A Transaction-Level Pure-Jump Model Yielding Cointegration

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Cointegration: Background

- A contemporaneous linear combination of two or more time series is less persistent than the individual series.

- Engle and Granger (1987) allowed for both standard and fractional cointegration.

- Under standard cointegration, the memory parameter is reduced from 1 to 0, while under fractional cointegration the level of reduction need not be an integer thus is more general.

- Empirical examples include: interest rates of different horizons, implied versus realized volatility, pairs trading strategy, etc.
Cointegration: Gold versus Oil Prices
Motivations

- A limitation of most existing models is that they are based on a particular fixed sampling interval $\Delta t$, e.g., one day, one month, etc. and therefore do not reflect the dynamics at all levels of aggregation.

- The entire observed price series are at transaction-level, which fully determine observations that sampled at any aggregation level $\Delta t$. In addition, the transaction-level observations are now available.

- How could the transaction-level price series generate clock-time cointegration? Is it possible to construct a transaction-level model such that any discretization of the process to an equally-spaced sampling grid produces fractional or standard cointegration?
Challenges

- How to impose cointegration under the nonsynchronous trading?

- The generated price and return series must have reasonable properties: Martingale-type behaviors for prices and Martingale-difference-type behaviors for returns.

- The proposed model must be flexible to capture various empirical stylized effects: persistence in durations, volatility clustering, leverage effect, nonsynchronous trading effect, etc.
Most paper study the price movement by a diffusion-type model, e.g. Brownian motion. But, price data, in continuous time, yield pure-jump realizations. In addition, a sampling scheme needs to be specified in the diffusion-type model, which is not usually done in literature.

Instead, we choose to develop models for prices in transaction level with no diffusion component: a transaction-level pure-jump perspective. Empirically, price remains constant between transactions and if there is a diffusion component underlying the price, it is not directly observable.

Price dynamics are completely determined by the waiting times between transactions (the durations, \( \{ \tau_k \} \)) and the jump sizes. The price series is a marked point process.

The durations convey information, see e.g., Dufour and Engle (2000).
Clock Time vs. Tick Time

- Two natural ways to measure time:
  - Clock time (e.g., 1 unit = 5 minutes)
  - Tick time (from one transaction to the next).

- Clock-time price dynamics, even long-term behavior, are completely determined by tick-time properties: duration dynamics, efficient price shocks and market microstructure noise.

- Therefore, tick time is more fundamental.
Tick-Time Source of Cointegration

- Cointegration is generally viewed solely as a clock-time phenomenon. Clock-time models for cointegration involve a common persistent component that is removed by the linear combination and require the choice of a fixed aggregation interval $\Delta t$, which is a limitation.

- But since the assets trade nonsynchronously, it is impossible to impose a common trend in tick time. Here, we will build a model for cointegration based on tick-time interactions, and then study its clock-time properties.

- We cover both

  **standard cointegration** $I(1) \rightarrow I(0)$

  **fractional cointegration** $I(1) \rightarrow I(d), \quad d \in (0, 1)$. 
The durations $\{\tau_k\}$ determine a **counting process:**

$$N(t) = \text{Number of transactions in } (0, t].$$

Simple Example: Compound Poisson Process, see Press (1967).

$$\log P(t) = \sum_{k=1}^{N(t)} e_k,$$

where $P(t) = \text{Price at time } t \geq 0$ and $N(t) = \text{Poisson Process}$. $\{e_k\}$ are *i.i.d.*, independent of $N(\cdot)$. 
Market Microstructure Consideration

- We include market microstructure noise \( \{\eta_k\} \) in the log-price model:

\[
\log P(t) = \sum_{k=1}^{N(t)} e_k + \sum_{k=1}^{N(t)} \eta_k,
\]

where \( \{e_k\} \) are i.i.d., independent of \( N(\cdot) \), \( \{\eta_k\} \) are the microstructure noise, independent of \( N(\cdot) \), \( E(e_k) = E(\eta_k) = 0 \).

