A Principal-Agent Model of Sequential Testing

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Motivation

In many situations a principal hires an agent to acquire information for her:

- A financial institution hires an expert to perform costly tasks (make experiments, read the literature)
- A politician hires an expert to perform interviews and assess the popularity of policies
- A sports team seeks the advice of a scout to obtain information about a young soccer player
All these situations share common similarities:

- The process of information acquisition is dynamic
- The level of effort chosen by the agent is private information
- The signal observed by the agent is private information
- The agent may start with better information.

Goal of this paper: study optimal mechanisms for agency problems with the above features
Two states of the world $\omega \in \{B, G\}$
The Model with Symmetric Initial Information

- Two states of the world $\omega \in \{B, G\}$
- The principal takes an action $A \in \{B, G\}$
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Action $G(B)$ is preferred if the state is $G(B)$
The Model with Symmetric Initial Information

- Two states of the world $\omega \in \{B, G\}$
- The principal takes an action $A \in \{B, G\}$
- Action $G(B)$ is preferred if the state is $G(B)$
- Prior $p_0$ that the state is $G$
The principal hires an agent at $t = 0$ to perform costly tests ($c > 0$) for (at most) $T$ periods.
The Testing

- The principal hires an agent at \( t = 0 \) to perform costly tests \( (c > 0) \) for (at most) \( T \) periods
- (At most) 1 test per period
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A test yields a signal $s \in \{B, G\}$ distributed according to:

$$\Pr(s = G \mid \omega = G) = \alpha$$

$$\Pr(s = G \mid \omega = B) = 0$$
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- If $s = G$ then $\Pr(\omega = G \mid s = G) = 1$
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- If $s = G$ then $\Pr(\omega = G \mid s = G) = 1$
- After sequence of $t$ signals $B$:
  \[
  p_t = \frac{p_0 (1 - \alpha)^t}{p_0 (1 - \alpha)^t + 1 - p_0}
  \]
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- After sequence of $t$ signals $B$:
  \[
  p_t = \frac{p_0 (1 - \alpha)^t}{p_0 (1 - \alpha)^t + 1 - p_0}
  \]
- The true state is public observable after the principal takes an action
The agent chooses to acquire a signal ($e$) or shirk ($ne$)
The Agency Problem

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- The agent’s action and the signal are private information
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- The agent’s action and the signal are private information
- The agent is risk neutral and has limited liability
- Common discount factor \(\delta \in (0, 1]\)
The principal commits to a contract to motivate the agent to acquire information in every period and to reveal it truthfully until the agent announces $s = G$ or until period $T$ is reached.
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- Payments may depend on the sequence of messages and on the realized state of the world.
- Payments are at the end.
- The agent receives 0 if he announces \( s = G \) and the state is \( B \).
A contract is \( w = \left( (w(t))_{t=0}^{T-1}, w(G), w(B) \right) \), where
The Contract (II)

- A contract is  
  $$w = \left( (w(t))^{T-1}_{t=0}, w(G), w(B) \right),$$  
  where 

  - $w(t)$ is the payment in period $t$ if the agent announces $G$ and $\omega = G$
The Contract (II)

- A contract is \( w = \left( (w(t))_{t=0}^{T-1}, w(G), w(B) \right), \) where
  1. \( w(t) \) is the payment in period \( t \) if the agent announces \( G \) and \( \omega = G \)
  2. \( w(G) \) is the payment in period \( T - 1 \) if the agent announces \( (B, \ldots, B) \) and \( \omega = G \)
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3. \( w(B) \) is the payment in period \( T - 1 \) if the agent announces \( (B, \ldots, B) \) and \( \omega = B \)
The set of the agent’s private histories at the beginning of period $t$ is:

$$H^t = \{ne, G, B\}^t$$
Histories and Strategies

- The set of the agent’s private histories at the beginning of period $t$ is:

$$H^t = \{ ne, G, B \}^t$$

- A pure strategy for the agent is $\sigma = (\sigma^A_t, \sigma^M_t)_{t=0}^{T-1} \in \Sigma$, where:

$$\sigma^A_t : H^t \rightarrow \{ e, ne \}$$

$$\sigma^M_t : H^{t+1} \rightarrow \{ G, B \}$$
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- $\sigma^*$ is the strategy consisting of always acquiring information and always revealing it truthfully.
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$$\sigma_M^t : H^{t+1} \to \{G, B\}$$

- $\sigma^*$ is the strategy consisting of always acquiring information and always revealing it truthfully.

