Computing Dynamic Optimal Mechanisms When Private Shocks Are Persistent

Kenichi Fukushima and Yuichiro Waki

University of Minnesota and FRB of Minneapolis

Econometric Society NASM 2009
Dynamic Mechanism Design

- Prescribes an efficient way to provide insurance over time, when private information prevents full risk sharing.
- Applications: wage contract, social insurance, etc.
- Makes sense to quantify the implications.
Difficulty: Computational

- Shocks, e.g. to income, wage, are
  - Persistent, and
  - Take many different values in reality.

- Solving a DMDP under such shocks is difficult.
  - High-dimensional, continuous state variable in DP;
  - The state space has to be computed using a set operator.

- We address this issue.
Outline

- Set up a general DMDP with Markov private shocks,
- Illustrate the difficulty,
- Describe our approach,
- Discuss its usefulness.
Dynamic Mechanism Design Problem: A Setup
Setup

- Time is discrete and infinite: $t = 0, 1, \ldots, \infty$.
- A planner and an agent.
- In each period, the planner gives the agent an "outcome" $x \in X(\text{convex, compact}) \subset \mathbb{R}^L$. 
Setup

- Agent's type $\theta_t \in \Theta$ (finite, $N = |\Theta|$).
- First order Markov with transition prob. $\pi(\theta|\theta_-)$.
- Full support: $\pi(\theta|\theta_-) > 0$ for all $\theta, \theta_-$. 
- $\theta_s = (\theta_s, \ldots, \theta_t)$ and $\theta^t = \theta^t_0$.
- $\theta_0$ is drawn from $\pi(.|\theta_{-1})$.
- $\theta_{-1}$ is public info.
- $\theta_t$, $t \geq 0$ is private info.
Setup

- An allocation is $x = \{x_t\}_{t=0}^{\infty}$ s.t. $\forall t$, $x_t : \Theta^{t+1} \rightarrow X$.
- Agent’s utility from an allocation $x$:
  \[ U(x; \theta_{-1}) = \sum_{t=0}^{\infty} \sum_{\theta^t} \beta^t u(x_t(\theta^t); \theta_t) \Pr(\theta^t|\theta_{-1}). \]
- Planner’s cost from an allocation $x$:
  \[ C(x; \theta_{-1}) = \sum_{t=0}^{\infty} \sum_{\theta^t} q^t c(x(\theta^t)) \Pr(\theta^t|\theta_{-1}). \]
- $u$, $c$ continuous. $u$ concave, $c$ convex.
Mechanism design problem

\[
\min_x C(x; \theta_{-1})
\]

s.t. two constraints:

1. Promise-keeping: \( U(x; \theta_{-1}) \geq U_0 \).

2. Incentive-compatibility:
   - A reporting strategy: \( r = \{r_t\}_{t=0}^{\infty} \) s.t. \( \forall t, r_t : \Theta^{t+1} \rightarrow \Theta \).
   - \( U(x; \theta_{-1}) \geq U(x \circ r; \theta_{-1}), \forall r \in R \).
Illustration of the Difficulty
Problem: "Curse of Dimensionality"

- In a recursive form of the planner’s problem:
  - Generally an $N$-dimensional ($N = |\Theta|$) continuous state variable necessary. (Fernandes-Phelan (2000))
  - The state space is non-rectangular and needs be computed using a set operator.
- Impractical unless $N \approx 2, 3$. 
Continuation Utility Profile as a State Variable

- Need to track a “continuation utility profile.”
  - Same report history ⇒ same continuation allocation.
  - But different types derive different utilities from it.

- A continuation utility of type $\hat{\theta}_{t-1}$ after report history $\theta^{t-1}$:

$$U_t(\theta^{t-1}; \hat{\theta}_{t-1}) = \sum_{\theta_t} \left[ u(x_t(\theta^t); \theta_t) + \sum_{s \geq t+1} \sum_{\theta_{t+1}^s} \beta^{s-t} u(x_s(\theta^s); \theta_s) \Pr(\theta_{t+1}^s | \theta_t) \right] \pi(\theta_t | \hat{\theta}_{t-1}).$$

- A continuation utility profile:

$$U_t(\theta^{t-1}; .) = \left( U_t(\theta^{t-1}; \hat{\theta}_{t-1}) \right)_{\hat{\theta}_{t-1} \in \Theta} \in \mathbb{R}^N.$$

\[ J^{FP}(U(.), \theta_-) = \min_{(x, U^+)} \sum_{\theta} \left\{ c(x(\theta)) + qJ^{FP}(U^+(\theta, .), \theta) \right\} \pi(\theta|\theta_-) \]

s.t.

\[ u(x(\theta); \theta) + \beta U^+(\theta, \theta) \geq u(x(\theta'); \theta) + \beta U^+(\theta', \theta), \quad \forall \theta, \theta', \]

\[ U(\hat{\theta}_-) = \sum_{\theta} \left\{ u(x(\theta); \theta) + \beta U^+(\theta, \theta) \right\} \pi(\theta|\hat{\theta}_-), \quad \hat{\theta}_- \]

\[ (x(\theta), U^+(\theta, .)) \in X \times V, \quad \forall \theta. \]
Our Approach
Our Approach