- We consider \( N(t) \) process that is induced by the underlying duration process \( \{\tau_k\} \). Therefore, properties of \( N(t) \) are determined by those of \( \{\tau_k\} \). Specifically, we consider \( N(t) \) being generated by a long-memory duration model with memory parameter \( d_\tau \in (0, \frac{1}{2}) \). Why? The resulting realized volatility has persistence. See Deo, Hurvich, Soulier and Wang (2008).

- \( \{\eta_k\} \) are assumed to be \( I(d_\eta) \), with \( d_\eta \in [-1, 0) \). The second summation represents a microstructure component.
Pure-Jump Model for Bivariate Series

- Bivariate Series: Asset 1, Asset 2.

- Transaction times \( \{t_{1,k}\}, \{t_{2,k}\} \). → Durations \( \{\tau_{1,k}\}, \{\tau_{2,k}\} \). → Counting processes \( N_1(\cdot), N_2(\cdot) \).

- Two kinds of disturbances: Efficient Shocks and Microstructure Noise.

- Efficient Shocks \( \{e_{1,k}\}, \{e_{2,k}\} \) are \( i.i.d. \). Microstructure Noise \( \{\eta_{1,k}\}, \{\eta_{2,k}\} \) has long memory with negative memory parameter \( d_\eta \).

- Fractional Cointegration Case: \( d_\eta \in (-1, 0) \). Standard Cointegration Case: \( d_\eta = -1 \).
Return Interactions and Changes in Prices

- In tick time, the return on each asset accumulates all disturbances ("own" and "other") originating since its own previous trade. (play the animation)

- Asset 1 Return = Current Asset 1 Disturbances + Intervening Asset 2 Disturbances (Weighted), as of previous Asset 1 Trade.

  Asset 2 Return = Current Asset 2 Disturbances + Intervening Asset 1 Disturbances (Weighted), as of previous Asset 2 Trade.
The Model

The model is,

\[
\log P_{1,t} = \sum_{k=1}^{N_1(t)} (e_{1,k} + \eta_{1,k}) + \sum_{k=1}^{N_2(t_1,N_1(t))} (\theta e_{2,k} + g_{21} \eta_{2,k})
\]

\[
\log P_{2,t} = \sum_{k=1}^{N_2(t)} (e_{2,k} + \eta_{2,k}) + \sum_{k=1}^{N_1(t_2,N_2(t))} \left( \frac{1}{\theta} e_{1,k} + g_{12} \eta_{1,k} \right)
\]

The quantity \(N_2(t_1,N_1(t))\) represents the total number of trades of Asset 2 occurring up to the time \((t_1,N_1(t))\) of the most recent trade of Asset 1.

The \(\theta\) versus \(\frac{1}{\theta}\) constrain is necessary to yield cointegration. But not required in general if there is no cointegration.

This model is identifiable, as each parameter can be estimated.
Components of the Model

- To exhibit the various components of our model, we rewrite the model as

\[
\log P_{1,t} = \sum_{k=1}^{N_1(t)} e_{1,k} + \sum_{k=1}^{N_2(t)} \theta e_{2,k} + \sum_{k=1}^{N_1(t)} \eta_{1,k} + \sum_{k=1}^{N_2(t)} g_{21} \eta_{2,k} \\
\text{common component} \\
\text{microstructure component}
\]

\[
- \sum_{k=\max(N_2(t_1, N_1(t)) + 1)}^{N_2(t)} (\theta e_{2,k} + g_{21} \eta_{2,k}) \\
\text{end effect due to nonsynchronous trading}
\]

- The same is true for \( \log P_{2,t} \).
Properties of the Model

- The common component is a Martingale, and is therefore $I(1)$. The microstructure components are random sums of $I(d_\eta)$, yielding $I(1 + d_\eta)$, less persistent than the common component.