- For a contract $w$, $u(\sigma, p_0; w)$ is the expect utility of the agent from the strategy $\sigma$. 
The Principal’s Problem

The optimal contract is a solution to:

$$\min_{w \geq 0} u(\sigma^*, p_0; w)$$

s.t.  $$u(\sigma^*, p_0; w) \geq u(\sigma, p_0; w)$$ for every $$\sigma \in \Sigma$$
Lemma 1: A contract $w$ is incentive compatible if and only if it satisfies:

$$w(t) \geq \delta w(t+1), \quad t = 0, \ldots, T - 2$$  \hspace{1cm} (1)

(Discounted payments are weakly decreasing)

$$u(0, p_0; w) \geq p_0 w(0)$$  \hspace{1cm} (2)

(No guessing immediately)

$$u(t, p_t; w) \geq \delta u(t+1, p_t; w), \quad t = 0, \ldots, T - 1$$  \hspace{1cm} (3)

(No one-period shirking)

- $u(t, p; w)$ is the continuation utility of the agent who starts period $t$ with beliefs $p$ and always acquires information and announces it truthfully.
Intuition for Lemma 1

- **Discounted payments are weakly decreasing (1):** Report immediately $G$
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- **No guessing immediately (2):** Since payments are decreasing there is no point in acquiring some information and/or waiting before guessing

- **No one-period shirking (3):**

\[
\begin{align*}
\text{One-period shirker} \\
0 & \quad \uparrow & \quad p_t & \quad \quad p_{t-1} & \quad \quad p_0 \\
& & & & & 1
\end{align*}
\]

\[
w(0) \geq \delta w(1) \geq \ldots \delta^t w(t) \geq \ldots \geq \delta^{T-1} w(T-1)
\]
**Proposition 1:** The optimal contract is

\[
w^*(t) = \frac{c}{\alpha p_t} + c \sum_{t'=t+1}^{T-1} \delta^{t'-t} \left( \frac{1}{p_{t'}} - 1 \right), \quad t = 0, \ldots, T - 1
\]

\[
w^*(G) = 0
\]

\[
w^*(B) = \frac{c}{\alpha (1-p_0)\delta^{T-1}} + c \sum_{t'=0}^{T-2} \delta^{-t'}
\]

(4)
**Proposition 1:** The optimal contract is

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\[ w^*(G) = 0 \tag{4} \]

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**Information Rent** (see 8)

\[ u(\sigma^*, p_0; w) = \frac{c}{\alpha} + c (1 - p_0) \sum_{t=1}^{T-1} \left( \frac{\delta}{1-\alpha} \right)^t \tag{5} \]
Model with Asymmetric Initial Information

- With probability $\rho$ the agent is a high type and has a prior $p_0^h \in (0, 1)$
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With probability $1 - \rho$ the agent is a low type and has a prior $p_0^l \in (0, p_0^h)$
With probability $\rho$ the agent is a high type and has a prior $p_0^h \in (0, 1)$

With probability $1 - \rho$ the agent is a low type and has a prior $p_0^l \in (0, p_0^h)$

The principal commits to a pair of mechanisms: $(w^h, w^l)$ where

$$w^k = \left( \left( w^k(t) \right)_{t=0}^{T-1}, w^k(G), w^k(B) \right)$$
A contract $w^k$ is suitable for type $k \in (h, \ell)$ if

$$u\left(\sigma^*, p_0^k; w^k\right) \geq u\left(\sigma, p_0^k; w^k\right) \text{ for every } \sigma \in \Sigma$$
Incentive Compatibility

1. A contract $w^k$ is suitable for type $k \in (h, \ell)$ if

$$u\left(\sigma^*, p_0^k; w^k\right) \geq u\left(\sigma, p_0^k; w^k\right) \quad \text{for every } \sigma \in \Sigma$$

