis.

The distribution of transaction prices can be fat-tailed when holes frequently emerge in the book. This is because the frequency of submission of the most aggressive limit orders correlates with the frequency of quote-rebounding, and because quote-rebounding, in turn, correlates with the probability of emerging holes. However, this relation is valid only under a balanced order flow. For example, an equilibria for $\alpha = 1$ and $\beta = 2/3$ under the two-period expiration in Table 3 indicates that large kurtosis does not relate with the probability of hole emergence. When the order flow is not balanced, traders on the big side of the market more likely queue while traders on the small side of the market more likely submit jumping quotes because the gain in execution probability is considerable. Queuing decreases kurtosis and holes in the book while quote-jumping increases kurtosis and holes. The size of holes is larger under the shorter expiration period of limit orders, which makes kurtosis large for $\alpha = 1$ and $\beta = 2/3$ under the two-period expiration of limit orders in Table 3.

6.7 Quotes on both sides of the book

We conjecture that the book has both limit sell orders and limit buy orders at the same time on the equilibrium path when the tick size is large, and when the expiration periods of limit orders is long. The reason is as follows. In an Edgeworth cycle equilibrium, traders choose quotes of limit orders to which traders on the opposite side of the market submit market orders. Conversely, when a seller (buyer) gives up to trade with the next buyer (seller), the book can hold both limit sell orders and limit buy orders at the same
time. The large tick size makes the cost in price of quote-cutting large, and induces traders to give up to trade with the next trader on the opposite side of the market. The longer expiration periods raise the execution probability of limit orders, which makes it difficult for traders to submit market orders. As a result, the large tick size and the long expiration periods can lead to the book with limit orders on both sides.

7 Concluding remarks

We investigate how traders submit orders, how quotes change, and how transactions take place in limit order markets by assuming that limit orders expire in two periods after their submission. This assumption simplifies the state space of the book, but does not eliminate competition among limit orders.

We present cyclic quote dynamics with quote-jumping and quote-rebounding. A trader undercut the best quote by more than one tick if such an order can prevent future quote-cutting because its gain in execution probability compensates for its cost in price. When the book has very aggressive limit orders, a trader can submit a limit order behind the best quote because its gain in price offsets its loss in execution probability. These order submission strategies cause holes to emerge in the book.

An Edgeworth cycle equilibrium yields the following original insight on limit order markets. (1) Limit orders submitted behind the best quote are reasonable, and such orders make widening the bid-ask spread faster than its narrowing. (2) Holes appear when quotes jump and rebound while holes disappear during one-tick quote-cutting. (3) The sizes of holes are large if the trader arrival rate is high and if the order flow is balanced. (4) The depth is greater by queueing if the tick size is greater, if the trader arrival rate is higher, and if the order flow is less balanced. In addition, numerical examples suggest that (5) the distributions of transaction prices can be fat-tailed when holes are frequently observed. Investigating the empirical relevance of these features for limit order markets is an interesting topic for future empirical studies.

Finally, we conclude this article with a discussion of one issue left for future research. We assume limit orders are canceled exogenously, and resubmission is not considered in the model. However, traders can choose when to cancel their limit orders, and some limit orders are resubmitted after their cancellation. The exact nature of dynamic order
submission strategies with cancellation and resubmission is an important issue to be addressed in a future study.
Appendix

In this Appendix, we first describe the set of the state of the book and some notations to specify the model. After that, we present the definitions of variables in Theorem 2, Theorem 2 itself, along with some relevant remarks, and then its proof. Next, we define queuing, and prove Theorems 3 and 4. Finally, we prove the propositions.

A.1 The set of states of the book

The set of states of the book under the $T$-period expiration of limit orders is defined as follows. When limit orders expire in $T$ periods, the book has at most $T$ limit orders. The possible limit orders in the book in period $\tau$ are limit orders submitted from $\tau - T$ to $\tau - 1$. Let $A_{-\tau'} (B_{-\tau'})$ be the ask (bid) of a limit sell (buy) order submitted in $\tau'$ periods before $\tau$. $y_{s-\tau'}$ takes the value $A_{-\tau'} \in (v_L, v_H) \cap N_k$ if a limit sell order is submitted in $\tau - \tau'$ and if it is still listed in the book in $\tau$, but otherwise takes $v_H$. Similarly, let $y_{b-\tau'}$ take $B_{-\tau'} \in (v_L, v_H) \cap N_k$ if a limit buy order submitted in $\tau - \tau'$ is in the book in $\tau$, but $v_L$ otherwise. Thus, $y_{s-\tau'} \in (v_L, v_H) \cap N_k$ and $y_{b-\tau'} \in (v_L, v_H) \cap N_k$. By these notations the book is represented by $\omega = (y_{b-1}, y_{b-2}, y_{s-1}, y_{s-2})$. Take care that the book satisfies $Max[y_{s-T}, \cdots, y_{s-2}, y_{s-1}] < Min[y_{s-T}, \cdots, y_{s-2}, y_{s-1}]$ because marketable limit orders are executed immediately. For example, the set of states of the book under the two-period expiration of limit orders is

$$\Omega = \{(v_L, v_L, A_{-2}, A_{-1}) : A_{-1}, A_{-2} \in (v_L, v_H) \cap N_k\} \cup \{(B_{-2}, B_{-1}, v_H, v_H) : B_{-1}, B_{-2} \in (v_L, v_H) \cap N_k \}$$

$$\cup \{(B_{-2}, v_H, A_{-1}) : A_{-1}, B_{-2} \in (v_L, v_H) \cap N_k, B_{-2} < A_{-1}\}$$

$$\cup \{(v_L, B_{-1}, A_{-2}, v_H) : A_{-2}, B_{-1} \in (v_L, v_H) \cap N_k, B_{-1} < A_{-2}\}.$$ 

A.2 Notations

First, to indicate traders, let $t_0$ be a trader in a given period. $t_s$ and $t_b$ denote a seller and a buyer in the following period, respectively. A seller and a buyer after $t_s$ ($t_b$) are denoted by $t_{ss}$ ($t_{bb}$) and $t_{sb}$ ($t_{bs}$), respectively. If no trader arrives following $t_0$, a seller and a buyer in the two periods ahead of $t_0$ are denoted by $t_{ns}$ and $t_{nb}$.

Next, let the decision process of a trader in period $\tau$ be divided into two stages. At stage 1, he decides whether or not he will submit a market order. If he submits a market order, period $\tau$ terminates and the next period begins. If he does not, he proceeds to
stage 2 and decides whether or not he will submit a limit order. After his submission of a limit order or no order, period \( \tau \) terminates and the next period begins.

Finally, to denote the execution probability, let \( \pi^*_m = \pi_b(1+\pi_n+\pi_s) \), \( \pi^*_f = \pi_b(1+\pi_n) \), and \( \pi^*_s = \pi_b(\pi_b + \pi_n) \). \( \pi^*_m \) is the execution probability of a limit sell order to an empty book if the next buyer, the buyer after the next seller, and the buyer after no trader will submit market buy orders, and if the next seller does not undercut the best ask on the book. This is the highest possible execution probability of a limit sell order. On the other hand, \( \pi^*_f \) is the execution probability of a limit sell order to an empty book if the next seller undercuts the best ask and if the next buyer and the buyer after no trader will submit market buy orders. \( \pi^*_s \) is the execution probability of a limit sell order to which the buyer after no trader and the buyer after the next buyer submit market buy orders. \( \pi^*_m > \pi^*_f > \pi^*_s \). Similarly, let \( \pi^*_b = \pi_s(1+\pi_n+\pi_b) \), \( \pi^*_f = \pi_s(1+\pi_n) \), and \( \pi^*_s = \pi_s(\pi_s+\pi_n) \).

### A.3 Definitions for Theorem 2

The variables for an Edgeworth cycle equilibrium in Theorem 2 are defined as follows. There are three cases of equilibria according to the trader arrival \( \pi \in \Pi \). Let \( g_1 = (1-\pi_s)^2 - \pi_s \pi_b(1-\pi_b) \) and \( g_2 = (1-\pi_b)^2 - \pi_s \pi_b(1-\pi_s) \). We define \( \Pi_1 = \{ \pi : g_1 \geq 0, g_2 \geq 0 \} \cap \Pi \), \( \Pi_2 = \{ \pi : g_1 < 0 \} \cap \Pi \), and \( \Pi_3 = \{ \pi : g_2 < 0 \} \cap \Pi \). The set \( \{\Pi_1, \Pi_2, \Pi_3\} \) is the partition of \( \Pi \). The arrival rate of buyers \( \pi_b \) for \( \pi \in \Pi_2 \) is relatively small, as is that of sellers \( \pi_s \) for \( \pi \in \Pi_3 \). These disproportions cause the differences in equilibrium among \( \pi \in \Pi_1, \Pi_2, \) and \( \Pi_3 \) in critical price levels and in strategies off the equilibrium path.