- The end-effect terms are random sums over time periods that are $O_p(1)$ as $t \to \infty$, and hence are negligible compared to all other terms.

- Both $\log P_{1,t}$ and $\log P_{2,t}$ are $I(1)$. The linear combination $\log P_{1,t} - \theta \log P_{2,t}$ is $I(1 + d_\eta)$. Therefore, log-price series are cointegrated, with cointegrating parameter $\theta$. And the memory parameter goes from 1 to $(1 + d_\eta) < 1$.

- For any fixed integer $k > 0$, the lag-$k$ autocorrelation of returns tends to zero as sampling interval $\Delta t \to \infty$. 
Estimation of the Cointegrating Parameter

Assume that the log-price series are observed at integer multiples of a fixed sampling interval $\Delta t$. The proposed model becomes

\[
\log P_{1,j} = \sum_{k=1}^{N_1(j\Delta t)} (e_{1,k} + \eta_{1,k}) + \sum_{k=1}^{N_2(t_{1,N_1(j\Delta t)})} (\theta e_{2,k} + g_{21}\eta_{2,k})
\]

\[
\log P_{2,j} = \sum_{k=1}^{N_2(j\Delta t)} (e_{2,k} + \eta_{2,k}) + \sum_{k=1}^{N_1(t_{2,N_2(j\Delta t)})} \left( \frac{1}{\theta} e_{1,k} + g_{12}\eta_{1,k} \right)
\]

Estimate $\theta$ by OLS regression of $\{\log P_{1,j}\}_{j=1}^n$ on $\{\log P_{2,j}\}_{j=1}^n$ without intercept.
Consistency of the OLS Estimator

- **Theorem 1** The cointegrating parameter \( \theta \) can be consistently estimated by \( \hat{\theta} \), the ordinary least squares estimator obtained by regressing \( \{ \log P_{1,j} \}_{j=1}^{n} \) on \( \{ \log P_{2,j} \}_{j=1}^{n} \) without intercept. For all \( \delta > 0 \), as \( n \to \infty \), we have

  - **Case I:** \( d_\eta \in (-\frac{1}{2}, 0) \), \( n^{-d_\eta-\delta}(\hat{\theta} - \theta) \xrightarrow{p} 0 \),
  - **Case II:** \( d_\eta \in (-1, -\frac{1}{2}) \), \( n^{\frac{1}{2}-\delta}(\hat{\theta} - \theta) \xrightarrow{p} 0 \),
  - **Case III:** \( d_\eta = -1 \), \( n^{1-\delta}(\hat{\theta} - \theta) \xrightarrow{p} 0 \).

- The rate of convergence of \( \hat{\theta} \) improves as \( d_\eta \) decreases.

- In the standard cointegration case \( d_\eta = -1 \), the rate is arbitrarily close to \( n \). The \( n \)-consistency (super-consistency) of the OLS estimator of the cointegrating parameter in the standard cointegration case has been shown for time series in discrete clock time. See Phillips and Durlauf (1986) and Stock (1987).
The Advantage of Modeling at Tick-time

- Uniform to any sampling interval $\Delta t$.

- A tick-level cointegrating parameter estimator is superior to any OLS estimator that constructed based on clock-time sampled prices.

- Able to investigate tick-level dynamics of the price series.

- Possible scenarios that fit into our model include: (1) buy (bid) and sell (ask) prices of a single security; (2) prices of two classified stocks (with different voting rights) from a given company; (3) prices of two different stocks within the same industry; (4) stock and option prices of a given company; (5) corporate bond prices at different maturities for a given company, etc.
Empirical Example: Tiffany

- We considered buy and sell prices of daily transactions between 9:30 AM to 4:00 PM from January 25, 2000 to July 20, 2000.
Empirical Example: Tiffany

- Evidence of strong fractional cointegration is observed between buy versus sell prices.

- Model parameters are estimates using method of moments based on tick-level prices.