- Allow $N$ to be large.
- Identify the subclasses of $\pi$’s which permit a recursive formulation with a lower dimensional state variable.
Identifying the subclasses

Recursive formulation when $\pi \in \Pi_K$
Identifying the subclasses of $\pi$’s

- $K \in \{1, 2, ..., N\}$.
- $\Pi_K$ is a set of $\pi$’s such that
  - $\exists$ a $K$ set of densities over $\Theta$, $\{p_k\}_{k=1}^K$, and
  - a set of non-negative weights on these densities, $\{\omega(\theta_-)\}_{\theta_-}$

such that

$$
\pi(\theta|\theta_-) = \sum_{k=1}^{K} p_k(\theta)\omega_k(\theta_-), \quad \forall \theta, \theta_-.
$$
Recursive formulation when $\pi \in \Pi_K$
Recursive formulation when $\pi \in \Pi_K$

- Show that if $\pi \in \Pi_K$, the continuation utility profile $U \in \mathbb{R}^N$ can be summarized by a vector $a \in \mathbb{R}^K$.
- Hence $a$ can serve as a state variable.
Recursive formulation when \( \pi \in \Pi_K \)

- Recall \( \pi(\theta|\theta_-) = \sum_{k=1}^{K} p_k(\theta)\omega_k(\theta_-) \).
- Think that \( \theta \) is drawn in two steps:
  1. One of the distributions \( \{p_k\} \) is drawn according to \( \{\omega_k(\theta_-)\} \).
  2. \( \theta \) is drawn from \( p_k \).
- \( U \) is a continuation utility profile before (1) happens.
- Consider a continuation utility at a fictitious interim state \( k \) - \( p_k \) is drawn already but \( \theta \) is not.
Continuation utility at an interim state $k$

Continuation utility at an interim state $k$ is

$$
\sum_{\theta_t} \left[ u(x_t(\theta^t); \theta_t) + \sum_{s \geq t+1} \sum_{\theta_{t+1}^s} \beta^{s-t} u(x_s(\theta^s); \theta_s) \Pr(\theta_{t+1}^s | \theta_t) \right] p_k(\theta_t)
$$
Continuation utility at an interim state $k$

Continuation utility at an interim state $k$ is

$$a_{k,t}(\theta^{t-1}) = \sum_{\theta_t} \left[ u(x_t(\theta^t); \theta_t) + \right.$$  

$$\sum_{s \geq t+1} \sum_{\theta_{t+1}^s} \beta^{s-t} u(x_s(\theta^s); \theta_s) \Pr(\theta_{t+1}^s | \theta_t) \right] p_k(\theta_t)$$
Recursive formulation when $\pi \in \Pi_K$

By law of iterated expectations,

$$U_t(\theta^{t-1}, \hat{\theta}) = \sum_{k} a_{k,t}(\theta^{t-1})\omega_k(\hat{\theta}) = a_t(\theta^{t-1}) \cdot \omega(\hat{\theta}).$$
Recursive formulation when $\pi \in \Pi_K$

By law of iterated expectations,

$$U_t(\theta^{t-1}, \hat{\theta}) = \sum_{k} a_{k,t}(\theta^{t-1})\omega_k(\hat{\theta}) = a_t(\theta^{t-1}) \cdot \omega(\hat{\theta}).$$

A $K$-dimensional vector $a_t(\theta^{t-1})$ summarizes an $N$-dimensional object $U_t(\theta^{t-1}, .)$. 
Recursive formulation when $\pi \in \Pi_K$

$a_t$ also satisfies a recursion:

$$a_{k,t}(\theta^{t-1}) = \sum_{\theta_t} \left\{ u(x(\theta^t); \theta_t) + \beta a_{t+1}(\theta^t) \cdot \omega(\theta_t) \right\} p_k(\theta_t)$$
a serves as a state variable

\[ J(a, \theta_-) = \min_{(x, a^+)} \sum_{\theta} \left\{ c(x(\theta)) + qJ(a^+(\theta), \theta) \right\} \pi(\theta|\theta_-) \]

s.t.

\[ u(x(\theta); \theta) + \beta a^+(\theta) \cdot \omega(\theta) \geq u(x(\theta'); \theta) + \beta a^+(\theta') \cdot \omega(\theta), \quad \forall \theta, \theta', \]

\[ a_k = \sum_{\theta} \left\{ u(x(\theta); \theta) + \beta a^+(\theta) \cdot \omega(\theta) \right\} p_k(\theta), \quad \forall k, \]

\[ (x(\theta), a^+(\theta)) \in X \times A, \quad \forall \theta. \]
Discussion/Conclusion
Discussion

- One can solve a DMDP with large $N$, as long as $K$ is low.
- Q: Is setting $K$ low too restrictive?
- A: Not necessarily.
- In the paper, we show
  - $\Pi_K$ with low $K$ can give a good statistical approximation for some persistent processes, and
  - one can perturb a solution under $\pi$, to obtain a near-optimal allocation in a neighborhood of $\pi$.
- Useful if the process of interest is sufficiently close to some $\pi \in \Pi_K$, and it is not implausible.
Conclusion

This paves the way for quantitative research using a DMDP.