Robust Properties of Stock Return Tails

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Overview

➡ Why tails?

➡ Tools
  ➕ Tail exponents
  ➕ OLS Hill estimator
  ➕ Gaussian crossing points

➡ Monte-carlo

➡ Stock returns

➡ Summary
Why Stock Return Tails?

➡ Quantitative measure
  ✤ Stylized fact
  ✤ Model fitting
  ✤ Not kurtosis

➡ Approximate distributions
  ✤ Risk measures

➡ Moment failure
Connections to Volatility

- Conditional variances changing
  - Drives fat tails
- Why bother with unconditional measures?
  - Stable long-range distribution estimates
  - Testing volatility models
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Power Law Tail

\[ \Pr(R < r) = F(r) \approx A |r|^{-\alpha} \]

\[ \log(F(r)) \approx \log(A) - \alpha \log(|r|) \]

\( \alpha = \text{Tail exponent} \)

\( \gamma = 1/\alpha = \text{Shape parameter} \)
GM Daily Returns
Pooled Tails and Student-t

![Graph showing the relationship between daily return and probability of return exceeding a certain value, with lines for different tails labeled T3, T4, and T5. The graph has a logarithmic scale for probability on the y-axis and daily return on the x-axis.]
Hill Estimator

\[ \hat{\gamma}_{n_k,n} = \frac{1}{n_k} \sum_{i=1}^{n_k} \left( \log(x_{(n-i+1)}) - \log(x_{(n-n_k)}) \right) \]

\[ x_{(n)} < x_{(n-1)} < \ldots < x_{(1)} \]

\[ k = \frac{n_k}{n} \]

\[ \hat{\alpha} = \frac{1}{\hat{\gamma}_{n_k,n}} \]
Hill Estimator

➡ Easy to estimate
➡ Difficult to decide on tail region
+ Very sensitive to k
+ Common for all tail estimators
Hill Estimator: Bias and Variance

\[ E(\hat{\gamma}) \approx \gamma + Bk^\rho \]

\[ \text{var}(\hat{\gamma}) \approx \frac{\gamma^2}{n_k} = \frac{\gamma^2}{k \cdot n} \]
OLS Hill:
Huisman et al., 2001, Rev. Econ. and Statistics

\[ E(\hat{\gamma}) \approx \gamma + Bk^\rho \]
Assume \( \rho=1 \)
Estimate over range of \( k \):
\( (k_j, \hat{\gamma}(k_j)) \)
OLS
\( \hat{\gamma}(k_j) = a + bk_j + \epsilon_j \)
\( \hat{\gamma}_{OLS} = \hat{a} \)
Other Approaches

- Estimate bias theoretically
- Get optimal k from bootstrap/subsampling method
Gaussian Crossing Points

⇒ Quantiles where empirical CDF crosses Gaussian CDF

\[ r : F(r) = F_G(r) \]
Gaussian Crossing Points

\[ \text{Prob}(X<x) = F(x) \]

Gaussian

Student–t(3)
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Monte-Carlo Tests

➤ How does the OLS Hill compare to Hill?
➤ How sensitive is it to k?
➤ Sample size = 20,000
➤ Student-t, 3 degrees of freedom
OLS Hill Implementation

- OLS range
  - 5 to kn incremented by 1
- Weighted least squares
Bias Estimate: Shape Estimators

$E(\gamma) - \gamma$

Tail Fraction

Hill

OLS Hill
Variance Estimate: Shape Estimators

Estimated Shape Variance vs Tail Fraction

- Hill
- OLS Hill
Mean Squared Error

\[ MSE = \frac{1}{M} \sum_{j=1}^{M} (\hat{\gamma}_j - \gamma)^2 \]

\[ \gamma = \frac{1}{3} \]
MSE: Shape Estimators

Min(Hill) = 1.1E-3, Min(OLS Hill) = 1.8E-4
Robustness

- Bands for “reasonable” k
- Move to 1.25 optimal MSE
- Record k
- Range of acceptable k’s
- Sample lengths: 2,000-20,000
MSE Optimal Tail: Hill OLS
(+/- 25 Percent Bands)
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➡ Summary
Data

- CRSP daily log returns with dividends
- Small set (14 firms)
  - Near dow
  - Full sample
- Sample size
  - Period: Jan 1926 - Dec 2004
  - N = 21016 daily observations
- Demeaned, normalized by std.
## Summary Statistics

