Bounds on Revenue Distributions in Counterfactual Auctions with Reserve Prices

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North American Econometric Society Meeting
June 5, 2009
Structural models for first-price auctions (FPA)

- **Bids** in FPA $\rightarrow$ **parameters** $\rightarrow$ **counterfactual** revenues (CFR)
  - private value (PV): nonparametric (N.P.) identification
    - Guerre, Perrigne&Vuong (2000), Li, Perrigne&Vuong (2002)
  - common value (CV): N.P. unidentified
    - Laffont&Vuong (1996), Athey&Haile (2005)

- Open questions related to CV
  - distinguishing PV & CV
  - size of differences in $E(Rev)$ across auction formats
  - choice of optimal reserve prices (RP)
  - what if CV is analyzed as PV?
Solution proposed in this paper

- Distri. of bids $\rightarrow$ bounds on distr. of CFR
  - minimum restrictions on the structure
  - tight, sharp, intuitive

- contributions:
  - partial i.d. of counterfactual distribution
  - compare formats with different RP and for RA bidders
  - empirical illustration: U.S. municipal bonds
The benchmark model (Milgrom & Weber 1982)

- FPA with $n$ R.N. bidders, nonbinding reserve
  - signal distr. $F_X$ : affiliated, exchangeable
  - values $V_i = \theta(X_i, X_{-i})$: nonneg., bounded and cts, exch’ble & non-$\downarrow$ in $X_{-i}$, strictly $\uparrow$ in $X_i$.

- Notation:
  - Seller’s reservation value: $v_0$;
  - Equi. under reserve $r$: $b_r$; Distri. of $b_r(X)$: $G_B$
  - Expected payoff for winner: $v(x) \equiv E(V_i | X_i = x, Y_i \leq x)$ with $Y_i \equiv \max_{j \neq i} X_j$
  - Screening level: $x^*(r) \equiv v^{-1}(r)$
Main idea in a nutshell:
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- CFR distr. in FPA under reserve $r$:
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- CFR distr. in FPA under reserve $r$:
  - by monotonicity of $b_0$,

$$F_{RI}(r)(p) = 0 \quad \forall p < v_0$$

$$= \Pr(b_0(X^{(1)}) < b_0(x^*(r))) \quad \forall p \in [v_0, r)$$

$$= \Pr(b_0(X^{(1)}) \leq b_0(b_r^{-1}(p))) \quad \forall p \in [r, +\infty)$$
Main idea in a nutshell:

- CFR distr. in FPA under reserve $r$:
  - by monotonicity of $b_0$,
    \[ F_{R^I(r)}(p) = \begin{cases} 0 & \forall p < v_0 \\ \Pr(b_0(X^{(1)}) < b_0(x^*(r))) & \forall p \in [v_0, r) \\ \Pr(b_0(X^{(1)}) \leq b_0(b_r^{-1}(p))) & \forall p \in [r, +\infty) \end{cases} \]
  - $\exists \delta_r: b_0(X) \longrightarrow b_r(X)$ for $X \geq x^*(r)$
Main idea in a nutshell:

- CFR distr. in FPA under reserve $r$:
  - by monotonicity of $b_0$,
    \[
    F_{R\hat{r}(r)}(p) = 0 \quad \forall p < v_0
    \]
    \[
    = \Pr(b_0(X^{(1)}) < b_0(x^*(r))) \quad \forall p \in [v_0, r]
    \]
    \[
    = \Pr(b_0(X^{(1)}) \leq b_0(b_r^{-1}(p))) \quad \forall p \in [r, +\infty)
    \]

- $\exists \delta_r: b_0(X) \longrightarrow b_r(X)$ for $X \geq x^*(r)$
  - $F_{R\hat{r}(r)}$: identified if $\delta_r$ could be recovered from bid distr.
Main idea in a nutshell:

- CFR distr. in FPA under reserve \( r \):
  - by monotonicity of \( b_0 \),
    \[
    F_{R^I(r)}(p) = \begin{cases} 
    0 & \forall p < v_0 \\
    \Pr(b_0(X^{(1)}) < b_0(x^*(r))) & \forall p \in [v_0, r) \\
    \Pr(b_0(X^{(1)}) \leq b_0(b_r^{-1}(p))) & \forall p \in [r, +\infty)
    \end{cases}
    \]

