Turnout and Power Sharing

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Objective

- How do institutions: e.g. electoral rules, bicameralism, judicial review, federalism, separation of powers, committee chair assignments, and all other institutions determining the power of majority and minorities, affect electoral participation?

- Different institutions determine the reduced form ‘Map’

  \[
  \text{Vote Share} \rightarrow \text{Power Share}
  \]

  1. Electoral Law:

     \[
     \text{Vote Share} \rightarrow \text{Seat Share}
     \]

  2. Other power sharing institutions:

     \[
     \text{Seat Share} \rightarrow \text{Power Share}
     \]
or rather somewhere in between?
Main Question

1. How does the map affect turnout decisions by rational voters?
2. How does the map affect campaign spending decision by parties?

- Behavioral, proportional power sharing (PR) may be how voters think
- Voter paradox:
  - one of the reasons for turnout being higher than standard pivotal model predictions may be that voters think like in the proportional power model: e.g. in the US there is majority rule but 58 senators is better than 57 better than 56.....
Institutions inducing higher power/vote ratio for the leader also induce higher turnout if it is a close race. The opposite is true if the race has a wide ex-ante margin. Results help reconcile empirical and experimental evidence on turnout and voting rules. Suggest a general set of predictions regarding the impact of institutions that increase proportionality of influence.
Models of Turnout

1. **Rational Voter Model**
   - Palfrey & Rosenthal (PC 1983 & APSR 1985)
   - Myerson (IJGT 1998 & JET 2000)
   - Krishna & Morgan (2009 working paper)

2. **Voter Mobilization Models**
   - Shachar & Nalebuff (AER 1999)
   - Herrera & Mattozzi (JEEA, forthcoming)
MR (Red) versus PR (Blue)

- PR: always some marginal impact, but infinitesimal in the amount, "no wasted votes"
- MR: large marginal impact, but infinitesimal chance (of being pivotal), "most votes wasted"
- Is the turnout comparison predictable?
Voters have a preference for one of the two parties \((\text{wlog: } q \leq 1/2)\)

- Voters decide whether to vote or abstain \((\alpha, \beta)\)

\[
\begin{array}{c}
\text{Voter} \\
\downarrow \\
\text{A} \\
\downarrow \quad \quad \downarrow \\
q \quad 1-\alpha \\
\downarrow \quad \quad \downarrow \\
1-q \quad 1-\beta \\
\downarrow \\
\text{B} \\
\downarrow \\
\beta \\
\end{array}
\]

- Vote: \(q\alpha\)
- Abstain
- Vote: \((1-q)\beta\)
- Abstain
Possible Outcome of a Two Party Election

Turnout for each side

\[ T_A = q \alpha \]
\[ T_B = (1 - q) \beta \]

Abstainers

A Votes

B Votes

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Rational Voter Model

- Types: preference for party A or B, voting cost $c \in [0, 1]$
- Distribution of voting costs

\[ F(c), \quad c \in [0, 1] \]
Cost Cutoff

\[(c_\alpha, c_\beta) \rightarrow (\alpha = F(c_\alpha), \beta = F(c_\beta))\]

Equilibrium Threshold Condition

\[B_i^A(\alpha, \beta) = F^{-1}(\alpha), \quad B_i^B(\alpha, \beta) = F^{-1}(\beta)\]

Expected Turnout

\[T := q\alpha + (1 - q)\beta\]

Exogenous to Endogenous

\[(q, N) \rightarrow (\alpha, \beta)\]
Population Uncertainty

- **Poisson Games:** Myerson (IJGT 1998 & JET 2000)

- Large but finite population $n$

  $$n \sim \frac{e^{-N} (N)^n}{n!} \quad (\text{Poisson' with mean } N)$$

- ‘Independent Actions’ Property

  $$\alpha n \sim \frac{e^{-\alpha N} (\alpha N)^n}{n!} \quad (\text{Poisson' with mean } \alpha N)$$

- Binomial $\rightarrow$ Poisson
Majority Rule (MR)

- Chance that \( k \) agents vote for \( A \)
  \[
  e^{-(qN\alpha)} \frac{(qN\alpha)^k}{k!}
  \]

