Risk Aversion and Equilibrium Optimal Portfolios in Large Markets

Jaksa Cvitanic  
Caltech  

Semyon Malamud  
ETH Zurich

North American Summer Meeting of the Econometric Society  
Boston University  
June 5, 2009.
Survival and Price Impact

- Friedman (1953): Irrational traders do not survive
- Sandroni (2000) and Blume and Easley (2001) confirm this intuition.
Models with Heterogeneous Risk Aversion

- Wang (1996)
- Benninga and Mashar (2000)
- Malamud (2008)
- Bharma and Uppal (2009)
- Yan (2009)
The Model

- standard Brownian motion $B_t$, $t \in [0, T]$;
- one stock with price process $S_t$ and terminal dividend
  \[ D = D_T = e^{\rho T} + \sigma B_T \]
- constant interest rate $r$;
- $n$ agents $k = 1, \cdots, n$; agent $k$ is initially endowed with $\psi_k$ shares of
  the stock;
- agent $k$ maximizes
  \[ E \left[ \frac{W_k^{1-\gamma_k} T}{1 - \gamma_k} \right] \]
Equilibrium

- complete markets;
- equilibrium is equivalent to an Arrow-Debreu equilibrium;
- there is a unique (up to a constant multiple) stochastic discount factor $M$;
- the stock price is
  \[ S_t = e^{r(t-T)} \frac{E_t[MD]}{E_t[M]}; \]
- optimal terminal wealth
  \[ W_{kT} = \frac{\psi_k E[MD]}{E[M^{1-b_k}]} M^{-b_k}, \quad b_k = \gamma^{-1}_k \]
- equilibrium SDF $M$ solves
  \[ \sum_{k} W_{kT} = D \]
**Theorem** The agent $0$ whose risk aversion is closest to 1 dominates in the long run:

\[ \lim_{T \to \infty} \frac{W_{kT}}{W_{0T}} = 0 \]

for all $k \neq 0$. 
Relative Extinction

**Definition** (KRWW (2006)): agent $i$ experiences extinction relative to agent $j$ if

$$\lim_{t \to \infty} \frac{W_{iT}}{W_{jT}} = 0$$

**Theorem.** Even if agent $i$ experiences extinction relative to agent $j$, adding a third agent $k$ to the economy may reverse the situation and force the agent $j$ to experience extinction relative to agent $i$. 
Global Bounds for the Price

Proposition

\[ e^{r(t-T)} \frac{E_t[D^{1-\gamma_{\max}}]}{E_t[D^{-\gamma_{\max}}]} \leq S_t \leq e^{r(t-T)} \frac{E_t[D^{1-\gamma_{\min}}]}{E_t[D^{-\gamma_{\min}}]} \]

Related to:

- bubbles and crashes (Cao and Ou-Yang (2005));
Representative Agent

- $M = U'(D)$

- aggregate risk aversion

\[ \gamma_U(x) = -\frac{x U''(x)}{U'(x)} \]
Representative Agent’s Risk Aversion

Proposition $\gamma^U(x)$ is monotone decreasing in $x$ and satisfies

$$\gamma_{\text{max}} = \lim_{x \to +0} \gamma^U(x) \geq \gamma^U(x) \geq \lim_{x \to \infty} \gamma^U(x) = \gamma_{\text{min}}.$$
Explicit Formulae for Drift and Volatility

\[
S_t^{-1} dS_t = \mu_t \, dt + \sigma_t \, dB_t
\]

**Proposition** The drift and volatility of the stock price are given by

\[
\mu_t = r + \sigma \frac{E_t[M \gamma^U(D)]}{E_t[M]} \sigma_t
\]

\[
\sigma_t = \sigma \left( 1 - \frac{E_t[MD \gamma^U(D)]}{E_t[MD]} + \frac{E_t[M \gamma^U(D)]}{E_t[M]} \right)
\]
**Excess Volatility**

**Proposition** For all $t$

$$\sigma \leq \sigma_t \leq \sigma(1 + \gamma_{\text{max}} - \gamma_{\text{min}})$$
Market Price of Risk

Proposition

\[ \sigma \gamma_{\min} \leq \frac{\mu_t - r}{\sigma_t} \leq \sigma \gamma_{\max}. \]
Explicit Formulae for Optimal Portfolios

Proposition

\[ \pi_{\gamma t} = \left( (b - 1) \frac{E_t[M^{1-b} \gamma^U(D)]}{E_t[M^{1-b}]} + \frac{E_t[M \gamma^U(D)]}{E_t[M]} \right) \sigma_t^{-1} \]
Monotonicity of optimal portfolios

Proposition Optimal portfolio $\pi_{\gamma t}$ is monotone decreasing in $\gamma > 1$. 
Myopic and Hedging Components