- \( \exists \delta_r: b_0(X) \longrightarrow b_r(X) \) for \( X \geq x^*(r) \)
  - \( F_{R^I(r)} \): identified if \( \delta_r \) could be recovered from bid distr.
  - \( \delta_r \) solves:
    \[
    \delta_r'(b; G_B^0) = [\xi(b; G_B^0) - \delta_r(b; G_B^0)]g_{M|B}^0(b|b)/G_{M|B}^0(b|b) \tag{1}
    \]
    with unknown BC \( \delta_r(b_0(x^*(r))) = r \), where \( M_i \equiv \max_{j \neq i} B_j \) and \( \xi(b) = b + G_{M|B}^0(b|b)/g_{M|B}^0(b|b) \).
Main idea in a nutshell:

CFR distr. in FPA under reserve $r$:

- by monotonicity of $b_0$,
  \[ F_{R^I(r)}(p) = 0 \quad \forall p < v_0 \]
  \[ = \Pr(b_0(X^{(1)}) < b_0(x^*(r))) \quad \forall p \in [v_0, r) \]
  \[ = \Pr(b_0(X^{(1)}) \leq b_0(b_r^{-1}(p))) \quad \forall p \in [r, +\infty) \]

- $\exists \delta_r$: $b_0(X) \longrightarrow b_r(X)$ for $X \geq x^*(r)$
  - $F_{R^I(r)}$: identified if $\delta_r$ could be recovered from bid distr.
  - $\delta_r$ solves:
    \[
    \delta'_r(b; G_0^0) = [\xi(b; G_0^0) - \delta_r(b; G_0^0)]g_{M|B}^0(b|b) / g_{M|B}^0(b|b) \quad (1)
    \]
  with unknown BC $\delta_r(b_0(x^*(r))) = r$, where $M_i \equiv \max_{j \neq i} B_j$ and $\xi(b) = b + G_{M|B}^0(b|b) / g_{M|B}^0(b|b)$.

- bound $b_0(x^*(r)) \Longrightarrow$ envelope $\delta_r \Longrightarrow$ bound $F_{R^I(r)}$
Bounds on screening levels (SL)

- Bounding winner’s payoffs
  - $v(x) \leq v_h(x) \equiv E(V_i|X_i = Y_i = x)$: by affiliation
  - $v(x) \geq v_l(x) \equiv E(v_h(Y)|X_i = x, Y_i \leq x)$: by equi. cond. in 2nd-price auctions (SPA)
  - $v_h, v_l$ are both $\uparrow$ due to affiliation

- Bounds on SL
  - Just invert $v_l, v_h$ at $r$: $v_h^{-1}(r) \leq x^*(r) \leq v_l^{-1}(r)$
  - Exhaustive, tight
  - Lower bd: PV; upper bd: extreme case of CV
Bound $b_0(x^*(r))$ (HMB)
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  • from F.O.C. and use c.o.v. btw $b_0$ and $x$,

$$v_h(x) = \xi(b_0(x)) ; \quad v_l(x) = \xi_l(b_0(x)) \equiv \int_{b_0(x_L)}^{b} \xi(\tilde{b}) \frac{g_{M|B}(\tilde{b}|b)}{g_{M|B}(b|b)} d\tilde{b}$$

(2)
Bound $b_0(x^*(r))$ \( (HMB) \)

- from F.O.C. and use c.o.v. btw $b_0$ and $x$,

$$v_h(x) = \xi(b_0(x)); \quad v_l(x) = \xi_l(b_0(x)) \equiv \int_{b_0(x_L)}^b \xi(\tilde{b}) \frac{g^0_{M|B}(\tilde{b}|b)}{G^0_{M|B}(b|b)} d\tilde{b}$$

(2)

- note: $\xi(b_0(x_l(r))) = v_h(x_l(r)) = r$ and $\xi_l(b_0(x_h(r))) = v_l(x_h(r)) = r$
Bound $b_0(x^*(r))$ \((HMB)\)