- Chance that \( k \) agents vote for \( B \)
  \[
  e^{-((1-q)N\beta)} \frac{((1-q)N\beta)^k}{k!}
  \]

- In the winner take all system (MR)
  \[
  B_M^A = \frac{1}{2} \sum_{k=0}^{\infty} \left( e^{-qN\alpha} \frac{(Nq\alpha)^k}{k!} \right) \left( e^{-((1-q)N\beta)} \frac{((1-q)N\beta)^k}{k!} \right) \]
  \[
  \times \left( 1 + \frac{(1-q)N\beta}{k+1} \right)
  \]

- when voting either \textit{breaks} a tie where it would have lost the coin toss, or \textit{makes} a tie and wins the coin toss
Approximation for large N

- With Myerson’s Approximation, the system becomes

\[
B^A_M \simeq e^{-N\left(\sqrt{q\alpha} - \sqrt{(1-q)\beta}\right)^2} \frac{\sqrt{q\alpha} + \sqrt{(1-q)\beta}}{\sqrt{N}\sqrt{q\alpha}^{1/2} \sqrt{1-q}\beta^{1/2}} \left(\frac{1}{4\sqrt{\pi}(q(1-q)\alpha\beta)^{1/4}}\right) = F^{-1}(\alpha)
\]

\[
B^B_M \simeq e^{-N\left(\sqrt{q\alpha} - \sqrt{(1-q)\beta}\right)^2} \frac{\sqrt{q\alpha} + \sqrt{(1-q)\beta}}{\sqrt{N}\sqrt{(1-q)\beta}^{1/2}} \left(\frac{1}{4\sqrt{\pi}(q(1-q)\alpha\beta)^{1/4}}\right) = F^{-1}(\beta)
\]

- \(B_M\) can be expressed exactly as the sum of “Modified Bessel Functions”
Turnout Ratio for A and for B

- Ratio of the two equations

\[ \sqrt{q\alpha} F^{-1}(\alpha) = \sqrt{(1 - q)\beta} F^{-1}(\beta) \]

- Relation between \((\alpha, \beta)\)

\[ (q\alpha) \left( F^{-1}(\alpha) \right)^2 = ((1 - q)\beta) \left( F^{-1}(\beta) \right)^2 \]

- Allows to prove existence and uniqueness

- Note that

\[ q = 1/2 \implies \alpha = \beta \]
1 **Size Effect**

\[ T_M \sim \frac{e^{-N(q - \frac{1}{2})^2}}{\sqrt{N}} = \begin{cases} \frac{1}{\sqrt{N}} & q = 1/2 \\ e^{-N} & q \neq 1/2 \end{cases} \]

2 **Competition Effect**

\[ q < 1/2 \implies \frac{\partial T_M}{\partial q} > 0 \]

3 **Underdog Effect** (‘partial compensation’)

\[ q < 1/2 \implies \alpha > \beta \quad \text{but} \quad q\alpha < (1 - q)\beta \]
Why all this?

- Heterogenous Costs (Green)
- Homogeneous Costs (Black)
The condition

\[ q\alpha (F^{-1}(\alpha))^2 = (1 - q) \beta (F^{-1}(\beta))^2 \]

in case of homogenous cost for both sides simplifies because

\[ F^{-1}(\alpha) = F^{-1}(\beta) = c \]

which implies ‘full compensation’: 50% ex ante chance of victory

\[ q\alpha = (1 - q) \beta \]

- Heterogenous cost with \( q = 1/2 \) \( \Leftrightarrow \) Homogenous cost

- Krasa & Polborn (GEB 2007)
- Goeree & Grosser (JET 2007)
Proportional Influence System (PR)

- Chance that \( a \) agents vote for \( A \)

\[
e^{-qN\alpha} \frac{(qN\alpha)^a}{a!}
\]

- Chance that \( b \) agents vote for \( B \)

\[
e^{- (1 - q) N\beta} \frac{((1 - q) N\beta)^b}{b!}
\]

