Achievable Outcomes in Smooth Dynamic Contribution Games

by

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Two Classes of Dynamic Contribution Models

- **Threshold Models.** E.g., binary public good projects (bridge, church)
  - Backwards induction (BI) works. Few subgame perfect equilibria (SPE).

- **No-Threshold Models.** E.g., projects producing a divisible public good as a concave function of contributions (university building, public television, open-source software, capital investment in a firm)
  - BI does not apply (for infinite horizon). Many equilibria.
    The sets of equilibria and equilibrium payoffs are not well understood.
Some Literature on No-Threshold Models

Marx and Matthews (2000)  
- kinked-linear production, discounting  
- some efficient outcomes are the limits of SPE outcomes if $\delta$ is large enough  
- strategic gradualism

Lockwood and Thomas (2002)  
- $n = 2$, discounting, sym, kinked-linear prod  
- any efficient outcome is the limit of SPE outcomes as $\delta \to 1$, for alt or simult moves  
- strategic gradualism

Gale (2001)  
- no discounting, general payoff functions  
- any core-like outcome is a SPE outcome
This Paper in a Nutshell


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The Two Main Results

1. Necessary conditions for a profile to be achievable.

2. Sufficient conditions for a profile to be the limit of achievable profiles as \( \delta \to 1 \). This extends Gale’s result to a setting with discounting, but more restrictive payoff functions.
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The characterizations are in terms of a core-like structure:

“Definition”. A profile is in the **undercore** if it is not “blocked from below”. (The usual core is the efficient frontier of the undercore.)
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Consequences

1. No Folk Theorem if \( n > 2 \). No coalition of players can be induced to contribute an “unfairly” large share.

2. Any efficient profile that can be approximately achieved (for large \( \delta \)) must be in the core.
The Sufficiency Result. As $\delta \rightarrow 1$, every profile in the undercore is the limit of achievable profiles if (i) the move structure satisfies a weak "cyclicity" property, and (ii) the payoffs satisfy a "prisoners dilemma" property.

Indeed, for any neighborhood $\mathcal{N}$ of an undercore profile, a path exists that converges to a profile in $\mathcal{N}$ and which is a SPE path for all large $\delta$. 
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Consequences. When periods are short,

1. The core is the set of efficient profiles that can be approximately achieved.

2. The move structure (simultaneous, round-robin, . . . ) is irrelevant.

3. Real-time gradualism is unnecessary: any profile in the undercore (and hence core) can be approximately achieved “in a twinkling of the eye”.
The Game

\[ N = \{1, \ldots, n\} \] set of players

\[ x^t \in \mathbb{R}^n_+ \] cumulative contribution profile in period \( t \)

past contributions are publicly observed.

\[ \vec{x} = \{x^t\}_{t=0}^{\infty} \] outcome path (starting at \( x^0 := (0, \ldots, 0) \))

\[ N_t \subset N \] set of players able to move in period \( t \)

player \( i \in N_t \) can choose \( x^t_i \in [x^t_{i-1}, \infty) \)

\[ \vec{N} = \{N_t\}_{t=1}^{\infty} \] move structure

each \( i \) can move infinitely often

\[ U(\vec{x}, \delta) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u(x^t) \] payoff functions
Main Payoff Assumptions

**Quasilinearity:** \( u_i(x) = v_i(X) - x_i \), where \( X = \sum x_i \), \( v_i(0) = 0 \).
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**Neoclassical:** Each $v_i$ is continuous, increasing, strictly concave, differentiable.

One consequence of this is the **Positive Spillovers** property:

$$(PS)\quad u_i(\cdot, x_{-i}) \text{ increases in } x_{-i}.$$
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Some results assume the **Prisoner’s Dilemma** property: $v_i'(0) \leq 1$ for each $i$, equivalent now to

$$ (PD) \quad u_i(x_i, \cdot) \text{ decreases in } x_i. $$
**Possible Scenario 1** (Pitchford and Snyder, 2004)

Players consume only at the end of the game. The game ends at any date $T$ with probability $(1 - \delta)\delta^{T-1}$. $U(\bar{x}, \delta)$ is expected utility.
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**Possible Scenario 2** (Matthews and Marx, 2000)

Players consume public good each period.