- from F.O.C. and use c.o.v. btw $b_0$ and $x$,

\[
v_h(x) = \zeta(b_0(x)) \quad \text{and} \quad v_l(x) = \bar{\zeta}(b_0(x)) \equiv \int_{b_0(x_L)}^{b} \zeta(\tilde{b}) \frac{g^0_{M|B}(\tilde{b}|b)}{G^0_{M|B}(b|b)} \, d\tilde{b}
\]

- note: $\zeta(b_0(x_l(r))) = v_h(x_l(r)) = r$ and $\bar{\zeta}(b_0(x_h(r))) = v_l(x_h(r)) = r$

- invert $\zeta, \bar{\zeta}$ to get \textit{tight, sharp} bounds on $b_0(x^*(r))$ (denoted $b_0(x_k(r))$)
Bound $b_0(x^*(r))$ (HMB)

- from F.O.C. and use c.o.v. btw $b_0$ and $x$,

$$v_h(x) = \xi(b_0(x)) ;\ v_l(x) = \xi_l(b_0(x)) \equiv \int_{b_0(x_L)}^{b} \xi(b) \frac{g^0_{M|B}(b|b)}{G^0_{M|B}(b|b)} dB$$

(2)

- note: $\xi(b_0(x_l(r))) = v_h(x_l(r)) = r$ and $\xi_l(b_0(x_h(r))) = v_l(x_h(r)) = r$

- invert $\xi, \xi_l$ to get tight, sharp bounds on $b_0(x^*(r))$ (denoted $b_0(x_k(r))$)

Envelopes on $\delta_r$: ($\{\delta_{r,k}\}_{k=l,h}$)
Bound $b_0(x^*(r))$ (HMB)

- from F.O.C. and use c.o.v. btw $b_0$ and $x$,

$$v_h(x) = \zeta(b_0(x)) ; \; v_l(x) = \zeta_l(b_0(x)) \equiv \int_{b_0(x_L)}^b \zeta(\tilde{b}) \frac{g_0^0(b|\tilde{b})}{G_0^0(b|\tilde{b})} d\tilde{b}$$

(2)

- note: $\zeta(b_0(x_l(r))) = v_h(x_l(r)) = r$ and $\zeta_l(b_0(x_h(r))) = v_l(x_h(r)) = r$

- invert $\zeta, \zeta_l$ to get **tight, sharp** bounds on $b_0(x^*(r))$ (denoted $b_0(x_k(r))$)

- **Envelopes** on $\delta_r$: ($\{\delta_{r,k}\}_{k=l,h}$)

  - solves $DE$ in (1) with $BC\; \delta_{r,k}(b_0(x_k(r))) = r$
**Bound** \( b_0(x^*(r)) \) \((HMB)\)

- from F.O.C. and use c.o.v. btw \( b_0 \) and \( x \),

\[
\begin{align*}
    v_h(x) &= \xi(b_0(x)) \\
    v_l(x) &= \xi_l(b_0(x)) \equiv \int_{b_0(x_L)}^b \xi(\tilde{b}) \frac{g^0_{M|B}(\tilde{b}|b)}{G^0_{M|B}(b|b)} d\tilde{b}
\end{align*}
\]

(2)

- note: \( \xi(b_0(x_l(r))) = v_h(x_l(r)) = r \) and \( \xi_l(b_0(x_h(r))) = v_l(x_h(r)) = r \)
- invert \( \xi, \xi_l \) to get tight, sharp bounds on \( b_0(x^*(r)) \) (denoted \( b_0(x_k(r)) \))

**Envelopes** on \( \delta_r \): \( \{\delta_{r,k}\}_{k=l,h} \)

- solves \( DE \) in (1) with \( BC \) \( \delta_{r,k}(b_0(x_k(r))) = r \)
- \( \uparrow \) in bids

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• **Bound** $b_0(x^*(r))$ (*HMB*)
  
  • from F.O.C. and use c.o.v. btw $b_0$ and $x$,

  $$v_h(x) = \xi(b_0(x)) ; v_l(x) = \xi_l(b_0(x)) \equiv \int_{b_0(x_L)}^{b} \xi(\tilde{b}) \frac{g_{0|M|B}(\tilde{b}|b)}{G_{0|M|B}(b|b)} d\tilde{b}$$

  (2)