- Marginal Benefit of Voting

\[
B_P^A = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left( e^{-qN\alpha} \frac{(qN\alpha)^a}{a!} \left( \frac{a+1}{a+b+1} - \frac{a}{a+b} \right) \left( e^{- (1 - q) N\beta} \frac{((1 - q) N\beta)^b}{b!} \right) \right)
\]

\[
B_P^B = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left( e^{-qN\alpha} \frac{(qN\alpha)^a}{a!} \left( \frac{b+1}{a+b+1} - \frac{b}{a+b} \right) \left( e^{- (1 - q) N\beta} \frac{((1 - q) N\beta)^b}{b!} \right) \right)
\]
Computational Idea

- Renaming the variables

\[ B_P^A = e^{-R-S} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left( \frac{R^a}{a!} \right) \left( \frac{S^b}{b!} \right) \left( \frac{a+1}{a+b+1} - \frac{a}{a+b} \right) \]

- Take a term, differentiate and integrate the summand

\[ \sum_{b=0}^{\infty} \frac{S^b}{b!} \frac{a}{a+b} = \frac{a}{S^a} \sum_{b=0}^{\infty} \int_{0}^{S} \left( \frac{d}{dr} \left( \frac{1}{b!} \frac{r^{a+b}}{a+b} \right) \right) dr \]

- Then invert the series and integral operators

\[ \frac{a}{S^a} \int_{0}^{S} \sum_{b=0}^{\infty} \left( \frac{1}{b!} \frac{r^{a+b-1}}{a+b-1} \right) dr = \begin{cases} \frac{a}{S^a} \int_{0}^{S} r^{a-1} e^r dr & \text{for } a \geq 1 \\ 1/2 & \text{for } a = 0 \end{cases} \]

- Sum over ‘a’ by inverting the series and integral operators again
Exact Closed Form for the Benefit in PR

- **Type A**

  \[
  B_P^A = \frac{(1 - q)\beta}{NT^2} - e^{-NT}\left(\frac{((1 - q)\beta)^2 - (q\alpha)^2 + (1 - q)\beta\frac{1}{N}}{2T^2}\right)
  \]

- **Type B**

  \[
  B_P^B = \frac{q\alpha}{NT^2} + e^{-NT}\left(\frac{((1 - q)\beta)^2 - (q\alpha)^2 - q\alpha\frac{1}{N}}{2T^2}\right)
  \]

- **Turnout**

  \[
  T := q\alpha + (1 - q)\beta
  \]
Turnout Ratio for A and for B

- The sum of the marginal benefits for the two types

\[ B_P^A + B_P^B = \frac{1}{NT} \left( 1 - \frac{e^{-NT}}{2} \right) \approx \frac{1}{NT} \]

- NT must go to infinity

- For \( N \) large enough the two equilibrium conditions become

\[ \frac{q\alpha}{NT^2} = F^{-1}(\beta), \quad \frac{(1 - q)\beta}{NT^2} = F^{-1}(\alpha) \]

- From ratio of equations above find the relationship between \( \beta_P \) and \( \alpha_P \) and show fix point

\[ q\alpha F^{-1}(\alpha) = (1 - q)\beta F^{-1}(\beta) \]
1. Size Effect

\[ T_P \sim \frac{1}{N} \]

2. Competition Effect

\[ q < \frac{1}{2} \implies \frac{\partial T_P}{\partial q} > 0 \]

3. Underdog Effect (partial compensation):

\[ q < \frac{1}{2} \implies \alpha > \beta \quad \text{but} \quad q\alpha < (1 - q)\beta \]

stronger for most relevant distributions of voting costs
One Regime in PR

\[ T_P \sim \frac{1}{N} \]

Different Regimes in MR

\[ T_M \sim \begin{cases} 
\frac{1}{\sqrt{N}} & q = 1/2 \\
-e^{-N} & q \neq 1/2 
\end{cases} \]
Numerical Example: PR versus MR

- Parameters: \( F(c) = c^{1/z}, \quad z = 5, \quad N = 3000 \)
- \( q = 1/3 \)

<table>
<thead>
<tr>
<th>( q = 1/3 )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( T )</th>
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<tr>
<td>MR</td>
<td>7.1%</td>
<td>6.7%</td>
<td>6.8%</td>
</tr>
<tr>
<td>PR</td>
<td>24.8%</td>
<td>22%</td>
<td>23%</td>
</tr>
</tbody>
</table>

- \( q = 1/2 \)

