Optimal City Hierarchy: A Dynamic Programming Approach to Central Place Theory

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This paper is about

- Given the existence of cities, why should there be a hierarchy?
- What is the optimal spacings of cities?
- Formalize central place theory via a social planner’s problem.
- Is the optimal solution decentralizable?
Central Place Theory

- It describes how a city hierarchy emerges from a uniformly populated plane (of farmers). (Christaller, 1933)

- Cities provide a wide variety of goods to farmers; goods differ in their scale economies.

- Hierarchy property: if a city provides a good with certain degree of scale economies, it provides all goods with lower scale economies.

- Put it differently, larger cities provide all goods that are provided by smaller ones.

- Theory? (See criticisms by Berliant (2005) and Fujita, Krugman, and Venables (2001))
A central place hierarchy on the plane
A central place hierarchy is a city system that satisfies the hierarchy and central place properties.
Why central place theory?

- Empirically plausible description of city systems
  1. Mori et al (2008a, b, c): the hierarchy property holds in Japanese data.
  2. Spatial pattern plausible: case studies (Berry, 1958, 1967)

- A route to explain two related empirical regularities between which the link is the hierarchy property (Mori et al., 2008)
  1. Zipf’s law (rank-size rule). (Beckmann, 1958; Hsu, 2008)
  2. Number-Average-Size (NAS) rule. (Mori et al., 2008; Hsu, 2008)
Zipf’s law

- For $\zeta \neq 1$

\[
\ln R_i = \alpha - \zeta \ln S_i,
\]

- Also known as the rank-size rule:

rank $\times$ size = constant.

- Robust across time and countries. (Gabaix and Ioannides, 2004)
Zipf’s law

Log Size versus Log Rank of All (362) the MSA’s (Metropolitan Statistical Areas)

Zipf’s Plot for All MSA’s (2000 US Census)
OLS: $y = -0.95x + 17, \ R^2 = 0.9857$
The Number-Average Size (NAS) Rule

- $N_n$: Number of cities that industry $n$ is located in.
- $A_n$: Average size of cities that industry $n$ is located in.
- NAS rule:
  \[
  \ln N_n = \alpha - \beta \ln A_n.
  \]
- First documented in Japanese data by Mori et al. (2007).
- I document that it also holds for the US.
The Number-Average Size (NAS) Rule

Log Average-Size versus Log Number of 77 3-digit NAICS (2000 County Business Pattern Data)

NAS Plot (3-digit NAICS)
slope = −0.7477, $R^2=0.9991$
Central place hierarchy as a socially optimal solution

- Quinzii and Thisse (1990, Econometrica) is the only other paper that takes a similar aim.
- A spatial competition model with firms of different goods agglomerate under certain conditions.
- Got the hierarchy property, but what about the spacing?
- This paper imposes the hierarchy property and asks

  What is the optimal configuration of a city hierarchy?
Model setup

- **Geographic space:** Real line.

- **Population:** Uniformly spread out with density 1.

- **Commodity and Demand:** There exists a continuum of goods indexed by $x \in [0, z_1]$, and every individual eats one unit of each good.

- **Transportation cost:** Linear in distance: $t$ per unit per mile traveled, for any good.
Model setup

- **Marginal cost:** Constant $c$, for any good.

- **Heterogenous fixed costs:** Let $\phi(x)$ denote the fixed cost of producing $x$. Assume continuity and rank all the goods so that we can define a nondecreasing function $\phi : [0, z_1] \rightarrow \mathbb{R}_+$. Define total fixed cost at a location of $[0, z]$ as $\Phi(z) = \int_0^z \phi(x)dx$.

- **Hierarchy property:** Any location of production must produce $[0, z]$ for some $z$. 
Social planner’s problem

Start with only one layer:

- Cities producing everything in \([0, z_1]\) must be equal distance apart, and let the distance be \(2L > 0\).

- Note that the average variable cost can be ignored.

- The total transportation cost involved in delivering goods of measure \([0, z]\) from a city with a market area being \(2L\) is \(ztL^2\) since

\[
2z \int_0^L tx \, dx = ztL^2.
\]
Social planner’s problem

- Given cities of \([0, z]\), we might or might not want to have smaller cities in between.
- Suppose the distance between two \(z\)-cities is \(2L\).
- Radius of market area is \(L\).
- If doing nothing, then the cost per capita for the goods \([0, z]\) is

\[
C_f(L, z) \equiv \frac{1}{2L} [\Phi(z) + ztL^2]
\]
City Planting
Consider the possibility of having \( n' \ z' \)-cities in between two \( z \)-cities (\( 0 < z' \leq z \)).

Given \( z > 0 \) and \( L > 0 \), the social planner solves the following dynamic programming problem.

\[
(FE_f) \\
C(L, z) = \min \{C_f(L, z), \min_{n', z'} \frac{1}{2L} [\Phi(z) - \Phi(z') + (z - z') tL^2] + C(L', z') \} \\
\text{s.t. } L' = \frac{L}{n' + 1}, \ n' \in \mathbb{N}, \ z' \in [0, z].
\]
Social Planner’s Problem: Two Stages

- The two stages:

  **Stage 1** Pick $2L$.

  **Stage 2** Given $2L$ and $z_1$, solve for the optimal city system.

- So, solving backward, one solves the Bellman’s equation first, and then solve the optimal radius for the $z_1$-cities:

  $$L_1 = \arg \min_L C(L, z_1).$$
Characterization

- If $\phi(0) = 0$, then $C_f(L, z)$ is never the optimal choice.
- Thus, there are infinitely many layers if $\phi(0) = 0$.
- The problem is then reduced to

\[(FE)\]

\[
C(L, z) = \min_{n', z'} \frac{1}{2L} [\Phi(z) - \Phi(z') + (z - z')tL^2] + C(L', z')
\]

s.t. $L' = \frac{L}{n' + 1}$, $n' \in \mathbb{N}$, $z' \in [0, z]$. 


Characterization

- The policy function for $z'$ is a simple rule given by

$$z'^o = \phi^{-1} \left( \frac{tL^2}{n'^o + 1} \right) = \phi^{-1} \left( \frac{tL^2}{g(L, z) + 1} \right).$$

- This is obtained by combining the first-order condition and the envelope condition.

$$\frac{\partial C(L, z)}{\partial z} = \frac{1}{2L} [\phi(z) + tL^2].$$

- So, what is left is to solve is $n'(L, z)$. 
In general, the solution depends on $\phi(.)$.

However, for $\phi(x) = abx^{b-1}$ (the functional form that supports Zipf’s law in Hsu (2008)), we can prove that $n'(L, z) = 1$, regardless of values of $L, z$.

Use guess-and-verify technique to show it.
The same structure of city hierarchy: Zipf’s law and the NAS rule hold.

Why? Solution: \( n_i^* = n_i^o = 1 \) for all \( i \).

There is a continuum of equilibrium in Hsu (2008).

**Decentralizable:** The optimal solution is always in the continuum.