  • note: $\xi(b_0(x_l(r))) = v_h(x_l(r)) = r$ and
  
  $\xi_l(b_0(x_h(r))) = v_l(x_h(r)) = r$

  • invert $\xi, \xi_l$ to get *tight, sharp* bounds on $b_0(x^*(r))$ (denoted $b_0(x_k(r))$)

  • **Envelopes** on $\delta_r$: (\{\delta_{r,k}\}_{k=l,h})
    
    • solves DE in (1) with BC $\delta_{r,k}(b_0(x_k(r))) = r$
    
    • ↑ in bids
    
    • inverses at $p$: *tight, sharp* bounds on $b_0(b_r^{-1}(p)) \forall p > r$
• Bounds on $F_{R^I(r)}(p)$: for $k = l, h$

\[
F^k_{R^I(r)}(p) = \Pr(b_0(X^{(1)}) < b_0(x_k(r)) \quad \forall p \in [v_0, r)
\]

\[
= \Pr(b_0(X^{(1)}) \leq \delta_{r,k}^{-1}(p)) \quad \forall p \in [r, +\infty)
\]
Main idea:

In SPA with $r$

- $\beta_r(x) = \nu_h(x) = \tilde{\zeta}(b_0(x))$ for $x \geq x^*_r$, $\beta_r(x) < r$ for $x < x^*_r$
- id. of $F_{R^I(r)}(t)$ for $t > r$ if $\nu_h(x^*_r)$ were known
- Sol'n: bound $\nu_h(x^*_r)$ by $\nu_h(x_l(r)) = r$ and $\nu_h(x_h(r)) = \tilde{\zeta}(b_0(x_h(r)))$
• Replace population distri. with empirical analogs
  • estimate bounds on $b_0(x^*(r))$ by inverting kernel estimates $\hat{\xi}$ and $\hat{\xi}_l$ at $r$
  • estimate the envelope on $\delta_r$ by plugging $\hat{\xi}^{-1}(r)$ and $\hat{\xi}_l^{-1}(r)$ into the solution of (1)
  • inverting the envelope at $p$ to estimate bounds on $b_0(b_r^{-1}(p))$
  • Roadmap of consistency proof:
    • smoothness of $G^0_B$, and then $\hat{\xi}_l, \xi \xrightarrow{P} \xi_l, \xi$ unif. over $\hat{C}_\delta(B)$. 
Replace population distri. with empirical analogs

- estimate bounds on $b_0(x^*(r))$ by inverting kernel estimates $\hat{\xi}$ and $\hat{\xi}_l$ at $r$
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  - inverses at $r$ ($\hat{b}^0_{k,r} \xrightarrow{P} b_0(x_k(r))$ for $k = l, h$ and interesting $r > 0$
Replace population distri. with empirical analogs

- estimate bounds on \( b_0(x^*(r)) \) by inverting kernel estimates \( \hat{\xi} \) and \( \hat{\xi}_l \) at \( r \)
- estimate the envelope on \( \delta_r \) by plugging \( \hat{\xi}^{-1}(r) \) and \( \hat{\xi}_l^{-1}(r) \) into the solution of (1)
- inverting the envelope at \( p \) to estimate bounds on \( b_0(b_r^{-1}(p)) \)
- Roadmap of consistency proof:
  - smoothness of \( G_B^0 \), and then \( \hat{\xi}_l, \hat{\xi} \xrightarrow{p} \xi_l, \xi \) unif. over \( \hat{C}_\delta(B) \)
  - inverses at \( r \) (\( \hat{b}_{k,r}^0 \xrightarrow{p} b_0(x_k(r)) \) for \( k = l, h \) and interesting \( r > 0 \))
  - \( \hat{\delta}_{k,r}(.; \hat{b}_{k,r}^0) \xrightarrow{p} \delta_{k,r}(.; b_{k,r}^0) \) unif. over \( \hat{C}_\delta(B) \) and \( \hat{\delta}_{r,k}^{-1}(t) \xrightarrow{p} \delta_{r,k}^{-1}(t) \) for all \( t \)
Replace population distri. with empirical analogs