The first ask \( A_f \) and the first bid \( B_f \) submitted to an empty book are defined as follows. For \( i \in \{1, 2, 3\} \), let \( A_{fi} \) and \( B_{fi} \) be

\[
A_{f1} = \frac{(1-\pi_s)^2 v_H + \pi_s(2-\pi_s)(1-\pi_b)^2 v_L}{1 - \pi_s \pi_b (2 - \pi_s)(2 - \pi_b)},
\]

\[
B_{f1} = \frac{\pi_b(1-\pi_s)^2(2-\pi_b)v_H + (1-\pi_b)^2 v_L}{1 - \pi_s \pi_b (2 - \pi_s)(2 - \pi_b)};
\]

\[
A_{f2} = \frac{((1-\pi_s)^2 + \pi_s \pi_b^2)v_H + \pi_s(2-\pi_s)(1-\pi_b)v_L}{1 - \pi_s \pi_b (2 - \pi_s)(2 - \pi_b)},
\]

\[
B_{f2} = \frac{\pi_b((1-\pi_s)^2 + \pi_s \pi_b)v_H + (1-\pi_b)v_L}{1 - \pi_s \pi_b (2 - \pi_s)(2 - \pi_b)};
\]

\[
A_{f3} = \frac{(1-\pi_s)v_H + \pi_s((1-\pi_b)^2 + \pi_s \pi_b)v_L}{1 - \pi_s \pi_b (2 - \pi_s)(2 - \pi_b)};
\]
\[ B_{f3} = \frac{\pi_b(1 - \pi_s)(2 - \pi_b)v_H + ((1 - \pi_b)^2 + \pi_s^2 \pi_b)v_L}{1 - \pi_s \pi_b(2 - \pi_s - \pi_b)}. \]

Furthermore, let \( A_f = A_{fi} \) and \( B_f = B_{fi} \) if \( \pi \in \Pi_i \) for \( i \in \{1, 2, 3\} \). \( A_f \) and \( B_f \) are continuous in \( \pi \in \Pi \) because \( A_{f1} = A_{f2} \) and \( B_{f1} = B_{f2} \) for \( \pi \in \Pi \) satisfying \( g_1 = 0 \) and because \( A_{f1} = A_{f3} \) and \( B_{f1} = B_{f3} \) for \( \pi \in \Pi \) satisfying \( g_2 = 0 \).

We obtain \( A_f \) and \( B_f \) for \( \pi \in \Pi_1 \) by solving Equations (11) and (12). For \( \pi \in \Pi_2 \), the definitions of \( A_f \) and \( B_f \) are slightly different from those for \( \pi \in \Pi_1 \) because \( \pi_b \) is relatively small, and the execution probability of a limit sell order is low. Suppose the book has a limit buy order at \( B_f \) submitted in \( \tau = -1 \). At stage 2, a seller \( t_0 \) in \( \tau = 0 \) submits a limit sell order at a higher ask \( A_l \) instead of \( A_f \) to compensate for the low execution probability due to such a small \( \pi_b \). By the limit sell order at \( A_l \), the seller \( t_0 \) can trade with \( t_{sb}, t_{nb}, \) and \( t_{bb} \) because \( t_s \) submits a market sell order to the limit buy order at \( B_f \) and \( t_b \) submits a limit buy order at \( B_f \). That is, the seller \( t_0 \) gets the expected payoff \( \pi_b(A_l - v_L) \) at stage 2. To attract a market sell order of \( t_0 \), the buyer in \( \tau = -1 \) submits a limit buy order at \( B_f = v_L + \pi_b(A_l - v_L) \), leading to

\[ B_f - v_L = \pi_b(\Delta - \pi^b_j(v_H - B_f)). \]  

Solving Equations (12) and (14) yields \( A_f \) and \( B_f \) for \( \pi \in \Pi_2 \). The symmetric equation to Equation (14) is

\[ v_H - A_f = \pi_s(\Delta - \pi^s_j(\Delta - v_L)). \]  

We obtain \( A_f \) and \( B_f \) for \( \pi \in \Pi_3 \) by solving Equations (11) and (15).

In Section 4.3, we present the discretized conditions of critical quotes for \( \pi \in \Pi_1 \). For \( \pi \in \Pi_2 \), Condition (3) is replaced by

\[ B_f - v_L \geq \pi_b(A_l - v_L) + nk \geq B_f - v_L - k, \quad n \in \{0, 1\}. \]  

For \( \pi \in \Pi_3 \), Condition (4) is replaced by

\[ v_H - A_f \geq \pi_s(v_H - B_l) + nk \geq v_H - A_f - k, \quad n \in \{0, 1\}. \]  

The critical quotes of every equilibrium with quote-cutting on both sides of the market we have found numerically satisfy these conditions if \( k/(v_H - v_L) \) is smaller than \( 1/7 \) as mentioned in Section 4.3. Though Conditions (14) and (15) suggest Conditions (16) and (17) with \( n = 0 \), the critical quotes of some equilibria satisfy the conditions with \( n = 1 \).
Once $A_f$ and $B_f$ are defined, the other variables are defined as follows. $v_f^b = \pi_f^b(v_H - B_f)$ is the expected payoff a buyer gets by submitting a limit buy order at $B_f$ to an empty book on the equilibrium path. $v_f^b$ is the maximum payoff for a buyer to submit a limit buy order. So, a buyer would submit a market buy order to a limit sell order at or below $A_l = v_H - v_f^b$. By submitting a limit sell order at $A_l$, a seller can get the reservation payoff $v_s^l = \pi_s^l(A_l - v_L)$ in any circumstance. $A_m = v_L + v_s^l / \pi_m^s$ is the ask for a seller to get the reservation payoff with the highest possible execution probability $\pi_m^s$. $A_u = v_L + v_s^l / \pi_f^s$ is the ask for a seller to get the reservation payoff with the execution probability $\pi_f^s$. A seller gets the reservation payoff by submitting a market sell order to a limit buy order at $B_e = v_L + v_s^l$. Let $A_d(B)$ be

$$A_d(B) = \arg\max A \in N_k \ s.t. \ v_L + (B - v_L)/\pi_f^s + k \geq A.$$\n
Given the bid at $B$ in the book, $A_d(B)$ is the maximum ask that induces the next seller not to undercut $A_d(B)$ by one tick but rather submit a market sell order to a limit buy order at $B$ because

$$B - v_L \geq \pi_f^s(A_d(B) - k - v_L).$$

Let $B_c = v_L + v_s^l \pi_f^s / \pi_m^s$. As shown by $v_f^s = \pi_f^s(A_d(B_c) - v_L)$ under $k = 0$, $B_c$ is the critical bid in the book that equates $v_f^s$ with the expected payoff from a limit sell order at $A_d(B)$ with the highest possible execution probability.

The other variables are defined symmetrically. $v_f^s = \pi_f^s(A_f - v_L)$ is the expected payoff a seller gets by submitting a limit sell order at $A_f$ to an empty book on the equilibrium path. A seller would submit a market sell order to a limit buy order at or above $B_l = v_L + v_f^s$, and the reservation payoff of a buyer is $v_f^b = \pi_f^b(v_H - B_l)$. $B_m = v_H - v_f^b / \pi_m^b$ is the bid for a buyer to get the reservation payoff with the highest possible execution probability $\pi_m^b$. $A_u = v_H - v_f^b / \pi_f^b$ is the bid for a buyer to get the reservation payoff with the execution probability $\pi_f^b$. A buyer gets the reservation payoff by submitting a market buy order to a limit sell order at $A_e = v_H - v_f^b$. $B_d(A)$ is defined as

$$B_d(A) = \arg\min B \in N_k \ s.t. \ v_H - (v_H - A)/\pi_f^b - k \leq B.$$\n
We define $A_c = v_H - v_f^b \pi_f^b / \pi_m^b$, which satisfies $v_f^b = \pi_f^b(v_H - B_d(A_c))$ under $k = 0$.  

37
We indicate variables for \( \pi \in \Pi \) by a subscript \( i \in \{1,2,3\} \). For example, \( v^b_{f_1} = \pi_s(1 + \pi_n)(v_H - B_{f_1}) \) and \( A_{k_1} = v_H - v^b_{f_1} \) denote the variables for \( \pi \in \Pi_1 \). Let \( D_1 = 1 - \pi_s\pi_0(2 - \pi_s)(2 - \pi_0) \) and \( D_2 = D_3 = 1 - \pi_s\pi_b(1 + \pi_n) \). By definition, \( D_1 \in (0,1) \) and \( D_2 = D_3 \in (0,1) \) for \( \pi \in \Pi \).

A.4 Theorem 2

\textbf{Theorem 2} Suppose that limit orders expire in two periods after their submission, and that \( A_j, B_j \in N_k \) for \( j \in \{l, f, h, c\} \), \( A_u + 1 < A_f \), and \( B_f \leq B_u - k \). The following profile of strategies is a pure strategy Markov perfect equilibrium for \( \pi \in \Pi \).

(s1): For \( (v_{lL}, v_{lL}, v_{hH}, v_{hH}) \) or \( (v_{lL}, v_{lL}, A_{-2}, v_{hH}) \), not depending on \( A_{-2} \), a seller submits a limit sell order at \( A_f \).

(s2): For \( (B_{-2}, v_{lL}, v_{hH}, v_{hH}) \), a seller submits a limit sell order at \( A_f \) if \( B_{-2} < B_l \) and a market sell order if \( B_l \leq B_{-2} \).

(s3): For \( (v_{lL}, B_{-1}, A_{-2}, v_{hH}) \) or \( (v_{lL}, B_{-1}, v_{hH}, v_{hH}) \), not depending on \( A_{-2} \), a seller submits a limit sell order at \( A_f \) if \( B_{-1} < B_c \), a limit sell order at \( A_d(B_{-1}) \) if \( B_c \leq B_{-1} < B_l \), a limit sell order at \( A_f(A_l) \) if \( B_l \leq B_{-1} < B_f \) and if \( \pi \in \Pi_1 \cup \Pi_3 \) \( (\pi \in \Pi_2) \), and a market sell order if \( B_f \leq B_{-1} \).

(s4): For \( (B_{-2}, B_{-1}, v_{hH}, v_{hH}) \), a seller submits a limit sell order at \( A_f \) if \( B_{-1} < B_c \) and if \( B_{-2} < B_l \), a limit sell order at \( A_d(B_{-1}) \) if \( B_c \leq B_{-1} < B_l \) and if \( B_{-2} < \pi_n(2 - \pi_b)(A_2(B_{-1}) - v_{lL}) + v_{lL} \), and a limit sell order at \( A_f(A_l) \) if \( B_l \leq B_{-1} < B_f \), if \( B_{-2} < B_f \), and if \( \pi \in \Pi_1 \cup \Pi_3 \) \( (\pi \in \Pi_2) \). In the other cases, a seller submits a market sell order.

