Change Points in Term-Structure Models: Pricing, Estimation and Forecasting

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We construct and estimate an arbitrage-free model of the term structure of zero-coupon bond yields with unknown change points.

In the model, the parameters in

- the dynamics of one latent and two macro factors evolutions
- the model of market price of factor risks, and
- the process of the stochastic discount factor

are all subject to change at unknown time points.

Basic pricing model is closely related to that of Duffie and Kan (1996) but modified and extended to deal with the possibility of change points.
In previous work, the question of breaks in the term structure has been handled within the context of Markov switching models (for example, Ang, Bekaert and Wei (2008), Bansal and Zhou (2002), and Dai, Singleton and Yang (2007)).

We focus on the change point formulation to capture the fundamental changes in the US economy over time.

In our view, it is appropriate to assume that if a regime is vacated it is never visited again.

In fact, the filtered regime probabilities in the empirical analysis of Dai, Singleton and Yang (2007) strongly suggest that the same regime has prevailed after 1986, which is more suggestive of a change-point rather than a Markov switching process.
1. Pricing bonds
   - We show that the bonds in our set-up can be priced straightforwardly
     once the change point model is re-formulated in the manner of Chib
     (1998)

2. Estimation
   - We consider models up to 4 change-points to 16 yields, the largest
     collection considered in this context
   - Our approach to inference is fully Bayesian, with priors set up to reflect
     the assumption of a positive term-premium
   - We summarize the resulting posterior distributions by tuned Markov
     chain Monte Carlo methods

3. Forecasting
   - Finally, we demonstrate that our proposed model outperforms no
     change-point model in terms of out-of-sample prediction, which none of
     the previous literatures has attempted.
Main Findings

2. The differences in the distribution of the term-structure are considerable across regimes, which are mainly driven by the changes in:
   - mean and volatility
   - term premium
   - factor loadings
3. The predictive performance of our best model is substantially better than that of the other models we consider
Model

- Let \( s_t \in \{1, 2, \ldots, q + 1\} \) denote the current state of the economy.
- We follow Chib (1998) and assume that \( s_t \) is Markov with evolution governed by the transition probability matrix

\[
P = \begin{bmatrix}
p_{11} & 1 - p_{11} & 0 & \ldots & 0 \\
0 & p_{22} & 1 - p_{22} & \ldots & 0 \\
0 & 0 & p_{33} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & p_{q+1,q+1}
\end{bmatrix}
\]

where \( p_{jk} = \Pr[s_{t+1} = k|s_t = j] \)
- The \( j \)th change occurs at \( t_j \) if \( s_{t_j} = j \) and \( s_{t_j+1} = j + 1 \).
Let $f_t = (u_t, m_t)$ denote a collection of 3 exogenous factors (one latent and two macro-economic variables) whose dynamics are responsible for determining the dynamics of bond prices

- $f_t$ is assumed to follow the regime-specific Gaussian VAR process

$$f_{t+1} = \left( \begin{array}{c} u_{t+1} \\ m_{t+1} \end{array} \right) | f_t, s_{t+1}, s_t \sim \mathcal{N}_3(\mu_{s_{t+1}} + G_{s_{t+1}} (f_t - \mu_s), \Omega_{s_{t+1}})$$

where $\mathcal{N}_3(., .)$ denotes the 3-dimensional normal distribution, and $\mu_j$ is a $3 \times 1$ vector, $G_j$ and $\Omega_j$ are $3 \times 3$ matrices
For the SDF, we assume the process

\[ \kappa_{t,t+1} = \exp \left( -r_{t,s_t} - \frac{1}{2} \gamma'_{t,s_t} \gamma_{t,s_t} - \gamma'_{t,s_t} \omega_{t+1} \right) \]

where \( r_{t,s_t} \) is the short rate which is assumed to be an affine function of the factors

\[ r_{t,s_t} = \delta_{1,s_t} + \delta'_{2,s_t} (f_t - \mu_{s_t}) \]

and \( \gamma_{t,s_t} \) is the market price of factor risk which is assumed to be time-varying within regimes and mean-reverting and to follow the process

\[ \gamma_{t,s_t} = \tilde{\gamma}_{s_t} + \Phi_{s_t} (f_t - \mu_{s_t}) \]
Model Solution

Let $P_t(s_t, \tau)$ denote the bond price at date $t$ that has $\tau$ periods left until maturity given $s_t$.