Contributions are made at the beginning of each period, and become non-depreciating capital. Public good is produced in period $t$ at rate $f(X^t)$. Period length is $\Delta$, discount rate is $r$, and $\delta = e^{-r\Delta}$. This yields the model, with

$$v_i(X) = r^{-1}\hat{v}_i(f(X)).$$

In this scenario, $\delta \to 1$ must be interpreted as $\Delta \to 0$, not $r \to 0$. 
Observation (Finger Exercise)

If (PD) holds locally at $x \gg 0$, then no NE path $\bar{x}$ converges to $x$ in finite time.
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*If (PD) holds locally at \( x \gg 0 \), then no NE path \( \vec{x} \) converges to \( x \) in finite time.*

**Proof.** Suppose \( x^{T-1} \neq x^T = x^t \) for all \( t > T \).
Previous and Upcoming Results: Nature of equilibria for $n = 2$ and $\delta = 1$
(or as $\delta \to 1$)

In this case, a payoff vector is an equilibrium payoff vector if and only if it is feasible and individually rational.

Issues. Folk Theorem? Real-time radualism?
Example: No Folk Theorem

\( n = 3, \quad v_i(X) = 1 - (X + 1)^{-1}. \)

The efficient profiles are those with the aggregate \( X = \sqrt{3} - 1 =: Y_N. \)

\( x^* = (.5Y_N, .5Y_N, 0) \) is efficient and strictly individually rational.

No other profile generates the payoff vector \( u(x^*). \)
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The efficient profiles are those with the aggregate \( X = \sqrt{3} - 1 =: Y_N. \)

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**Claim.** No equilibrium path converges to any profile near \( x^*. \)

**Proof.** On such a path, \( v_1(X) + v_2(X) - X \) would eventually decrease. So for some \( t \) and \( i \in \{1, 2\} \), \( u_i(x^t) > u_i(x^*). \) This player should have stopped contributing earlier.
Equilibrium Paths and Achievable Profiles: Preliminaries

\( Y_i \), the **standalone contribution** of player \( i \), is the maximizer of \( u_i(x_i) - x_i \).

\[ \bar{Y} := \max(Y_1, \ldots, Y_n). \] (\( \bar{Y} = 0 \) iff PD holds).

\[ u^*_i(x) := \max_{x'_i \geq x_i} u_i(x'_i, x_{-i}). \]

**Definition.** \( x \) is a **satiation profile** iff \( u^*(x) = u(x) \) \( \iff X \geq \bar{Y} \).

\[ U^t(\bar{x}, \delta) := (1 - \delta) \sum_{s \geq t} \delta^{s-t} u(x^s), \text{ the continuation payoff.} \]
Equilibrium Paths and Achievable Profiles: Lemmas

Lemma 1. Any equilibrium path satisfies

\[ u_i(x_{t}^{t-1}, x_{-i}^{t}) \leq U_i^{t}(\bar{x}, \delta) \text{ for all } t \geq 1, \ i \in N, \]  

(1)

\[ u_{i}^{*}(x_{t}^{t-1}, x_{-i}^{t}) \leq U_i^{t}(\bar{x}, \delta) \text{ for all } t \geq 1, \ i \in N_t. \]  

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If (PD) holds, then any feasible path is an equilibrium path iff it satisfies (1).
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\[ u_i^*(x_{i}^{t-1}, x_{-i}^{t}) \leq U_i^t(\bar{x}, \delta) \quad \text{for all } t \geq 1, \ i \in N_t. \]  

If (PD) holds, then any feasible path is an equilibrium path iff it satisfies (1).

Lemma 2. Every equilibrium path converges to a satiation profile $x$. The convergence does not occur in finite time if $X > \bar{Y}$. 
The Underlying Coalitional Game

The coalitional game is defined to reflect two features of the dynamic game:

1. Equilibrium paths are nondecreasing. Thus, a coalition should be able to block the achieved profile only from below.

2. When she is able to move, a player can always increase her own contribution any amount. Thus, the payoff function $u_i^*$ should be used to evaluate the profile used for blocking.
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Coalition $S \subseteq N$ blocks $x$ using $z$ if for all $i \in S$,

$$z_{-S} = 0 \text{ and } u^*_{S}(z) > u_{S}(x).$$

Coalition $S$ underblocks $x$ if it blocks $x$ using a profile $z \leq x$. 
The Underlying Coalitional Game (con’t)

The **core** is

\[ C := \{ x \in \mathbb{R}^n_+ : \text{no coalition blocks } x \}. \]

The **undercore** is

\[ D := \{ x \in \mathbb{R}^n_+ : \text{no coalition underblocks } x \}. \]

Hence, \( C \subset D \).
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**Lemma 5.** Any \( x \in D \) is an individually rational satiation profile.
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Hence, \( C \subset D \).