(s5): For \( (v_{lL}, v_{lL}, v_{hH}), A_{-1} \) or \( (v_{lL}, v_{lL}, A_{-2}, A_{-1}) \), not depending on \( A_{-2} \), a seller submits a limit sell order at \( A_f \) if \( A_{-1} < A_m \), a limit sell order at \( A_m \) if \( A_m < A_{-1} < A_{u} + k \), a limit sell order at \( A_{-1} - k \) if \( A_u + k < A_{-1} \leq A_f \), and a limit sell order at \( A_f \) if \( A_f < A_{-1} \).

(s6): For \( (B_{-2}, v_{lL}, v_{hH}, A_{-1}) \), a seller submits a limit sell order at \( A_f \) if \( A_{-1} < A_m \) and if \( B_{-2} < B_c \), a limit sell order at \( A_m \) if \( A_m < A_{-1} < A_{u} + k \) and if \( B_{-2} < B_c \), a limit sell order at \( A_{-1} - k \) if \( A_u + k < A_{-1} \leq A_f \) and if \( B_{-2} < v_{lL} + \pi_0(1 + \pi_n)(A_{-1} - k - v_{lL}) \), and a limit sell order at \( A_f \) if \( A_f < A_{-1} \) and if \( B_{-2} < B_l \). In the other cases, a seller submits a market sell order.

(b1): For \( (v_{lL}, v_{lL}, v_{hH}, v_{hH}) \) or \( (B_{-2}, v_{lL}, v_{hH}, v_{hH}) \), not depending on \( B_{-2} \), a buyer submits a limit buy order at \( B_f \).

(b2): For \( (v_{lL}, v_{lL}, A_{-2}, v_{hH}) \), a buyer submits a market buy order if \( A_{-2} \leq A_f \) and a limit buy order at \( B_f \) if \( A_f < A_{-2} \).

(b3): For \( (B_{-2}, v_{lL}, v_{hH}, A_{-1}) \) or \( (v_{lL}, v_{lL}, A_{-1}, v_{hH}) \), not depending on \( B_{-2} \), a buyer submits a market buy order if \( A_{-1} \leq A_f \), a limit buy order at \( B_f(2B_f - B_{-1}) \) if \( A_f < A_{-1} \leq A_l \) if \( \pi \in \Pi_1 \cup \Pi_2 \) \( (\pi \in \Pi_3) \), a limit buy order at \( B_d(A_{-1}) \) if \( A_l < A_{-1} \leq A_c \), and a limit buy order at \( B_f \) if \( A_c < A_{-1} \).

(b4): For \( (v_{lL}, v_{lL}, A_{-2}, A_{-1}) \), a buyer submits a limit buy order at \( B_f \) if \( A_f < A_{-1} \leq A_l \), if \( A_f < A_{-2} \), and if \( \pi \in \Pi_1 \cup \Pi_2 \) \( (\pi \in \Pi_3) \). A buyer submits a limit buy order at \( B_d(A_{-1}) \) if \( A_l < A_{-1} \leq A_c \) and if \( v_{hH} - \pi_0(2 - \pi_s)(v_{hH} - B_d(A_{-1})) < A_{-2} \), and a limit buy order at \( B_f \) if \( A_c < A_{-1} \) and if \( A_f < A_{-2} \). In the other cases, a buyer submits a market buy order.

(b5): For \( (v_{lL}, B_{-1}, v_{hH}, v_{hH}) \) or \( (B_{-2}, B_{-1}, v_{hH}, v_{hH}) \), not depending on \( B_{-2} \), a buyer submits a limit buy order at \( B_f \) if \( B_{-1} < B_f \), a limit buy order at \( B_{-1} + k \) if \( B_f \leq B_{-1} \leq B_u - k \), a limit buy order at \( B_m \) if \( B_u - k < B_{-1} = B_m \), and a limit buy order at \( B_f \) if \( B_m \leq B_{-1} \).
For \((v_L, B_{-1}, A_{-2}, v_H)\), a buyer submits a limit buy order at \(B_f\) if \(B_{-1} < B_f\) and if \(A_l < A_{-2}\), a limit buy order at \(B_{-1} + k\) if \(B_f \leq B_{-1} \leq B_u - k\) and if \(v_H - \pi_s(1 + \pi_n)(v_H - B_{-1} - k) < A_{-2}\), a limit buy order at \(B_m\) if \(B_u - k < B_{-1} < B_m\) and if \(A_e < A_{-2}\), and a limit buy order at \(B_l\) if \(B_m \leq B_{-1}\) and if \(A_e < A_{-2}\). In the other cases, a buyer submits a market buy order.

**Remark 1:** Theorem 2 requires that the critical quotes satisfy some conditions for simplicity of representations of strategies. If critical quotes fail on the pricing grid, their multiple substitutes on the pricing grid may lead to multiple equilibria. To avoid this kind of complexity from discreteness in price, conditions \(A_j, B_j \in \mathbb{N}_k\) for \(j \in \{l, f, h, c\}\) are required. On the other hand, the condition \(A_u + k \leq A_f\) is required for \(A_{-1} \in [A_u + k, A_f]\) to exist for strategies in (s5) and (s6). Then, there is at least one one-tick quote-cutting. The condition \(B_f \leq B_u - k\) is required for reasons of symmetry. These conditions are satisfied if the tick size \(k\) is appropriately small. For example, \(k = 1/n\) for some integer \(n\) meets the conditions under the parameter values \(v_H = 21, v_L = 0, \alpha = 1,\) and \(\beta = 1/2\).

**Remark 2:** The difference in strategies between \(\pi \in \Pi_1\) and \(\pi \in \Pi_2\) is the limit price when \(B_{-1} \in [B_l, B_f]\) in (s3) and (s4). The difference in strategies between \(\pi \in \Pi_1\) and \(\pi \in \Pi_3\) is the limit price when \(A_{-1} \in (A_f, A_l]\) in (b3) and (b4).

**Remark 3:** Strategies in (s1) and (s5) relate to the quote dynamics on the ask side in Corollary 1.

### A.5 Proof of Theorem 2

Theorem 2 is proved simply by checking the optimality of a strategy for each type of trader and for every state of the book. The proof is constructed in Steps 1 and 2. In Step 1, we show that for \(\pi \in \Pi_1\), optimal strategies for a seller \(t_0\) are strategies from (s1) to (s6) in Theorem 2 if sellers and buyers arriving after \(t_0\) follow strategies in Theorem 2. A symmetric argument is applied to strategies from (b1) to (b6). In Step 2, we prove the case for \(\pi \in \Pi_2 \cup \Pi_3\).

Before Step 1, we present Lemmas 1, 2, and 3 to prove the theorem. The proofs of lemmas are straightforward.

**Lemma 1:** (1) For \(\pi \in \Pi, 0 < v^s_i < v^s_f < \Delta\) and \(0 < v^b_i < v^b_f < \Delta\). (2) For \(\pi \in \Pi, v_L < A_m < A_u < A_f < A_l < A_c \leq A_e < v_H\) and \(v_L < B_c \leq B_e < B_l < B_f < B_u < B_m < v_H\). (3) For \(\pi \in \Pi, B_f < A_f, B_e < A_m, B_m < A_c,\) and \(A_u < B_u\).
Lemma 2: For \( \pi \in \Pi \), \( \pi^b_s(A_c - v_L) < v^b_f \) and \( \pi^b_s(v_H - B_c) < v^b_f \).

Lemma 3: (1) For \( \pi \in \Pi_1 \), \( \pi^a_m(A_f - v_L) \geq \pi^b(A_l - v_L) \) and \( \pi^b_m(v_H - B_f) \geq \pi_s(v_H - B_l) \). (2) For \( \pi \in \Pi_2 \), \( \pi^a_m(A_f - v_L) < \pi^b(A_l - v_L) \) and \( \pi^b_m(v_H - B_f) > \pi_s(v_H - B_l) \). (3) For \( \pi \in \Pi_3 \), \( \pi^a_m(A_f - v_L) > \pi^b(A_l - v_L) \) and \( \pi^b_m(v_H - B_f) < \pi_s(v_H - B_l) \).

Step 1: the case for \( \pi \in \Pi_1 \).

(s1) Under the book \( (v_L, v_L, v_H, v_H) \) or \( (v_L, v_L, A_{-2}, v_H) \), if \( t_0 \) submits a limit sell order at \( A_0 \) at stage 2, the book becomes \( (v_L, v_L, v_H, A_0) \). If \( A_0 \leq A_f \), \( t_b \) submits a market buy order according to (b3). The possibility for \( t_{bb} \) to submit a market buy order as is follows: If \( A_f < A_0 \leq A_l \), \( t_b \) submits a limit buy order at \( B_f \) according to (b3). In this case, \( t_{bb} \) faces \( (v_L, B_f, A_0, v_H) \) and submits a market buy order according to (b6). If \( A_l < A_0 \leq A_c \), \( t_b \) submits a limit buy order at \( B_d(A_0) \), and \( t_{bb} \) faces \( (v_L, B_d(A_0), A_0, v_H) \). For \( A_0 \in (A_l, A_c] \), because \( A_c \leq A_c \) and because \( B_d(A) \) is non-decreasing in \( A \),

\[
B_f - k = v_H - (v_H - A_l)/\pi^b_f - k < v_H - (v_H - A_0)/\pi^b_f - k \leq B_d(A_0) \\
B_d(A_c) < v_H - (v_H - A_c)\pi^b_f = B_u,
\]