Arbitrage-free bond pricing requires that

$$P_t(s_t, \tau) = \mathbb{E} [\kappa_{t,t+1} P_{t+1}(s_{t+1}, \tau - 1) | f_t, s_t]$$

where the expectation is over the factor shocks in $(t + 1)$ and the two possible values that $s_{t+1}$ can take given $s_t$, and $\kappa_{t,t+1}$ is the pricing kernel.
We solve the risk-neutral pricing formula by assuming that $P_t(s_t, \tau)$ is a regime-dependent exponential affine function of $f_t$

$$P_t(s_t, \tau) = \exp(-\tau R_{\tau t})$$

where

$$R_{\tau t} = \frac{1}{\tau} a_{s_t}(\tau) + \frac{1}{\tau} b_{s_t}(\tau)'(f_t - \mu_{s_t})$$

is the continuously compounded yield and $a_{s_t}(\tau)$ and $b_{s_t}(\tau)$ are functions depending on $s_t$ and $\tau$.

By the usual techniques, we find the expressions for the latter functions by the method of undetermined coefficients.

### Model

### Estimation

SSM
Identifying restrictions
Prior
Posterior

### Results

The Change-Points
Median and Volatility
Term Premium
Factor loadings
Forecasting

### Conclusion
This procedure produces the following recursive system for the unknown functions

\[
a_j(\tau) = \left( \begin{array}{cc}
p_{jj} & p_{jk} \\
p_{kj} & p_{kk} \\
\end{array} \right) \left( \begin{array}{c}
\delta_{1,j} - \bar{\gamma}_j L_j' b_j(\tau - 1) - b_i(\tau - 1)' L_j L_j' b_j(\tau - 1)/2 + a_j(\tau - 1) \\
\delta_{1,j} - \bar{\gamma}_j L_k' b_k(\tau - 1) - b_k(\tau - 1)' L_k L_k' b_k(\tau - 1)/2 + a_k(\tau - 1) \\
\end{array} \right)
\]

\[
b_j(\tau) = \left( \begin{array}{cc}
p_{jj} & p_{jk} \\
p_{kj} & p_{kk} \\
\end{array} \right) \left( \begin{array}{c}
\delta_{2,j} + \left( G_j - L_j \Phi_j \right)' b_j(\tau - 1) \\
\delta_{2,j} + \left( G_k - L_k \Phi_j \right)' b_k(\tau - 1) \\
\end{array} \right)
\]

where \( j \in \{1, 2, \ldots, q\} \) and \( k = j + 1 \), \( L_j \) is the Cholesky decomposition of \( \Omega_j \) and \( \tau \) runs over the positive integers.

These recursions are initialized by \( a_{st}(0) = 0 \) and \( b_{st}(0) = 0_{3 \times 1} \) for all \( s_t \).
Regime-specific Term Premium

- Under risk-neutral pricing, after adjusting for risk, agents are indifferent between holding a $\tau$-period bond and a risk-free bond for one period. The risk adjustment is the term premium.

- In our model it can be calculated as

  \[
  \text{Term premium} = (\tau - 1) \text{Cov} (\ln \kappa_{t,t+1}, R_{\tau-1,t+1}|f_t, s_t = j) \\
  = -p_{jj} b_j (\tau - 1) L_j \gamma_{t,j} - p_{jk} b_k (\tau - 1) L_k \gamma_{t,j}
  \]

- We calculate this regime-specific term premium for each time period in the sample.
We fit quarterly data on sixteen U.S. treasury yields ranging from 1972:I to 2007:IV, denoted by

\[(R_{1t}, R_{2t}, \ldots, R_{16t})\]

where \(R_{it} = R_{\tau_i, t}\) and \(\tau_i\) is the \(i\)th maturity.

In the application, the set of maturities is given by

\[\{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 16, 20, 24, 28, 36, 40\}\]

Following the previous works, we assume that one basis yield \((R_{8t}, \text{the eighth in the above list})\) is priced exactly by the model.

This implies that conditioned on \(m_t, u_t\) can be expressed in terms of \(R_{8t}\)
Let the remaining 15 yields (which are measured with error) be denoted by $R_t$

Define $\bar{a}_{i,s_t} = a_{s_t}(\tau_i)/\tau_i$ and $\bar{b}_{i,s_t} = b_{s_t}(\tau_i)/\tau_i$

Let $\bar{a}_{s_t}$ and $\bar{b}_{s_t}$ be the corresponding intercept and factor loadings for $R_t$