Myopic Component

$$\pi_{\gamma t}^{\text{myopic}} = \frac{\mu t - r}{\gamma \sigma_t^2} = \gamma^{-1} \pi_{1t}.$$  

Decomposition:

$$\pi_{\gamma t} = \pi_{\gamma t}^{\text{myopic}} + \pi_{\gamma t}^{\text{hedging}},$$

There is no short selling.
Myopic and Hedging Components

Myopic Component

\[ \pi_{\gamma t}^{\text{myopic}} = \frac{\mu t - r}{\gamma \sigma_t^2} = \gamma^{-1} \pi_{1t}. \]

Decomposition:

\[ \pi_{\gamma t} = \pi_{\gamma t}^{\text{myopic}} + \pi_{\gamma t}^{\text{hedging}}, \]

There is no short selling.
Hedging and risk aversion

Proposition

\[ \pi_{\gamma t}^{\text{hedging}} > 0 \text{ if } \gamma > 1; \]

\[ \pi_{\gamma t}^{\text{hedging}} < 0 \text{ if } \gamma < 1. \]
Large population limit

Assumption. Risk aversions densely cover an interval \([1, \Gamma]\).
**Long run dynamics**

**Definition.** Given a random process $X_t$, $t \in [0, T]$, we define

$$X(\lambda) = \lim_{T \to \infty} X_{\lambda T}$$

for $\lambda \in (0, 1)$. 
Long run drift and volatility

Theorem

\[ \mu(\lambda) = \begin{cases} r + (1 + \lambda^{-1})^2 \sigma^2, & \lambda \geq (\Gamma - 1)^{-1} \\ r + \Gamma^2 \sigma^2, & \lambda < (\Gamma - 1)^{-1} \end{cases} \]

and

\[ \sigma(\lambda) = \begin{cases} \sigma (1 + \lambda^{-1}), & \lambda \geq (\Gamma - 1)^{-1} \\ \sigma \Gamma, & \lambda < (\Gamma - 1)^{-1} \end{cases} \]
The special role of risk aversion two.

For $\lambda$ close to one, drift and volatility

$$\lambda(\lambda) = 2\sigma, \quad \mu(\lambda) = r + 4\sigma^2$$

are determined by agent with risk aversion two

and not by the log agent!
The special role of risk aversion two.

For $\lambda$ close to one, drift and volatility

$$\lambda(\lambda) = 2\sigma, \quad \mu(\lambda) = r + 4\sigma^2$$

are determined by agent with risk aversion two
and not by the log agent!
Volatility and Sharpe Ratio are Decreasing

**Corollary** In the limit $T \to \infty$, the instantaneous drift, the volatility and the Sharpe ratio of the stock are monotone decreasing in $t = \lambda T$. 
Long run myopic portfolios

Proposition

\[ \pi^\text{myopic}_\gamma (\lambda) = \frac{1}{\gamma} \]

is independent of \( \lambda \).
Theorem. We have

- if \( \lambda > (\Gamma - 1)^{-1} \) then
  \[
  \pi_\gamma(\lambda) = \gamma^{-1} + \frac{\gamma - 1}{(\lambda + 1) \gamma (1 + \lambda (\gamma - 1))} ;
  \]

- if \( \lambda < (\Gamma - 1)^{-1} \) then
  \[
  \pi_\gamma(\lambda) = \gamma^{-1} + (\gamma - 1) \frac{(\Gamma - 1) (1 + \lambda (\gamma - 1)) - (\gamma - 1)}{\Gamma \gamma (1 + \lambda (\gamma - 1))}.
  \]
Monotonicity properties

Proposition. Let $\lambda > (\Gamma - 1)^{-1}$. Then,

- the hedging portfolio $\pi^\text{hedging}_\gamma(\lambda)$ is monotone decreasing in $\lambda$ for each fixed $\gamma$;
- for each fixed $\lambda$, $\pi^\text{hedging}_\gamma(\lambda)$ is monotone increasing in $\gamma$ for $\gamma < 1 + \lambda^{-1/2}$ and is monotone decreasing for $\gamma > 1 + \lambda^{-1/2}$. 
Optimal portfolio as a function of $\lambda = t/T$

Figure 1: Long run portfolio weights for gamma=2, as lambda varies.

Risk Aversion and Equilibrium Optimal Portfolios in Large Markets
Hedging portfolio as a function of risk aversion

Figure 2: Hedging portfolio values for $\lambda=0.5$, as $\gamma$ varies between 1 and 10.
Conclusions

- With more than two agents, agents impact relative extinction of each other;
- Long run drift and volatility are determined by the agent with risk aversion two;
- Hedging demand never vanishes and may exhibit unexpected patterns in terms of risk aversion;
- Close to $t = T$, agent with risk aversion two has the highest hedging demand.
Thank You!