which implies that \( A_0 \leq v_H - \pi^a_f(v_H - B_d(A_0) - k) \) and \( B_f \leq B_d(A_0) \leq B_u - k \). Thus, \( t_{bb} \) submits a market buy order if \( A_l < A_0 \leq A_c \) according to (b6). If \( A_c < A_0 \), \( t_b \) submits a limit buy order at \( B_f \) and \( t_{bb} \) faces \( (v_L, B_f, A_0, v_H) \). In that case,

\[
v_H - \pi^a_f(v_H - B_f - k) = A_l + \pi^b_f k \leq A_l + k \leq A_c < A_0
\]

because \( A_l, A_c \in N_k \) and because \( A_l < A_c \) implies \( A_l + k \leq A_c \). Thus, \( t_{bb} \) submits a limit buy order at \( B_f + k \) according to (b6). On the whole, \( t_{bb} \) submits a market buy order if \( A_f < A_0 \leq A_c \). \( t_{sb} \) faces \( (v_L, v_L, A_0, v_H) \) and submits a market buy order if \( A_0 \leq A_l \) according to (b2). \( t_0 \)'s limit sell order at \( A_0 \) is matched with \( t_{sb} \)'s market buy order if \( A_0 \leq A_m \). The reason is as follows: if \( A_m < A_0 \), \( t_s \) submits a limit sell order strictly lower than \( min[A_0, A_f + k] \) according to (s5). Then, \( t_{sb} \) faces \( (v_L, v_L, A_0, A_l) \) with \( A_l \leq A_f \) and submits a market buy order according to (b4). \( t_{sb} \)'s market buy order is matched with \( t_s \)'s limit sell order because of price precedence. If \( A_0 \leq A_m \), \( t_s \) submits a limit sell order at \( A_l \) according to (s5). Then, \( t_{sb} \) faces \( (v_L, v_L, A_0, A_l) \) and submits a market buy order, which is matched with \( t_0 \)'s limit sell order.
In brief, if an ask \( A_0 \) at which \( t_0 \) submits a limit sell order is \( A_0 \leq A_m \), its execution probability is \( \pi^*_m \) because he can trade with \( t_b \), \( t_{nb} \), and \( t_{sb} \). If \( A_m < A_0 \leq A_f \), its execution probability is \( \pi_f^* \) because he can trade with \( t_b \) and \( t_{nb} \). If \( A_f < A_0 \leq A_l \), its execution probability is \( \pi_l^* \) because he can trade with \( t_{bb} \) and \( t_{nb} \). If \( A_l < A_0 \leq A_c \), its execution probability is \( \pi_c^*_b \) because he can trade with \( t_{bb} \). If \( A_c < A_0 \), its execution probability is zero.

The optimal asks among those with the same execution probability are \( A_m \), \( A_f \), \( A_l \), and \( A_c \), which yield the expected payoffs \( v_l^* \), \( v_f^* \), \( v_l^* \), and \( \pi_b^2(A_c - v_L) \), respectively. From Lemmas 1 and 2, it is optimal for \( t_0 \) to obtain \( v_f^* \) by submitting a limit sell order at \( A_f \).

(s2) Under the book \((B_{-2}, v_L, v_H, v_H)\) the decision at stage 2 is the same as that in (s1). At stage 1, if \( B_1 \leq B_{-2}, v_f^* \leq B_{-2} - v_L \) and \( t_0 \) submits a market sell order. In the other cases, he submits a limit sell order at \( A_f \).

(s3) Under the book \((v_L, B_{-1}, A_{-2}, v_H)\) or \((v_L, B_{-1}, v_H, v_H)\), if \( t_0 \) submits a limit sell order at \( A_0 > B_{-1} \), the book becomes \((B_{-1}, v_L, v_H, A_0)\). For the same reason in (s1), it can be matched with \( t_0 \)'s market buy order if \( A_0 \leq A_f \) and with \( t_{bb} \)'s market buy order if \( A_f < A_0 \leq A_c \). \( t_{nb} \) submits a market buy order if \( A_0 \leq A_l \) according to (b2). When the book is \((B_{-1}, v_L, v_H, A_0)\), \( t_s \) follows (s6). In the case where \( t_s \) submits a market sell order, \( t_{sh} \) faces \((v_L, v_L, A_0, v_H)\) and submits a market buy order if \( A_0 \leq A_l \) according to (b2). In the case where \( t_s \) does not submit a market sell order, if \( A_0 \leq A_m \), \( t_s \) submits a limit sell order at \( A_l \) and \( t_{sb} \) submits a market buy order, which is matched with \( t_0 \)'s limit sell order. In the other cases, \( t_0 \)'s limit sell order will not be executed.

When \( B_{-1} < B_c \), because of \( B_c < A_m \) from Lemma 1(3) and for the same reason as in (s1), it is optimal for \( t_0 \) to obtain \( v_f^* \) by submitting a limit sell order at \( A_f \). He does not submit a market sell order because \( B_c - v_L < v_f^* \).

When \( B_c \leq B_{-1} < B_l \), the optimal asks among those with the same execution probability are \( A_d(B_{-1}) \), \( A_f \), \( A_l \), and \( A_c \), which yield the expected payoffs \( \pi_m^*(A_d(B_{-1}) - v_L) \), \( v_f^* \), \( v_l^* \), and \( \pi_b^2(A_c - v_L) \), respectively. Because of Lemmas 1 and 2, the optimal asks are either \( A_f \) or \( A_d(B_{-1}) \). From the definition of \( A_d(B) \), we have

\[
v_L + (B_c - v_L)/\pi_f^* < A_d(B_c),
\]

\[
A_d(B_c - k) \leq v_L + (B_c - k - v_L)/\pi_f^* + k,
\]
which imply $v_f^s < \pi_m^s(A_d(B_e) - v_L)$ and

$$\pi_m^s(A_d(B_e - k) - v_L) \leq v_f^s + \pi_m^s(1 - 1/\pi_f^s)k < v_f^s.$$ 

Since $A_d(B)$ is non-decreasing in $B$, it is optimal for $t_0$ to submit a limit sell order at $A_f$ if $B_e \leq B_{-1} < B_f$ (if $B_e = B_c$, there is no $B_{-1}$ satisfying this condition) and to submit a limit sell order at $A_d(B_{-1})$ if $B_e \leq B_{-1} < B_l$. He does not submit a market sell order because $B_c - v_L < v_f^s$ and $B_l - v_L = v_f^s < \pi_m^s(A_d(B_{-1}) - v_L)$ for $B_e \leq B_{-1} < B_l$.

When $B_l \leq B_{-1} < A_f$, the optimal asks among those with the same execution probability are $A_f$, $A_l$, and $A_c$, which yield the expected payoffs $\pi_m^s(A_f - v_L)$, $\pi_b(A_l - v_L)$, and $\pi_c^s(A_c - v_L)$, respectively. Because of Lemmas 2 and 3, it is optimal for $t_0$ to receive $\pi_m^s(A_f - v_L)$ by submitting a limit sell order at $A_f$. However, if $B_f \leq B_{-1} < A_f$, $\pi_m^s(A_f - v_L) \leq B_{-1} - v_L$ and $t_0$ submits a market sell order. In short, $t_0$ submits a limit sell order at $A_f$ if $B_l \leq B_{-1} < B_f$, and a market sell order if $B_f \leq B_{-1} < A_f$.

When $A_f \leq B_{-1}$, $t_0$ submits a market sell order because it yields a higher expected payoff than a limit sell order at $A_f$ or $A_c$.

(s4) Under the book $(B_{-2}, B_{-1}, v_H, v_H)$, the decision at stage 2 is the same as that in (s3). At stage 1, $t_0$ submits a market sell order in the following cases. When $B_{-1} < B_c$, because he obtains $v_f^s$ at stage 2, he submits a market sell order if $B_{-2} \geq v_L + v_f^s = B_l$. When $B_c \leq B_{-1} < B_l$, because he obtains $\pi_m^s(A_d(B_{-1}) - v_L)$ at stage 2, he submits a market sell order if $B_{-2} \geq v_L + \pi_m^s(A_d(B_{-1}) - v_L)$. When $B_l \leq B_{-1} < B_f$, because he obtains $\pi_m^s(A_f - v_L)$ at stage 2, he submits a market sell order if $B_{-2} \geq B_f$. When $B_f \leq B_{-1}$, he submits a market sell order for the same reason as in (s3).

(s5) Under the book $(v_L, v_L, v_H, A_{-1})$ or $(v_L, v_L, A_{-2}, A_{-1})$, if $t_0$ submits a limit sell order at $A_0$, the book becomes $(v_L, v_L, A_{-1}, A_0)$. When $A_l < A_{-1}$, it is optimal for $t_0$ to receive $v_f^s$ by submitting a limit sell order at $A_f$ because the limit sell order at $A_f$ is optimal under the book $(v_L, v_L, v_H, v_H)$ from (s1), and because a higher price priority is assigned to $t_0$’s limit sell order than to the limit sell order at $A_{-1}$.

When $A_f < A_{-1} \leq A_l$, to get the execution probability $\pi_{\tilde{b}}$, $t_0$ has to submit a limit sell order at an ask lower than $A_c$ because of strategies in (b4). The execution probabilities for other asks are the same as those for (s1). Since the optimal ask is not $A_c$ but $A_f$ in (s1), it is optimal for $t_0$ to submit a limit sell order at $A_f$.

Next consider the case where $A_{-1} \leq A_f$. $t_b$ submits a market buy order according to (b4). If $A_0 < A_{-1}$, $t_0$’s limit sell order can be matched with $t_b$’s market buy order.
If $A_{-1} \leq A_0$, a limit sell order at $A_{-1}$ is matched with $t_b$’s market buy order. Then, $t_{bb}$ faces the book $(v_L, v_L, A_0, v_H)$ and submits a market buy order if $A_0 \leq A_l$ according to (b2). $t_{nb}$ submits a market buy order if $A_0 \leq A_l$ according to (b2). $t_s$ follows (s5). To get a higher price priority over $t_s$’s limit sell order, $t_0$ must submit a limit sell order such as $A_0 \leq A_m$. In this case, $t_{nb}$ submits a market buy order according to (b4), which is matched with $t_0$’s limit sell order.