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</tr>
</thead>
<tbody>
<tr>
<td>MR</td>
<td>40.9%</td>
<td>40.9%</td>
<td>40.9%</td>
</tr>
<tr>
<td>PR</td>
<td>23.5%</td>
<td>23.5%</td>
<td>23.5%</td>
</tr>
</tbody>
</table>

- Homogeneous \( c \), \( T_M > T_P \) always
Turnout: MR (Red), PR (Blue)
Higher Turnout with More Parties

- Intuition: smaller parties obtain a higher turnout

1. First, fixing the number of votes for the other parties $z$, the vote share increase for party $A$ is

$$\left( \frac{a+1}{a+z+1} - \frac{a}{a+z} \right) = \left( \frac{a^2 + a}{z} + 2a + 1 + z \right)^{-1}$$

which is larger for smaller values of the random variable $a$

2. A smaller party assigns in the marginal benefit $B^A_P$ a larger Poisson weight $\left( \frac{e^{-A}A^a}{a!} \right)$ to small values of $a$
Citizens have the same intensity of preference $Z$ for their party, vote if: $Z > c$

Campaign spending reduces voting costs (FOSD): $F_A(c; s_A) = c^{\frac{1}{1+s_A}}$

Simultaneous move game determining mobilization vector $(s_A, s_B)$

For any spending profile $(s_A, s_B)$, the vote share for party $A$ is

$$x(s_A, s_B) = \frac{qZ^{\frac{1}{1+s_A}}}{qZ^{\frac{1}{1+s_A}} + (1 - q)Z^{\frac{1}{1+s_B}}}$$

Turnout

$$T = qZ^{\frac{1}{1+s_A}} + (1 - q)Z^{\frac{1}{1+s_B}}$$

$$= q\alpha + (1 - q)\beta$$
Party A maximizes

\[ U_A = P_A(x) - I_A(s_A) \]

\( I_A(s_A) \) is increasing and convex cost of campaign

\( P_A(x) \) is the power share induced by the vote share \( x \)

\[ P_A(x, \gamma) = \begin{cases} 
\frac{1}{2} (2x)^\gamma & x < \frac{1}{2} \\
1 - \frac{1}{2} (2(1 - x))^\gamma & x \geq \frac{1}{2}
\end{cases} \]

\( \gamma \) power sharing regime
Mapping Vote to Power

- $\gamma = 1$ (Blue) ‘pure PR’
- $\gamma = 5$ (Red) ‘approaching MR’
- $\gamma \to \infty$ (Green) ‘full MR’
An interior equilibrium $s^*$ exists for every $\gamma \geq 1$

Equilibrium spending level: $s_A = s_B = s$

$$s_A = s_B = s : l'(s)(1+s)^2 = \frac{\gamma q (-\ln Z)}{2} (2(1-q))^\gamma$$

1. $q = 1/2$ Turnout is strictly increasing in $\gamma$

2. $q < \hat{q} < 1/2$, Turnout is maximal at $\gamma = 1$ (PR)
\[ \gamma = 1 \text{ (Blue) \ 'pure PR'} \]
\[ \gamma = 5 \text{ (Red) \ 'approaching MR'} \]
**Intuition**

- \( q = 1/2 \)

\[
 s_A = s_B = s : \quad l'(s)(1 + s)^2 = \frac{\gamma (-\ln Z)}{4}
\]

- higher \( \gamma \) \( \implies \) higher marginal benefit/impact of campaigning

- \( q \ll 1/2 \)

- higher \( \gamma \) \( \implies \) lower marginal benefit/impact of campaigning (lower chance of a tie)
1. Uncertainty in $q$ (Chamberlain & Rothschild, JET 1980)
2. ‘Combined Model’: Party Mobilization + Rational Voter
Empirical and Experimental Evidence

**Cross Country Evidence: Higher Turnout in PR**
- ‘Turnout boost from PR between 9-12% in most salient elections’

**Experimental Evidence: Higher Turnout in MR**
- Schram & Sonnemans (IJGT 1996) (but $q = 1/2$)
- Explanation: in the experiment they have $q = 1/2$ while empirical results include all kinds of situations, so no contrast between the two findings in light of our results
- Levine & Palfrey (APSR 2007) ($q \neq 1/2$, but only MR, no PR!)