<table>
<thead>
<tr>
<th></th>
<th>Skewness</th>
<th>Kurtosis</th>
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<tbody>
<tr>
<td>ATT</td>
<td>-0.11</td>
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<td>Coke</td>
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<td>17.9</td>
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<td>Sears</td>
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<td>US Steel</td>
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<td>Woolworth</td>
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<td>VW Index</td>
<td>-0.13</td>
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Left Tail Exponents

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<thead>
<tr>
<th></th>
<th>k=0.15</th>
<th>k=0.25</th>
<th>k=0.20</th>
<th>Std.</th>
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<td>Coke</td>
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<td>3.91</td>
<td>3.64</td>
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<tr>
<td>GM</td>
<td>3.31</td>
<td>3.69</td>
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<td>0.13</td>
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<td>3.47</td>
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<tr>
<td>VW Index</td>
<td>3.03</td>
<td>3.40</td>
<td>3.15</td>
<td>0.10</td>
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</table>
Gaussian Comparison (k=0.20)
Monte-carlo: 1000, N=20,000

<table>
<thead>
<tr>
<th>Mean</th>
<th>Std</th>
<th>Min</th>
<th>Max</th>
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<tbody>
<tr>
<td>7.93</td>
<td>0.45</td>
<td>6.82</td>
<td>9.55</td>
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## Tail Exponent Differences ($k = 0.20$)

<table>
<thead>
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<th></th>
<th>Left</th>
<th>Right</th>
<th>P-value</th>
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<td>3.58</td>
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<td>Coke</td>
<td>3.53</td>
<td>3.92</td>
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<tr>
<td>Exxon</td>
<td>3.64</td>
<td>3.60</td>
<td>0.60</td>
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<tr>
<td>GM</td>
<td>3.45</td>
<td>3.51</td>
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<td>IBM</td>
<td>3.26</td>
<td>3.50</td>
<td>0.10</td>
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<tr>
<td>VW Index</td>
<td>3.15</td>
<td>2.82</td>
<td>0.98</td>
</tr>
</tbody>
</table>
Tail Exponent Differences (Right/Left)

- Observations for $p < 0.05$: 2
- Observations for $0.05 < p < 0.95$: 12
- Observations for $p > 0.95$: 1
First - Last 5000 (daily returns)

<table>
<thead>
<tr>
<th>Company</th>
<th>First</th>
<th>Last</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ATT</td>
<td>4.06</td>
<td>3.45</td>
<td>0.91</td>
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<tr>
<td>Coke</td>
<td>3.18</td>
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<tr>
<td>Exxon</td>
<td>3.58</td>
<td>3.81</td>
<td>0.27</td>
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<tr>
<td>GM</td>
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<td>0.09</td>
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<tr>
<td>IBM</td>
<td>2.91</td>
<td>3.20</td>
<td>0.23</td>
</tr>
<tr>
<td>VW Index</td>
<td>3.34</td>
<td>4.36</td>
<td>0.00</td>
</tr>
</tbody>
</table>
First - Last (5,000 daily returns)

- $p < 0.05$: 4 observations
- $0.05 < p < 0.95$: 11 observations
- $p > 0.95$: 0 observations
<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ATT</td>
<td>0.026</td>
<td>0.032</td>
<td>0.05</td>
</tr>
<tr>
<td>Coke</td>
<td>0.023</td>
<td>0.038</td>
<td>0.00</td>
</tr>
<tr>
<td>Exxon</td>
<td>0.025</td>
<td>0.028</td>
<td>0.21</td>
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<tr>
<td>GM</td>
<td>0.022</td>
<td>0.034</td>
<td>0.00</td>
</tr>
<tr>
<td>IBM</td>
<td>0.024</td>
<td>0.032</td>
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<tr>
<td>VW Index</td>
<td>0.030</td>
<td>0.018</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Crossing points

Observations

- $p < 0.05$: 7
- $0.05 < p < 0.95$: 7
- $p > 0.95$: 1
Summary

➡ Magnitude
   ✤ Near 3
   ✤ 4th moments and beyond??

➡ Stability
   ✤ Relatively stable

➡ Symmetry
   ✤ Mixed
Tail Methodology

- OLS (Huisman et al.) Hill estimator
- Augment kurtosis measures
- Gaussian crossing point
Future

➡ Volatility
  ➕ Range-based estimators
  ➕ Multi-horizon/long memory
➡ Long term risk measures
➡ Correlations and moment failure
  ➕ Beta/hedge ratios
➡ Robust/stylized facts and other series