When $A_{-1} \leq A_m$, the optimal asks among those with the same execution probability are $A_{-1} - k$, $A_m$, and $A_l$, which yield the expected payoffs $\pi^s_m(A_{-1} - k - v_L) < v^s_l$, $\pi^s_b(A_m - v_L) < v^s_l$, and $v^s_l$, respectively. Thus, $t_0$ submits a limit sell order at $A_l$ to obtain $v^s_l$.

When $A_m < A_{-1} < A_u + k$, the optimal asks among those with the same execution probability are $A_m$, $A_{-1} - k$ if $A_{-1} - k > A_m$, and $A_l$, which yield the expected payoffs $v^s_l$, $\pi^s_f(A_{-1} - k - v_L) < \pi^s_f(A_u - v_L) = v^s_l$, and $v^s_l$, respectively. He then submits a limit sell order at $A_m$ to obtain $v^s_l$.

When $A_u + k \leq A_{-1} < A_f$, the optimal asks among those with the same execution probability are $A_m$, $A_{-1} - k$, and $A_l$, which yield the expected payoffs $v^s_l$, $\pi^s_f(A_{-1} - k - v_L) \geq \pi^s_f(A_u - v_L) = v^s_l$, and $v^s_l$, respectively. Then, he submits a limit sell order at $A_{-1} - k$ to receive $\pi^s_f(A_{-1} - k - v_L)$.

To summarize, $t_0$ submits a limit sell order at $A_l$ to obtain $v^s_l$ if $A_{-1} \leq A_m$, a limit sell order at $A_m$ to obtain $v^s_l$ if $A_m < A_{-1} < A_u + k$, a limit sell order at $A_{-1} - k$ to obtain $\pi^s_f(A_{-1} - k - v_L)$ if $A_u + k \leq A_{-1} \leq A_f$, and a limit sell order at $A_f$ to obtain $v^s_f$ if $A_f < A_{-1}$.

(s6) Under the book $(B_{-2}, v_L, v_H, A_{-1})$, the optimal limit sell order and its expected payoff at stage 2 are just the same as those in (s5). At stage 1, when $A_{-1} < A_u + k$, $t_0$ submits a market sell order if $v^s_l \leq B_{-2} - v_L$, that is, $B_e \leq B_{-2}$. When $A_u + k \leq A_{-1} \leq A_f$, he submits a market sell order if $\pi^s_f(A_{-1} - k - v_L) \leq B_{-2} - v_L$. When $A_f < A_{-1}$, he submits a market sell order if $v^s_f \leq B_{-2} - v_L$, i.e., $B_l \leq B_{-2}$.

The above argument proves that the strategies of sellers in the theorem are optimal for $\pi \in \Pi_1$. We can use a symmetric argument for the strategies of buyers. We have thus proved the theorem for $\pi \in \Pi_1$.

Step 2: the case for $\pi \in \Pi_2 \cup \Pi_3$.

Consider the case $\pi \in \Pi_2$. Between $\pi \in \Pi_1$ and $\pi \in \Pi_2$, strategies differ in (s3) and (s4)
due to Lemma 3. First, we will check the optimality of (s3) for \( \pi \in \Pi_2 \). Under the book \((v_L, B_{-1}, A_{-2}, v_H)\) or \((v_L, B_{-1}, v_H, v_H)\), the execution probabilities and optimal strategies for \( B_{-1} < B_l \) are just the same as those for \( \pi \in \Pi_1 \). When \( B_l \leq B_{-1} < B_f \), the optimal asks among those with the same execution probability are \( A_f, A_l, \) and \( A_c \), which yield the expected payoffs \( \pi_m^a(A_f - v_L), \pi_b(A_l - v_L), \) and \( \pi_b^a(A_c - v_L) \), respectively. It is optimal for \( t_0 \) to receive \( \pi_b(A_l - v_L) \) by submitting a limit sell order at \( A_l \) if \( B_l \leq B_{-1} < B_f \) because of Lemmas 2 and 3(2). He submits a market sell order if \( B_f \leq B_{-2} \) because \( \pi_b(A_l - v_L) = B_f - v_L \). Thus, strategies in (s3) are optimal for \( \pi \in \Pi_2 \). The optimality of (s4) for \( \pi \in \Pi_2 \) is similar.

Differences in the strategies of sellers in (s3) and (s4) do not affect sellers’ other strategies since the expected payoffs of sellers are affected by other sellers only through strategies in (s5) and (s6). In addition, it does not affect the optimal strategies of buyers, either. To verify this, consider the symmetric case \( \pi \in \Pi_3 \). Strategies in (b3) and (b4) for \( \pi \in \Pi_3 \) are different from those for \( \pi \in \Pi_1 \). These differences do not affect the strategies of sellers for the following reason. In proving the case \( \pi \in \Pi_1 \), strategies in (b3) and (b4) are explicitly used to confirm the optimality of strategies in (s1). Concerning (s1), when \( t_0 \) submits a limit sell order at \( A_0 \) such as \( A_f < A_0 \leq A_l, t_0 \) submits a limit buy order not at \( B_f \) but at \( B_l \) according to (b3) when \( \pi \in \Pi_3 \). Then, \( t_0 \) faces \((v_L, B_l, A_0, v_H)\) and submits a market buy order according to (b6). This does not change \( t_0 \)'s expected payoff. Similarly, for the other states of the book, the differences of strategies in (b3) and (b4) between \( \pi \in \Pi_1 \) and \( \pi \in \Pi_3 \) do not affect the expected payoffs of sellers. By symmetric argument, the differences of strategies in (s3) and (s4) between \( \pi \in \Pi_1 \) and \( \pi \in \Pi_2 \) do not affect the optimal strategies of buyers. We have thus proved the theorem for the case \( \pi \in \Pi_2 \), thereby completing the proof of Theorem 2 since the case for \( \pi \in \Pi_3 \) is symmetric. Q.E.D.

### A.6 Definition of queuing

Definition 1: Suppose limit orders expire in two periods after their submission. (1) Queuing on both sides of the market is defined by strategies of the following (q1) to (q8). (2) Queuing at the ask is defined by strategies of the following (q1), (q2), (q3), and (q4'). (3) Queuing at the bid is defined by strategies of the following (q5), (q6), (q7), and (q8'). Let \( A_q \in (v_L, v_H) \cap N_k \) be the ask and \( B_q \in (v_L, v_H) \cap N_k \) be the bid.
(q1) A seller submits a limit sell order at \( A_q \) to \((v_L,v_L,A_q,A_q)\).

(q2) A buyer submits a market buy order to \((v_L,v_L,A,y^b_+)\) where \( \forall A \in (v_L,A_q] \cap N_k \)
and \( \forall y^b_+ \in (v_L,v_H] \cap N_k \).

(q3) A seller submits a limit sell order at \( A' \in [A_q,v_H] \cap N_k \) to \((v_L,v_L,y^s_-,A_q-k)\)
where \( \forall y^s_- \in (v_L,v_H] \cap N_k \).

(q4) Let \( A_{q2} \) be
\[
A_{q2} = \max_{A \in (v_L,v_H] \cap N_k} A \text{ s.t. } v_H - A \geq \pi^b_m(v_H - B_q).
\]
When \( A_{q2} > A_q \), a seller submits a limit sell order at \( A' \in (v_L,A_{q2}) \cap N_k \) to \((v_L,v_L,A_q,A_{q2})\)
and a buyer submits a market buy order to \((v_L,v_L,A_{q2},v_H)\).

(q4') Let \( A_{q1} \) be
\[
A_{q1} = \max_{A \in (v_L,v_H] \cap N_k} A \text{ s.t. } v_H - A \geq \pi^b_f(v_H - B')
\]
for some \( B' \in (v_L,v_H - k) \cap N_k \). When \( A_{q1} > A_q \), a seller submits a limit sell order
at \( A' \in (v_L,A_{q1}) \cap N_k \) to \((v_L,v_L,A_q,A_{q1})\) and a buyer submits a market buy order to \((v_L,v_L,A_{q1},v_H)\).

(q5) A buyer submits a limit buy order at \( B_q \) to \((B_q,B_q,v_H,v_H)\).

(q6) A seller submits a market sell order to \((B,y^b_+v_H,v_H)\) where \( \forall B \in [B_q,v_H] \cap N_k \)
and \( \forall y^b_+ \in [v_L,v_H] \cap N_k \).

(q7) A buyer submits a limit buy order at \( B' \in (v_L,B_q] \cap N_k \) to \((y^b_-,B_q + k,v_H,v_H)\)
where \( \forall y^b_- \in [v_L,v_H] \cap N_k \).

(q8) Let \( B_{q2} \) be
\[
B_{q2} = \min_{B \in (v_L,v_H] \cap N_k} B \text{ s.t. } B - v_L \geq \pi^s_m(A_q - v_L).
\]
When \( B_{q2} < B_q \), a buyer submits a limit buy order at \( B' \in (B_{q2},v_H] \cap N_k \) to \((B_q,B_{q2},v_H,v_H)\)
and a seller submits a market sell order to \((B_{q2},v_L,v_H,v_H)\).