2. A pair of contracts is incentive compatible if and only if it is suitable for both types and

$$u\left(\sigma^*, p_0^h; w^h\right) \geq u\left(\sigma, p_0^h; w^h\right), \quad \sigma \in \Sigma$$

$$u\left(\sigma^*, p_0^\ell; w^\ell\right) \geq u\left(\sigma, p_0^\ell; w^h\right), \quad \sigma \in \Sigma$$
The Principal’s Problem

\[
\min_{w^h, w^\ell} \rho u \left( 0, p^h_0; w^h \right) + (1 - \rho) u \left( 0, p^\ell_0; w^\ell \right)
\]

s.t. \((w^h, w^\ell)\) is incentive compatible
Proposition 2: If \((w^h, w^\ell)\) is optimal then

\[ w^\ell(t) = w^* \left( t; p_0^\ell \right) \]

for every \(t = 0, \ldots, T - 1\), and

\[ w^\ell(G) = w^* \left( G; p_0^\ell \right) \]
Proposition 2: If \((w^h, w^l)\) is optimal then

\[ w^l(t) = w^* \left( t; p_0^l \right) \]

for every \( t = 0, \ldots, T - 1 \), and

\[ w^l(G) = w^* \left( G; p_0^l \right) \]

The principal should set the payments in the state \( G \) as low as possible for the low type agent.
If $w^l(t) > w^*(t; p^l_0)$ for some $t$ we can decrease some $w^l(\hat{t})$ and increase $w^l(B)$ in such a way that the low type is made indifferent, while the high type who imitates the low type is made worse off.
Intuition for Proposition 2

1. If $w^\ell(t) > w^* (t; p_0^\ell)$ for some $t$ we can decrease some $w^\ell(\hat{t})$ and increase $w^\ell(B)$ in such a way that the low type is made indifferent, while the high type who imitates the low type is made worse off.

2. Then we can find a way to decrease some payments in $w^h$. Therefore we construct a cheaper mechanism.
Proposition 3: If \((w^h, w^l)\) is incentive compatible then

\[ u(0, p^l_0; w^l) > p^l_0 w^l(0) \]
Proposition 3: If \((w^h, w^l)\) is incentive compatible then

\[ u \left( 0, p_0^l; w^l \right) > p_0^l w^l(0) \]

Therefore the low type receives an information rent.
We can write $u(0, p_0; w^h) = a(0, p_0^h; w^h) + p_0 b(0, p_0^h; w^h)$.
Intuition for the Low Type Rent

We can write \( u(0, p_0; w^h) = a(0, p_0^h; w^h) + p_0 b(0, p_0^h; w^h) \).
Lemma 2: If \((w^h, w^\ell)\) is optimal then

\[ w^h(0) = w^\ell(0) = w^*(0; p^\ell_0) \]  

\[ u(0, p^h_0; w^h) = p^h_0 w^h(0) \]  

Corollary: The high type rent is: 

\[ p^h_0 [w^*(0; p^\ell_0) - w^*(0; p^h_0)] > 0 \]
Lemma 2: If \((w^h, w^\ell)\) is optimal then

\[
w^h(0) = w^\ell(0) = w^*(0; p^\ell_0)
\]  

(6)

and

\[
u(0, p^h_0; w^h) = p^h_0 w^h(0)
\]

(7)

Corollary: The high type rent is:

\[
p^h_0 [w^* (0; p^\ell_0) - w^* (0; p^h_0)] > 0
\]

Higher payments are necessary to motivate the low type. This leads to rents for the high type.
Intuition for Lemma 2

- \( u(0, p^h_0; w^h) = p^h_0 w^h(0) \). If \( u(0, p^h_0; w^h) > p^h_0 w^h(0) \) we can increase \( w^h(0) \) and decrease \( w^h(B) \) in such a way that the high type is indifferent, while the low type who imitates the high type is worse off. Then we decrease \( w^\ell(B) \) and construct a cheaper mechanism.
Intuition for Lemma 2

- \( u(0, p^h_0; w^h) = p^h_0 w^h(0) \). If \( u(0, p^h_0; w^h) > p^h_0 w^h(0) \) we can increase \( w^h(0) \) and decrease \( w^h(B) \) in such a way that the high type is indifferent, while the low type who imitates the high type is worse off. Then we decrease \( w^l(B) \) and construct a cheaper mechanism.