<table>
<thead>
<tr>
<th>Period</th>
<th>Type</th>
<th>$\hat{\sigma}_{i,e}^2 [SE](\times 10^{-6})$</th>
<th>$\hat{\sigma}_{i,\eta}^2 [SE](\times 10^{-6})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: trading day 1 to 25</td>
<td>Buy</td>
<td>3.01 [1.42]</td>
<td>3.38 [1.70]</td>
</tr>
<tr>
<td></td>
<td>Sell</td>
<td>3.05 [0.89]</td>
<td>1.93 [0.97]</td>
</tr>
<tr>
<td>2: trading day 41 to 70</td>
<td>Buy</td>
<td>6.22 [1.41]</td>
<td>0.72 [1.06]</td>
</tr>
<tr>
<td></td>
<td>Sell</td>
<td>4.08 [0.79]</td>
<td>0.83 [0.63]</td>
</tr>
<tr>
<td>3: trading day 90 to 124</td>
<td>Buy</td>
<td>3.50 [0.87]</td>
<td>1.18 [0.83]</td>
</tr>
<tr>
<td></td>
<td>Sell</td>
<td>2.00 [0.50]</td>
<td>2.66 [0.95]</td>
</tr>
<tr>
<td>entire period</td>
<td>Buy</td>
<td>4.67 [1.07]</td>
<td>1.26 [0.96]</td>
</tr>
<tr>
<td></td>
<td>Sell</td>
<td>3.35 [0.83]</td>
<td>1.66 [0.88]</td>
</tr>
</tbody>
</table>
Several Interesting Findings

- Microstructure noise variance estimates ($\hat{\sigma}^2_{i,\eta}$) are smaller in subperiod where stock price was fluctuating than those in subperiods where price was either raising or declining.


- Efficient noise variance estimates ($\hat{\sigma}^2_{i,e}$) show an opposite pattern, i.e., larger in subperiod two but smaller in subperiods one and three.
Information Share Analysis

- In Hasbrouck (1995), "one security" is traded on several markets and different market prices share a common random-walk component.

- The information ratio proposed in Hasbrouck (1995) measures how market information that drives stock prices are distributed across difference exchanges. The information ratio of a specific market is defined as the proportion of the efficient price innovation variance that can be attributed to that market.
Information Share Analysis: Tiffany

In our analysis, we investigate how market information that drives stock prices are distributed between buy versus sell trades within three different market environments: price declining, price fluctuating and price increasing.

<table>
<thead>
<tr>
<th>Period</th>
<th>$\hat{S}_{buy}[SE]$</th>
<th>$\hat{S}_{sell}[SE]$</th>
<th>$(\hat{S}<em>{buy} - \hat{S}</em>{sell})[SE]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: Trading day 1 to 25</td>
<td>45.7% [13.8%]</td>
<td>54.3% [13.8%]</td>
<td>-8.6% [27.6%]</td>
</tr>
<tr>
<td>2: Trading day 41 to 70</td>
<td>51.5% [7.5%]</td>
<td>48.5% [7.5%]</td>
<td>3.0% [14.9%]</td>
</tr>
<tr>
<td>3: Trading day 90 to 124</td>
<td>57.5% [8.6%]</td>
<td>42.3% [8.6%]</td>
<td>15.2% [17.3%]</td>
</tr>
<tr>
<td>Entire Period</td>
<td>52.6% [8.4%]</td>
<td>47.4% [8.4%]</td>
<td>5.2% [16.8%]</td>
</tr>
</tbody>
</table>
Generalization of the Model and Future Work

- Nonsynchronous trading induced portfolio return autocorrelation; leverage effect

- Investigate the interplay between cointegration and option pricing, hedging, asset allocation, pairs trading and index tracking in the current pure-jump context.

- Generalizing the model to include more than two assets.
Acknowledgement

- This is a joint work with Cliff Hurvich at Stern School, NYU.

- The corresponding papers can be found at:
  www.stern.nyu.edu/~ywang2/research.html

Thank you for listening!