(q8') Let \( B_{q1} \) be
\[
B_{q1} = \min_{B \in (v_L,v_H] \cap N_k} B \text{ s.t. } B - v_L \geq \pi^s_m(A' - v_L)
\]
for some \( A' \in (v_L + k,v_H) \cap N_k \). When \( B_{q1} < B_q \), a buyer submits a limit buy order
at \( B' \in (B_{q1},v_H] \cap N_k \) to \((B_q,B_{q1},v_H,v_H)\) and a seller submits a market sell order to \((B_{q1},v_L,v_H,v_H)\).
Remark 1: Though the conditions characterizing queuing itself are (q1) and (q5), we require the other conditions in proving Theorems 3 and 4. (q2) demands that a buyer submits a market buy order to \((v_L, v_L, A_q, A_q)\). Under (q3), no successive quote-cutting occurs on the ask side, and a seller submits a limit sell order at \(A_q - k\) when he undercuts the best ask. Under (q4), a seller who submits a limit sell order at \(A_q\) can trade with the buyer after no trader and with the buyer after the next buyer by giving them the highest possible expected payoff \(\pi_b(v_H - B_q)\) given (q5) and (q6). (q4) demands that a seller submits a limit sell order at \(A_q\) if the previous seller overbids \(A_q\). (q4) is the condition for both sellers and buyers to queue while (q4') is the condition for only sellers to queue. (q5) to (q8) are the conditions for buyers to queue, and are analogous to the conditions (q1) to (q4) for the ask side.

Remark 2: The conditions from (q1) to (q8) are neither necessary nor sufficient for transactions to take place only at \(A_q\) or \(B_q\).

A.7 Proof of Theorem 3

Consider the case of queuing at the ask, and the case of queuing at the bid is analogous. First, we examine the condition where a seller has no incentive to overbid \(A_q\). Next, we examine the condition where a seller has no incentive to undercut \(A_q\). Finally, combining these conditions leads to the theorem.

The condition where a seller has no incentive to overbid \(A_q\) is as follows. Suppose \(A_{q1}\) in (q4') satisfies \(A_{q1} > A_q\). A seller \(t_0\) facing \((v_L, v_L, v_H, A_q)\) obtains the expected payoff \(\pi_b(A_q - v_L)\) by submitting a limit sell order at \(A_q\) because his limit sell order is matched with market buy orders of \(t_{sb}, t_{nb},\) and \(t_{bb}\) due to (q1) and (q2). Similarly, \(t_0\) facing \((v_L, v_L, v_H, A_q)\) gets the expected payoff \(\pi_b(\pi_n + \pi_b)(A_q - v_L)\) by submitting a limit sell order at \(A_{q1}\) due to (q2) and (q4'). A seller prefers \(A_q\) to \(A_{q1}\) if

\[
\pi_b(A_q - v_L) \geq \pi_b(\pi_n + \pi_b)(A_{q1} - v_L). \tag{18}
\]

As a result, if \(A_{q1} > A_q\), queuing at the ask satisfies Condition (18). If \(A_{q1} \leq A_q\), Condition (18) is always satisfied. Thus, Condition (18) is satisfied for every equilibrium with queuing at the ask. By the definition of \(A_{q1}\), \(\pi_f^b(v_H - B') > v_H - A_{q1} - k\) for some \(B'\). Applying this to Condition (18) yields

\[
A_q - v_L > (1 - \pi_s)(\Delta - k) - \pi_s(1 - \pi_s)(1 + \pi_n)(v_H - B'). \tag{19}
\]
for some $B'$ under which a seller facing the book $(v_L, v_L, v_H, A_q)$ has no incentive to overbid $A_q$.

Next, we examine the condition where a seller has no incentive to undercut $A_q$. If a seller $t_0$ facing $(v_L, v_L, v_H, A_q)$ submits a limit sell order at $A_q - k$, $t_0$ obtains the expected payoff $\pi^s_m(A_q - k - v_L)$ due to (q2) and (q3). Under (q3), a seller has no incentive to undercut more than one tick. Because a limit sell order at $A_q$ yields $\pi_b(A_q - v_L)$, $t_0$ does not undercut $A_q$ but queues at $A_q$ if

$$\pi_b(A_q - v_L) \geq \pi^s_m(A_q - k - v_L).$$

This condition is transformed to

$$A_q \leq v_L + (2 - \pi_b)k/(1 - \pi_b). \quad (20)$$

There is no $A_q$ satisfying Conditions (19) and (20) for $\forall B' > v_L$ if

$$v_L + (1 - \pi_s)(\Delta - k) - \pi_s(1 - \pi_s)(1 + \pi_n)(v_H - v_L) \geq v_L + (2 - \pi_b)/(1 - \pi_b)k.$$  

This condition is transformed to

$$k \leq (1 - \pi_s)(1 - \pi_b)(1 - \pi_s(1 + \pi_n))\Delta / \{2 - \pi_b + (1 - \pi_s)(1 - \pi_b)\},$$

which is the condition described in the theorem. Q.E.D.

### A.8 Proof of Theorem 4

The proof proceeds as follows. First, we consider the condition where a trader has no incentive to submit a limit order behind the best quote. Second, we consider the condition where a trader has no incentive to submit a limit order inside the bid-ask spread. Combining these conditions yields Theorem 4.

First, we examine the condition where a seller has no incentive to submit a limit order behind $A_q$. Suppose $A_{q2}$ in (q4) satisfies $A_{q2} > A_q$. A seller $t_0$ facing $(v_L, v_L, v_H, A_q)$ obtains the expected payoff $\pi_b(A_q - v_L)$ by submitting a limit sell order at $A_q$ under (q1) and (q2). Similarly, $t_0$ facing $(v_L, v_L, v_H, A_q)$ obtains the expected payoff $\pi_b(\pi_n + \pi_b)(A_{q2} - v_L)$ by submitting a limit sell order at $A_{q2}$ under (q2) and (q4). A seller prefers $A_q$ to $A_{q2}$ if

$$\pi_b(A_q - v_L) \geq \pi_b(\pi_n + \pi_b)(A_{q2} - v_L). \quad (21)$$
As a result, if $A_{q2} > A_q$, an equilibrium with queuing on both sides satisfies Condition (21). If $A_{q2} \leq A_q$, Condition (21) is always satisfied. Thus, Condition (21) is satisfied for every equilibrium with queuing on both sides.

By the definition of $A_{q2}$, $\pi^b_m(v_H - B_q) > v_H - A_{q2} - k$. Applying this to Condition (21) yields

$$A_q - v_L > (1 - \pi_s)(\Delta - k) - \pi_s(1 - \pi_s)(2 - \pi_s)(v_H - B_q).$$  \hfill (22)

The symmetric condition for a buyer is

$$v_H - B_q > (1 - \pi_b)(\Delta - k) - \pi_b(1 - \pi_b)(2 - \pi_b)(A_q - v_L).$$  \hfill (23)

Every equilibrium with queuing on both sides satisfies Conditions (22) and (23).

Next, we examine the condition where a seller has no incentive to undercut $A_q$. A seller facing the book $(v_L, v_L, v_H, A_q)$ obtains the expected payoff $\pi_b(A_q - v_L)$ by submitting a limit sell order at $A_q$ under (q1) and (q2). If a seller $t_0$ facing $(v_L, v_L, v_H, A_q)$ submits a limit sell order at $A_q - k$, $t_0$ obtains the expected payoff $\pi^s_m(A_q - k - v_L)$ under (q2) and (q3). Thus, $t_0$ does not undercut $A_q$ but queues at $A_q$ if

$$\pi_b(A_q - v_L) \geq \pi^s_m(A_q - k - v_L).$$

This condition is transformed to

$$A_q \leq v_L + (2 - \pi_b)k/(1 - \pi_b).$$  \hfill (24)

The symmetric condition for a buyer not to undercut $B_q$ is

$$B_q \geq v_H - (2 - \pi_s)k/(1 - \pi_s).$$  \hfill (25)

An equilibrium with queuing on both sides satisfies Conditions (22), (23), (24), and (25). Accordingly, if there are no $A_q$ and $B_q$ satisfying these four conditions, an equilibrium with queuing on both sides does not exist. There are no $A_q$ and $B_q$ satisfying these four conditions if

$$A_x - v_L \leq (1 - \pi_s)(\Delta - k) - \pi_s(1 - \pi_s)(2 - \pi_s)(v_H - B_x)$$  \hfill (26)

or

$$v_H - B_x \leq (1 - \pi_b)(\Delta - k) - \pi_b(1 - \pi_b)(2 - \pi_b)(A_x - v_L)$$  \hfill (27)

are satisfied at $A_x = v_L + (2 - \pi_b)k/(1 - \pi_b)$ and $B_x = v_H - (2 - \pi_s)k/(1 - \pi_s)$. Condition (26) is transformed to $k \leq k^s_{q2}\Delta$, and Condition (27) is transformed to $k \leq k^b_{q2}\Delta$. Thus, there is no equilibrium with queuing on both sides if $k \leq k^s_{q2}\Delta$ or $k \leq k^b_{q2}\Delta$. Q.E.D.
A.9 Proofs of Propositions

Proof of Proposition 1. Straightforward. Q.E.D.

Proof of Proposition 2. (1) The direct calculation shows \( \partial A_r/\partial \pi_s < 0 \). The bid side is symmetric.

(2) The direct calculation shows \( \partial A_r/\partial \pi_b > 0 \). The bid side is symmetric.

(3) \( \partial A_r/\partial \alpha = \psi_1(\alpha, \beta)\Delta/(1 - \alpha^2(1 - \beta))^2 \), where \( \psi_1(\alpha, \beta) = -1 + 2\alpha(1 - \beta) - \alpha^2\beta(1 - \beta) \). Thus, \( \partial A_r/\partial \alpha < 0 \) if \( \psi_1(\alpha, \beta) < 0 \). Since \( \psi_1(\alpha, \beta) < 0 \) if \( \psi_1(1, \beta) = 1 - 3\beta + \beta^2 < 0 \). Because \( \psi_1(1, \beta) < 0 \) for \( \beta \in (3 - \sqrt{5})/2, (3 + \sqrt{5})/2 \), \( \partial A_r/\partial \alpha < 0 \) for \( \beta > (3 - \sqrt{5})/2 \). The bid side is symmetric.