Then, the measurement equations have the form

$$
\begin{pmatrix}
R_t \\
f_t
\end{pmatrix} =
\begin{pmatrix}
\bar{a}_{s_t} \\
\mu_{s_t}
\end{pmatrix} +
\begin{pmatrix}
\bar{B}_{s_t} \\
I_3
\end{pmatrix}(f_t - \mu_{s_t}) +
\begin{pmatrix}
I_{15} \\
0_{3 \times 15}
\end{pmatrix} \varepsilon_t
$$

where $\varepsilon_t \sim \mathcal{N}_{15}(0, \Sigma_{s_t})$ and $\Sigma_{s_t}$ is diagonal
The transition equation is given by the evolution equation of the factors. As initial conditions of the factors, we assume that \( m_0 \) is known from the data, and \( u_0 \), the latent factor at time 0, follows the steady-state distribution in regime 0:

\[
u_0 \sim N(0, V_u)
\]

where \( V_u = (1 - G_{11,0})^{-1} \).
We impose the standard identifying restrictions on the parameters of our model

- \( \mu_{u,s_t} = 0 \) which means that \( \delta_{1,s_t} \) is a free parameter
- The (1,1) element of \( L_{s_t} \) is 1/400 for normalization
- The first element of \( \delta_{2,s_t} \), namely \( \delta_{21,s_t} \), is non-negative
- For stationarity of the factor process, eigenvalues of \( G_{s_t} \) are constrained to lie inside the unit circle
Model parameters

- $\theta$ denote the free parameters in $(G_{st}, \mu_{m,s}, \delta_{st}, \gamma_{st}, \Phi_{st}, L_{st}, P)$
- $u_0$ is the latent factor at time 0
- These along with $\sigma^2$ the diagonal elements of $\{\Sigma_{st}\}$ form the set of unknown parameters.
Our prior which we give in the paper is set up to reflect the apriori belief of a positive term premium.

**Figure:** *The implied prior term structure dynamics for 3 change point model*
Our posterior of interest is given by

$$\pi(\theta, \sigma^2, u_0, S_n | y) \propto f(y|\theta, \sigma^2, S_n)p(S_n|\theta, u_0)\pi(u_0|\theta)\pi(\theta)\pi(\sigma^2)$$

In the 4 change-point model, conditioned on $S_n$, this is a 209 parameter distribution.

In our high-dimensional models, it becomes virtually impossible to find the maximum likelihood estimates due to the multi-modalities of the likelihood surface.

For this reason, we use a tailored multiple-block MCMC methods (TaRB-MH) that is suitably designed to deal with such irregularities and high dimensionality.
We also calculate the marginal likelihood of each of our change-point models.

It is calculated as

\[
\ln \hat{m}(\mathbf{y}) = \ln f(\mathbf{y} | \theta^*, \sigma^*^2, u_0^*) + \ln \left\{ \pi(u_0^* | \theta^*) \pi(\theta^*) \pi(\sigma^*^2) \right\} \\
- \ln \hat{\pi}(\theta^*, \sigma^*^2, u_0^* | \mathbf{y})
\]

where the posterior ordinate is computed based on Chib (1995) and Chib and Jeliazkov (2001).
## Model Comparison

Table – Log likelihood (lnL), log marginal likelihood (lnML), posterior probability of each model (Pr[Cq|y]), and change point estimates

| Model | lnL     | lnML    | Pr[Cq|y] | change point               |
|-------|---------|---------|----------|---------------------------|
| C0    | -1487.0 | -1657.3 | 0.00     |                           |
| C1    | -1154.3 | -1507.5 | 0.00     | 1986:II                   |
| C2    | -774.5  | -1297.9 | 0.00     | 1985:IV, 1995:II          |
| C3    | -396.5  | -1121.9 | 1.00     | 1980:II, 1985:IV, 1995:II |
Posterior Probability of Regimes


**Figure: Posterior probability of** $s_t = 1$, $s_t = 2$, $s_t = 3$ **and** $s_t = 4$
The term structure behavior is notably different across regimes exhibiting positive average term premium, especially in median and volatility.

Figure: Sample term structure of interest rates
The term premium has shifted down since the Volker disinflation.

Figure: *Term premium of 2 year bond*
Factor loadings

Figure: Estimates of the factor loadings, $\bar{b}_{st}$
Predictive Densities

Figure: The MCMC forecasts of the yield curve
For a formal comparison, we compute the posterior predictive criterion (PPC).

PPC favors the model that minimizes the sum of two terms:

- The first term is a penalty on model complexity.
- The second term is a sum of squared residuals and measures goodness-of-fit how well the forecasts fit the actual observation.

### Table – Posterior predictive criterion

<table>
<thead>
<tr>
<th>forecast period</th>
<th>C0</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
</tr>
</thead>
</table>
We develop an affine term structure model with unknown multiple change points.


Incorporating the change points is essential to improve forecasting yields.

We think that this model has opened up new possibilities for understanding term-structure dynamics.