- \( w^h(0) = w^l(0) = w^*(0; p^l_0) \). If \( w^h(0) > w^l(0) = w^*(0; p^l_0) \) and \( u(0, p^h_0; w^h) = u(0, p^h_0; w^l) \) we can offer the same contract \((w^l, w^l)\) to both types. We will have \( u(0, p^h_0; w^h) > p^h_0 w^l(0) \). Thus we can decrease \( w^l(B) \) and construct a cheaper mechanism.
Discounted payments in state G

- Low Type
- High Type

(time)
Optimal Contracts with Heterogeneous Priors: Frontloading Payments to the High Type

- Distant initial beliefs \( p_0^\ell \leq p_1^h \)

![Discounted payments in state G](image_url)
Proposition 4: Suppose that $p_0^\ell \leq p_1^h$. There exists an optimal mechanism $(w^h, w^\ell)$ which satisfies

$$w^h (B) \in \left[ w^* (B; p_0^h), w^\ell (0) \right]$$

and one of the following two conditions:

(i) there exists $\hat{t} \in \{1, \ldots, T - 1\}$ such that

$$w^h (t) = \frac{w^* (0; p_0^\ell)}{\delta^t}, \quad t < \hat{t}$$

$$w^h (\hat{t}) \in \left( w^* (\hat{t}; p_0^\ell), \frac{w^h (\hat{t} - 1)}{\delta} \right]$$

$$w^h (t) = w^* (t; p_0^\ell), \quad t > \hat{t}$$

$$w^h (G) = 0$$

(ii)

$$w^h (t) = \frac{w^* (0; p_0^\ell)}{\delta^t}, \quad t = 0, \ldots, T - 1$$

$$w^h (G) \in \left[ 0, w^h (T - 1) - \frac{c}{\alpha p^h_{T-1}} \right]$$
Intuition for Proposition 4

- Large payments to the high type in initial periods are not attractive to the low type
Intuition for Proposition 4

- Large payments to the high type in initial periods are not attractive to the low type
- Therefore payments should be set as frontloaded as possible
Decomposing the Information Rent

- Decompose (5) in two components:
  - **Hidden Information rent:**
    $$\frac{c}{\alpha} + c (1 - p_0) \sum_{t=1}^{T-1} \delta^t$$
    \hfill (8)
Decomposing the Information Rent

- Decompose (5) in two components:
  - **Hidden Information rent:**
    \[
    \frac{c}{\alpha} + c (1 - p_0) \sum_{t=1}^{T-1} \delta^t
    \]  
    (8)
  - **Moral Hazard rent:**
    \[
    c (1 - p_0) \sum_{t=1}^{T-1} \left( \frac{\alpha \delta}{1 - \alpha} \right)^t
    \]  
    (9)
Comparative Statics on Information Rents

- **Hidden Information rent** \( \frac{c}{\alpha} + c (1 - p_0) \sum_{t=1}^{T-1} \delta^t : \)

Increasing in \( \delta \)

Decreasing in \( \alpha \) and \( p_0 \)

- **Moral Hazard rent** \( c (1 - p_0) \sum_{t=1}^{T-1} \alpha \delta^t : \)

Increasing in \( \delta \) and \( \alpha \)

Decreasing in \( p_0 \)
Comparative Statics on Information Rents

- **Hidden Information rent** $\frac{c}{\alpha} + c (1 - p_0) \sum_{t=1}^{T-1} \delta_t$:

  - Increasing in $\delta$

- **Moral Hazard rent** $c (1 - p_0) \sum_{t=1}^{T-1} \delta_t$:

  - Increasing in $\delta$ and $\alpha$

  - Decreasing in $p_0$
Comparative Statics on Information Rents

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  1. Increasing in \( \delta \)
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Comparative Statics on Information Rents

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Comparative Statics on Information Rents

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- **Moral Hazard rent** \( c (1 - p_0) \sum_{t=1}^{T-1} \left( \frac{\alpha \delta}{1-\alpha} \right)^t : \)
  1. Increasing in \( \delta \) and \( \alpha \)
  2. Decreasing in \( p_0 \)
**Literature Review**


Conclusion

- Analyzed optimal dynamic contracts to motivate an agent to acquire information
- Obtained closed-form solution to the model with symmetric initial information
- Characterized the optimal contract with asymmetric initial information
- Derived and analyzed new information rents