(4) The direct calculation shows that \( \partial A_r/\partial \beta < 0 \). The bid side is symmetric. Q.E.D.

(5) The set of the states of the book on the equilibrium path is \{empty, a book with a limit sell order at \( A_r \), a book with a limit buy order at \( B_r \}\}. Because the stationary distribution is \((1 - \pi_s)(1 - \pi_b)/(1 - \pi_s\pi_b), \pi_s(1 - \pi_b)/(1 - \pi_s\pi_b), \pi_b(1 - \pi_s)/(1 - \pi_s\pi_b)\)\), the expected bid-ask spread \( s_r \) is \( s_r = \{(1 - \pi_s)(1 - \pi_b)(1 + \pi_s + \pi_b - \pi_s\pi_b)\}/(1 - \pi_s\pi_b)^2 \).

Thus,

\[
\frac{\partial s_r}{\partial \alpha} = -2\alpha\Delta \frac{1 - \beta(1 - \beta)(4 - (2 - \alpha)(1 - \alpha\beta)(1 - \alpha(1 - \beta)))}{(1 - \alpha^2\beta(1 - \beta))^3} < 0
\]

because \( \beta(1 - \beta) \leq 1/4 \) and \( (2 - \alpha)(1 - \alpha\beta)(1 - \alpha(1 - \beta)) < 2 \).

(6) The direct calculation shows that \( \partial s_r/\partial \beta > 0 \) if \( \beta < 1/2 \), \( \partial s_r/\partial \beta = 0 \) if \( \beta = 1/2 \), and \( \partial s_r/\partial \beta < 0 \) if \( \beta > 1/2 \). Q.E.D.

Proof of Proposition 3. (1) \( A_f - B_f > 0 \) from Lemma 1(3) in the proof of Theorem 2.

(2) From Lemma 1(2) and (3) in the proof of Theorem 2, \( B_m > B_u > A_u > A_m \).

Q.E.D.

Proof of Proposition 4. (1) Straightforward.

(2) Straightforward.

(3) If \( \beta = 1/2 \) and \( \pi_s = \pi_b = \alpha/2, \pi \in \Pi_1 \). Let \( p = \alpha/2 \). For this case, \( \partial A_f/\partial p = -2(1 - p)^5\Delta/D_1^2 < 0 \), and

\[
\partial A_m/\partial p = -(1 - p)^4((1 - p)^4 + 4(1 - p)^2 + 4(2 - p)p^3)\Delta/D_1^2/(2 - p)^2 < 0.
\]

\( \partial A_l/\partial p = -\partial v_l^b/\partial p \), and \( \partial A_u/\partial p = -\partial v_l^b/\partial p/2 \). Because \( \partial v_l^b/\partial p = 2(1 - p)^4\Delta(1 - 2p - p^2)/D_1^2, \partial v_l^b/\partial p \) is positive for \( p \in (0, a_1/2) \) and negative for \( p \in (a_1/2, 1/2) \). The bid side is symmetric. Q.E.D.

49
(4) Straightforward.

**Proof of Proposition 5.** \( A_r - B_r > A_f - B_f \) for \( \pi \in \Pi \) because

\[
\begin{align*}
(A_r - B_r) - (A_f1 - B_f1) &= (1 - \pi_s)\Delta /D_1/(1 - \pi_s)\pi_b > 0, \\
(A_r - B_r) - (A_f2 - B_f2) &= \pi_s (1 - \pi_b)^2 + \pi_b (1 - \pi_s)\Delta /D_2/(1 - \pi_s)\pi_b > 0, \\
(A_r - B_r) - (A_f3 - B_f3) &= \pi_b (1 - \pi_s)^2 + \pi_s (1 - \pi_s)\Delta /D_3/(1 - \pi_s)\pi_b > 0.
\end{align*}
\]

Q.E.D.

**Proof of Proposition 6.** If \( \beta = 1/2 \) and \( \pi_s = \pi_b = \alpha/2, \pi \in \Pi_1 \). Let \( p = \alpha/2 \). For this case,

\[
\begin{align*}
\partial (A_u - A_m)/\partial p &= (1 - p)^4(1 + 5p^2 + 2p^3(1 - p))\Delta /d_2 > 0, \\
\partial (A_f - A_f)/\partial p &= 2(1 - p)^4p(1 + p)\Delta /d_1 > 0, \\
\partial (A_f - A_m)/\partial p &= (1 - p)^4(-3 + 12p - p^4)\Delta /d_1 > 0.
\end{align*}
\]

Let \( \psi_2(p) = -3 + 12p - p^4. \) \( \psi_2(p) > 0 \) if \( p \in (a_2/2, 1/2], \psi_2(p) = 0 \) if \( p = a_2/2, \) and \( \psi_2(p) < 0 \) if \( p \in (0, a_2/2). \) The bid side is symmetric. Q.E.D.

**Proof of Proposition 7.** Let \( \beta_{g1} \) be the solution between zero and one for \( g_1 = 0, \) and let \( \beta_{g2} \) be the solution between zero and one for \( g_2 = 0. \) \( \beta_{g1} = (\sqrt{5} - 1)/2 \approx 0.618 \) and \( \beta_{g2} = (3 - \sqrt{5})/2 \approx 0.381. \) If \( \alpha = 1, \pi \in \Pi_3 \) for \( \beta < \beta_{g2}, \pi \in \Pi_1 \) for \( \beta_{g2} \leq \beta < \beta_{g1}, \) and \( \pi \in \Pi_2 \) for \( \beta_{g1} < \beta. \) Let \( b_1 \) be the solution between zero and one for \( \partial (A_u3 - A_m3)/\partial \beta = 0. \) This equation is represented by

\[
1 - 3\beta + 3\beta^2 - 11\beta^3 + 6\beta^4 - 3\beta^6 + 3\beta^7 = 0,
\]

and has one solution between zero and one which is \( b_1 \approx 0.3326 < \beta_{g2}. \) The direct calculation shows that \( \partial (A_u3 - A_m3)/\partial \beta > 0 \) for \( \beta < b_1, \partial (A_u3 - A_m3)/\partial \beta < 0 \) for \( b_1 < \beta < \beta_{g2}, \partial (A_u1 - A_m1)/\partial \beta < 0 \) for \( \beta_{g2} \leq \beta < \beta_{g1}, \) and \( \partial (A_u2 - A_m2)/\partial \beta < 0 \) for \( \beta_{g1} < \beta. \) Thus, \( A_u - A_m \) is maximized at \( \beta = b_1, \) increases in \( \beta \) for \( \beta < b_1, \) and decreases in \( \beta \) for \( \beta > b_1. \)
Let $b_2$ be the solution between zero and one for $\partial(A_{q2} - A_{f2})/\partial \beta = 0$, which is represented by $-2 + 4\beta - 2\beta^2 + \beta^3 = 0$. Let $b_3$ be the solution between zero and one for $\partial(A_{f1} - A_{q1})/\partial \beta = 0$, which is represented by $1-4\beta + 6\beta^2 - 4\beta^3 + 6\beta^4 + 10\beta^5 - \beta^6 - 4\beta^7 = 0$. The similar argument for $A_u - A_m$ is applied to prove the cases for $A_f - A_f$ and $A_f - A_u$. The proof of the bid side is similar. Q.E.D.

Proof of Proposition 8. The direct calculation shows that $k_{q1}^s < (A_f - A_m)/\Delta$ and $k_{q1}^b < (B_m - B_f)/\Delta$ if $\alpha = 1$ or if $\beta = 1/2$. Q.E.D.

Proof of Proposition 9. (1) The direct calculation shows that $\partial k_{q1}^s/\partial \pi_s = (1-\pi_b)\psi_3(\pi_s, \pi_b)/(2-\pi_b + (1-\pi_s)(1-\pi_b))^2$ where $\psi_3(\pi_s, \pi_b) = -8 + 8\pi_b - 2\pi_b^2 + 18\pi_s - 18\pi_b\pi_s + 4\pi_b^2\pi_s - 12\pi_s^2 + 10\pi_b\pi_s^2 - \pi_b^2\pi_s^2 + 2\pi_s^3 - 2\pi_b\pi_s^3$. Because $\partial \psi_3(\pi_s, \pi_b)/\partial \pi_s = 2\{(1-\pi_s)(1-\pi_b)(6-3\pi_s - \pi_b) + 3 - 2\pi_b + \pi_b^2 - 3\pi_s\} > 0$ for $\pi \in \Pi$, $\psi_3(1 - \pi_b, \pi_b) = -\pi_b(2 + \pi_b - \pi_b^2) < 0$ means $\partial k_{q1}^s/\partial \pi_s < 0$ for $\pi \in \Pi$. Thus, $k \leq k_{q1}^s$ is more likely satisfied under the smaller $\pi_s$ given $k$ and $\pi_b$. The proof for queuing at the bid is similar.

(2) If $\beta = 1/2$, the direct calculation shows that $\partial k_{q1}^s/\partial \alpha = (2-\alpha)\psi_4(\alpha)/(3+(3-\alpha)^2)^2$ where $\psi_4(\alpha) = -36 + 62\alpha - 40\alpha^2 + 10\alpha^3 - \alpha^4$. Because $\psi_4(\alpha) < 0$ for $\alpha \in (0,1]$, $\partial k_{q1}^s/\partial \alpha < 0$. Thus, given $k$, $k \leq k_{q1}^s$ is more likely satisfied for the lower $\alpha$. The case for queuing at the bid is similar. Q.E.D.