- estimate bounds on $b_0(x^*(r))$ by inverting kernel estimates $\hat{\zeta}$ and $\hat{\zeta}_l$ at $r$
- estimate the envelope on $\delta_r$ by plugging $\hat{\zeta}^{-1}(r)$ and $\hat{\zeta}_l^{-1}(r)$ into the solution of (1)
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- Roadmap of consistency proof:
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  - inverses at $r$ ($\hat{b}^0_{k,r} \xrightarrow{P} b_0(x_k(r))$ for $k = l, h$ and interesting $r > 0$
  - $\hat{\delta}_{k,r}(\cdot; \hat{b}^0_{k,r}) \xrightarrow{P} \delta_{k,r}(\cdot; b^0_{k,r})$ unif. over $\hat{C}_\delta(B)$ and $\delta_{r,k}^{-1}(t) \xrightarrow{P} \delta_{r,k}^{-1}(t)$ for all $t$
  - Glivenko-Cantelli $ULLN$: consistency of $\hat{F}^l_{RL}(r)$ and $\hat{F}^u_{RL}(r)$
Extension: obs. hetero.

- Bounds in the benchmark model extend for $F_R(r,z)$
  - condition $v_h, v, v_l, G_B^0(z), \xi_l, \xi, \delta_r, \kappa$ on $z$
  - practical problem: dimension of $z$ is large

- HHS (2003) solution: "homogenization" of bids
  - $V_i = z'\gamma + \theta(X_i, X_{-i})$ and $\{X_i\}_{i \in I} \perp Z$ conditional on $n$.
  - $b_{0i}(x, z; n, \psi) = z'\gamma + \lambda(x; n, \psi) \forall x, z \forall i$
  - given $n, \forall (z, z; x), b_i(x, z) = b_i(x, z) - z'\gamma + R'\gamma \forall i$. 

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Extension: Binding reserve in data

- New challenges:
  - bids from screened bidders are uninformative about signals (v_l(.) unidentified from $G^r_B$)
  - $G^r_B$ may be "truncated": data only include auctions with $X^{(1)} \geq x^*(r)$

- With $G^r_B$ not truncated, bounds on $F_{R^l}(r')$ extends immediately
  - need to know $n$ and replace $v_l$ with
    \[ v_{l,r}(x) = E[\max\{r, v_h(Y)\}|X = x, Y \leq x] \]

- With truncated $G^r_B$: bound conditional distribution
  \[ F_{R^l}(r')|X^{(1)} \geq x^*(r). \]
  - Probability of truncation can be identified if signals are i.i.d.
U.S. municipal bonds

- Market size: $1.8 trillion outstanding as of 2005
- Initial issuance through FPA to IBs
- Value of bonds: resale price on secondary market
- Private signals: IB’s estimates of the resale prices
Data

The data (6,721 FPA from 2004-2006, *Thompson Financial Website*):

- issuers and features (rate, maturity, par, ratings)
- type of municipal support and dummy for bank-qualification
- all bids in *TIC* (total interest costs) : used for computing dollar bids ($/$100 in par)
- number and identity of syndicates

The reference auction: \( n = 4 \) with features

- \( WACR \) : 4%
- \( WAPN \) : 21 semiannual coupons
- *Total Par*: $4.84 million
- Backed by full credit of municipalities
### CF screening prob

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<th>$\hat{b}_0(x_h(r))$</th>
<th>$b.w.$ of $b_0(x^*_r)$</th>
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Percentiles in CFR distri.

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About optimal reserve

- compare formats for both RA and RN sellers
  - need measures of $v_0$.
    - $95.71$ : present value of cash flows from the reference bond

- Utility specifications for RA bidders
  - $DARA : u(t) = \log(t)$ and $CRRA \ u(t) = \frac{t^{1-\rho}}{1-\rho}$ with $\rho = 0.6$ and $0.9$
Summary of findings

- Rev. ranking confirmed

- RA seller’s optimal choice of format depends on utility specifications

- Very marginal increase in Exp. Rev. for RN bidders if switch from naive RP to the optimal RP
Conclusion

- Distr. of bids in FPA gives informative bounds on CFR distr.
  - robust, tight, efficient
  - extend for obs. hetero. and data from binding reserve (with \( n \) known)
  - consistent N.P. estimator with convincing finite-sample performance

- Some informative answers to policy questions in our application

- Future directions
  - Asymmetry, unknown \# of potential bidders