**Lemma 5.** Any \( x \in D \) is an individually rational satiation profile.

**Proposition 1.** \( D \) is the set of profiles satisfying a partially familiar set of inequalities.
Corollary. The core consists of the undercore profiles that are efficient:

\[ C = \{ x \in D \mid X = Y_N \}, \]

where \( Y_N := \arg \max_X \sum_{i \in N} v_i(X) - X. \)
**Corollary.** The core consists of the undercore profiles that are efficient:

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The following depicts \( C \) and \( D \) for the case \( n = 2 \) and (PD):
Necessary Conditions for Achievability

Proposition 2. Every achievable profile is in the undercore.

“Proof.” Let $x$ be the limit of an equilibrium path $\bar{x}$. If $S$ underblocks $x$ using a profile $z < x$, the coalition member who is supposed to be the last to raise her action above $z_i$ can do better by not doing so. ■
Necessary Conditions for Achievability

Proposition 2. *Every achievable profile is in the undercore.*

*Proof.* Let $x$ be the limit of an equilibrium path $\bar{x}$. If $S'$ underblocks $x$ using a profile $z < x$, the coalition member who is supposed to be the last to raise her action above $z_i$ can do better by not doing so. ■

Proposition 3. *Any achievable $x$ is inefficient.*

*Proof.* Messy but straightforward calculus proof. Second-order marginal benefits but first-order marginal costs. ■
Sufficient Conditions for Achievability

A cyclicity property:

\[(CY) \quad m > 0 \text{ exists such that } \forall i \in N : i \in N_{(nk+i)m} \forall k \geq 0.\]

Note that (CY) is satisfied, with \( m = 1 \), by both the simultaneous move structure and the round-robin structure \( \vec{N}^R \).
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**Proposition 4.** Suppose (PD) holds and \( \hat{N} \) satisfies (CY). Then an open dense subset \( D_s \) of \( D \) exists that satisfies the following property: for any \( x \in D_s \), there exists a path \( \hat{x} \) converging to \( x \), and a discount factor \( \delta < 1 \), such that \( \hat{x} \) is an equilibrium path for all \( \delta > \delta \).

“Proof.” Long and complicated (for me).
Synthesis: Achievable Profiles

The set of achievable profiles for a fixed move structure is

\[ A(\vec{N}) := \left\{ x \in \mathbb{R}_+^n : x \text{ is achievable for some } \delta < 1 \right\}. \]
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**Theorem 1.** Under the maintained assumptions,

(a) \( A(\vec{N}) \subset D, \) and

(b) \( \text{cl} A(\vec{N}) = D \) if (PD) and (CY) hold.
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In (b), any achievable profile can be approximately achieved instantaneously as the period length shrinks to zero. Real-time gradualism is unnecessary.
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In (b), the set of achievable profiles is essentially independent of the move structure.
Synthesis: Equilibrium Payoffs

The set of limits of equilibrium payoffs:

\[ W(\vec{N}) := \text{cl} \{ U(\vec{x}, \delta) : 0 < \delta < 1, \, \vec{x} \text{ an equilibrium path for } \delta \} . \]

Let \( P \) denote the set of efficient and individually rational payoff vectors.
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Let \( P \) denote the set of efficient and individually rational payoff vectors.

**Theorem 2.** Under the maintained assumptions,

(i) \( P \cap W(\vec{N}) \subseteq u(C') \), and

(ii) \( P \cap W(\vec{N}) = u(C') \) if (PD) and (CY) hold.

Any efficient payoff that is the limit of equilibrium payoffs as the period length shrinks is a core payoff. If (PD) and (CY) hold, any core payoff is such a limit.
Concluding Remarks

- The necessity result (only undercore profiles are achievable) is in my view the most novel and useful result. It is also robust – it does not depend on concavity, quasilinearity, or the discount factor (it holds for all $\delta \leq 1$).
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- The sufficiency result depends more on the assumptions. Without discounting it holds fairly generally (Gale, 2001), but with discounting it fails to hold if the payoffs are significantly non-concave, as in the case of a binary threshold public good.