Proof of Proposition 10. If $\beta = 1/2$, $k_{q2}^s = k_{q2}^b$, and the direct calculation shows that $\partial k_{q2}^s/\partial \alpha = 8(2-\alpha)\psi_5(\alpha)/(-48 - 8\alpha + 28\alpha^2 - 10\alpha^3 + \alpha^4)^2$ where $\psi_5(\alpha) = -56 + 52\alpha - 30\alpha^2 + 9\alpha^3 - \alpha^4$. Because $\psi_5(\alpha) < 0$ for $\alpha \in (0,1]$, $\partial k_{q2}^s/\partial \alpha < 0$. Thus, given $k$, $k \leq k_{q2}^s$ is more likely satisfied for the lower $\alpha$. Q.E.D.

Proof of Proposition 11. An equilibrium with queuing on both sides satisfies Conditions (24) and (25) in the proof of Theorem 4, leading to the inequality in the proposition. Q.E.D.
References


52

Harris, L., 1996, Does a Large Minimum Price Variation Encourage Order Exposure?, Mimeo

Hasbrouck, J. and G. Saar, 2002, Limit Orders and Volatility in a Hybrid Market: The Island ECN, Mimeo


Rosu, I., 2008a, A Dynamic Model of the Limit Order Book, Mimeo

Rosu, I., 2008b, Liquidity and Information in Order Driven Markets, Mimeo


Seppi, D. J., 1997, Liquidity Provision with Limit Orders and a Strategic Specialist, Review of Financial Studies 10, 103-150

Van Damme, E., R. Selten, and E. Winter, 1990, Alternating Bid Bargaining with a Smallest Money Unit, Games and Economic Behavior 2, 188-201
Table 1: Examples of equilibria under one-period expiration.

<table>
<thead>
<tr>
<th>Order in the book</th>
<th>Equilibrium A Strategies of a seller</th>
<th>Strategies of a buyer</th>
<th>Equilibrium B Strategies of a seller</th>
<th>Strategies of a buyer</th>
</tr>
</thead>
<tbody>
<tr>
<td>No order</td>
<td>Limit sell at 2</td>
<td>Limit buy at 1</td>
<td>Limit sell at 2</td>
<td>Limit buy at 2</td>
</tr>
<tr>
<td>Limit sell at 1</td>
<td>Limit sell at 2</td>
<td>Market buy</td>
<td>Limit sell at 2</td>
<td>Market buy</td>
</tr>
<tr>
<td>Limit sell at 2</td>
<td>Limit sell at 2</td>
<td>Market buy</td>
<td>Limit sell at 2</td>
<td>Market buy</td>
</tr>
<tr>
<td>Limit buy at 1</td>
<td>Market sell</td>
<td>Limit buy at 1</td>
<td>Limit sell at 2</td>
<td>Limit buy at 2</td>
</tr>
<tr>
<td>Limit buy at 2</td>
<td>Market sell</td>
<td>Limit buy at 1</td>
<td>Market sell</td>
<td>Limit buy at 2</td>
</tr>
</tbody>
</table>

Table 1: The parameter values are $v_H = 3$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. Limit orders are assumed to expire in one period after their submission. A row represents a state of the book which is distinguished by an unfilled limit order in the book. Equilibrium A and Equilibrium B are equilibrium profiles of strategies. For example, a seller submits a limit sell order at 2 to the book with no order under Equilibrium A.
Table 2: Effect of expiration period and trader arrival rate.

<table>
<thead>
<tr>
<th>Order imbalance ($\beta$)</th>
<th>1</th>
<th>1/2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expiration periods</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Trader arrival rate ($\alpha$)</td>
<td>1/3</td>
<td>2/3</td>
<td>1</td>
</tr>
<tr>
<td>First eight asks</td>
<td>10</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>Expected bid-ask spread</td>
<td>12.2</td>
<td>10.6</td>
<td>9.4</td>
</tr>
<tr>
<td>Order flow composition of sell orders (%)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Market order</td>
<td>21.7</td>
<td>30.8</td>
<td>33.7</td>
</tr>
<tr>
<td>Cutting by more than one tick</td>
<td>0</td>
<td>0.6</td>
<td>4.8</td>
</tr>
<tr>
<td>Cutting by one tick</td>
<td>13.0</td>
<td>22.3</td>
<td>25.7</td>
</tr>
<tr>
<td>At the market</td>
<td>7.3</td>
<td>5.1</td>
<td>0</td>
</tr>
<tr>
<td>Behind the best quote</td>
<td>1.5</td>
<td>2.8</td>
<td>3.2</td>
</tr>
<tr>
<td>Empty</td>
<td>56.5</td>
<td>38.4</td>
<td>32.6</td>
</tr>
<tr>
<td>Probability of holes (%)</td>
<td>0.0005</td>
<td>0.5</td>
<td>7.5</td>
</tr>
<tr>
<td>Kurtosis of transaction prices</td>
<td>1.1</td>
<td>1.3</td>
<td>3.6</td>
</tr>
</tbody>
</table>

Table 2: The parameter values are $v_H = 14$, $v_L = 0$, and $k = 1$. For all equilibria, strategies for sellers and buyers are symmetric. “First eight asks” denotes the asks submitted if the first seller arrives at an empty book and the other seven sellers arrive consecutively. In “Order flow composition of sell orders,” we designate market orders as “Market order,” limit orders undercutting the best quote by more than one tick as “Cutting by more than one tick,” limit orders undercutting the best quote by one tick as “Cutting by one tick,” limit orders at the best quote as “At the market,” limit orders behind the best quote as “Behind the best quote,” and limit orders submitted to an empty book as “Empty.” “Probability of holes” denotes the probability of holes emerging in the book.
Table 3: Effect of expiration period and order imbalance.

<table>
<thead>
<tr>
<th>Trader arrival rate($\alpha$)</th>
<th>Expired periods 2</th>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order imbalance ($\beta$)</td>
<td>$1/2$</td>
<td>$3/5$</td>
<td>$2/3$</td>
</tr>
<tr>
<td>First eight asks</td>
<td>8</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>First eight bids</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>Expected bid-ask spread</td>
<td>9.4</td>
<td>8.1</td>
<td>4.8</td>
</tr>
<tr>
<td>Order flow composition of sell orders (%)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Market order</td>
<td>33.7</td>
<td>20.3</td>
<td>8.54</td>
</tr>
<tr>
<td>Cutting by more than one tick</td>
<td>4.8</td>
<td>15.1</td>
<td>0</td>
</tr>
<tr>
<td>Cutting by one tick</td>
<td>25.7</td>
<td>22.0</td>
<td>0</td>
</tr>
<tr>
<td>At the market</td>
<td>0</td>
<td>0</td>
<td>74.5</td>
</tr>
<tr>
<td>Behind the best quote</td>
<td>3.2</td>
<td>12.7</td>
<td>0</td>
</tr>
<tr>
<td>Empty</td>
<td>32.6</td>
<td>30.0</td>
<td>16.9</td>
</tr>
<tr>
<td>Order flow composition of buy orders (%)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Market order</td>
<td>33.7</td>
<td>49.7</td>
<td>74.5</td>
</tr>
<tr>
<td>Cutting more than one tick</td>
<td>4.8</td>
<td>2.0</td>
<td>0.6</td>
</tr>
<tr>
<td>Cutting by one tick</td>
<td>25.7</td>
<td>17.3</td>
<td>7.6</td>
</tr>
<tr>
<td>At the market</td>
<td>0</td>
<td>0</td>
<td>0.05</td>
</tr>
<tr>
<td>Behind the best quote</td>
<td>3.2</td>
<td>1.1</td>
<td>0.3</td>
</tr>
<tr>
<td>Empty</td>
<td>32.6</td>
<td>30.0</td>
<td>16.9</td>
</tr>
<tr>
<td>Probability of holes (%)</td>
<td>7.5</td>
<td>16.6</td>
<td>0.3</td>
</tr>
<tr>
<td>Kurtosis of transaction prices</td>
<td>3.6</td>
<td>5.1</td>
<td>64.5</td>
</tr>
</tbody>
</table>

Table 3: The parameter values are $v_H = 14$, $v_L = 0$, and $k = 1$. “First eight asks” denotes the asks submitted if the first seller arrives at an empty book and the other seven sellers arrive consecutively. “First eight bids” denotes the bids submitted if the first buyer arrives at an empty book and the other seven buyers arrive consecutively. In “Order flow composition,” we designate market orders as “Market order,” limit orders undercutting the best quote by more than one tick as “Cutting by more than one tick,” limit orders undercutting the best quote by one tick as “Cutting by one tick,” limit orders at the best quote as “At the market,” limit orders behind the best quote as “Behind the best quote,” and limit orders submitted to an empty book as “Empty.” “Probability of holes” denotes the probability of holes emerging in the book.
Figure 1: Example of quote dynamics

Figure 1 illustrates an example of quote dynamics of an Edgeworth cycle equilibrium under $v_H = 21$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. The abscissa is the period and the ordinate the quoted price. The horizontal solid lines indicate the asks and the horizontal broken lines indicate the bids. Two lines in one period indicate that the book has two limit orders. The dot indicates a transaction. The figure depicts a case in which the sequence of the type of traders is sssss sssss bbbbbb bbbbbb sssss sssss sssssss sbbss, where ‘s’ denotes a seller and ‘b’ a buyer.
Figure 2: Asks and expected payoffs

Figure 2 depicts the relation of $A_m$, $A_u$, $A_f$, $A_l$, $v_f^s$, and $v_l^s$. The figure illustrates the case with the parameter values $v_H = 21$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. Points indicate asks submitted on the equilibrium path and the corresponding expected payoffs. Solid lines indicate the expected payoffs for a seller facing an empty book by submitting a limit sell at $A$. 

$$\pi_b /LParen1 /Plus\pi_n /Plus\pi_s /RParen1 /LParen1 A /Minusv L /RParen1 \pi_b /LParen1 \pi_n /Plus\pi_b /RParen1 /LParen1 A /Minusv L /RParen1$$