Recursive Smooth Ambiguity Preferences

Peter Klibanoff, Massimo Marinacci and Sujoy Mukerji

Northwestern University, Collegio Carlo Alberto, University of Oxford

May 2009
Propose and provide foundations for a preference model set in an explicitly dynamic framework with uncertainty where:

- DM is sensitive to ambiguity
- ambiguity attitude is separated from ambiguity
- flexibility in ambiguity attitude and in scope of ambiguity
- preferences are dynamically consistent
- discounted expected utility is a special case
- beliefs are updated over time
Propose and provide foundations for a preference model set in an explicitly dynamic framework with uncertainty where:

- DM is sensitive to ambiguity
- ambiguity attitude is separated from ambiguity
- flexibility in ambiguity attitude and in scope of ambiguity
- preferences are dynamically consistent
- discounted expected utility is a special case
- beliefs are updated over time

We obtain a recursive extension of our (single period) Smooth Ambiguity model (Klibanoff, Marinacci, Mukerji, 2005) to a setting where uncertainty unfolds through a dated event tree.
Related Literature

- **Recursive Ambiguity Preference Models**
  - MEU - Epstein and Schneider (2003), Wang (2003), Hayashi (2005)
  - Variational - Maccheroni, Marinacci, Rustichini (2006)
Related Literature

- Recursive Ambiguity Preference Models
  - MEU - Epstein and Schneider (2003), Wang (2003), Hayashi (2005)
  - Variational - Maccheroni, Marinacci, Rustichini (2006)

- Non-recursive approach to dynamically consistent updating of ambiguity averse preferences
Related Literature

- **Recursive Ambiguity Preference Models**
  - MEU - Epstein and Schneider (2003), Wang (2003), Hayashi (2005)
  - Variational - Maccheroni, Marinacci, Rustichini (2006)

- **Non-recursive approach to dynamically consistent updating of ambiguity averse preferences**

- **Strotzian approach to relaxing dynamic consistency in dynamic ambiguity models**
  - Siniscalchi (2006, 2009)
Related Literature

- Recursive Ambiguity Preference Models
  - MEU - Epstein and Schneider (2003), Wang (2003), Hayashi (2005)
  - Variational - Maccheroni, Marinacci, Rustichini (2006)

- Non-recursive approach to dynamically consistent updating of ambiguity averse preferences

- Strotzian approach to relaxing dynamic consistency in dynamic ambiguity models
  - Siniscalchi (2006, 2009)

- Applications of our model or variations
  - Ju and Miao (2007)
  - Chen, Ju and Miao (2009)
  - Collard, Mukerji, Tallon and Sheppard (2009)
  - Hansen (2007)
The decision maker prefers act $f$ to act $g$ if and only if
\[ E_\mu \phi (E_\pi u \circ f) \geq E_\mu \phi (E_\pi u \circ g) \]

$vN$ is a vN-M utility function and characterizes attitude towards pure risk, as usual.
\( \phi \) maps expected utilities to the reals and is shown to capture a notion of ambiguity attitude.
\( \phi \) concave = ambiguity averse;
\( \phi \) convex = ambiguity loving.
\( \mu \) is a subjective probability over probability measures (denoted \( \pi \)) on the state space.
Ambiguity of belief about an event is shown to be characterized in the model by disagreement about the probability of the event across the \( \pi \)'s in the support of \( \mu \).
The decision maker prefers act \( f \) to act \( g \) if and only if

\[
E_\mu \phi (E_\pi u \circ f) \geq E_\mu \phi (E_\pi u \circ g)
\]

\( u \) is a vN-M utility function and characterizes attitude towards pure risk, as usual.
The decision maker prefers act $f$ to act $g$ if and only if

$$E_\mu \phi (E_\pi u \circ f) \geq E_\mu \phi (E_\pi u \circ g)$$

- $u$ is a vN-M utility function and characterizes attitude towards pure risk, as usual.
- $\phi$ maps expected utilities to the reals and is shown to capture a notion of ambiguity attitude. $\phi$ concave $= \text{ambiguity averse}$; $\phi$ convex $= \text{ambiguity loving}$. 

\[ E_\mu \phi (E_\pi u \circ f) \geq E_\mu \phi (E_\pi u \circ g) \]
Review of (Timeless/1-period) Smooth Ambiguity Model (KMM, 2005)

- The decision maker prefers act $f$ to act $g$ if and only if
  \[ E_\mu \phi (E_\pi u \circ f) \geq E_\mu \phi (E_\pi u \circ g) \]

- $u$ is a vN-M utility function and characterizes attitude towards pure risk, as usual.

- $\phi$ maps expected utilities to the reals and is shown to capture a notion of ambiguity attitude. $\phi$ concave = ambiguity averse; $\phi$ convex = ambiguity loving.

- $\mu$ is a subjective probability over probability measures (denoted $\pi$) on the state space,
Review of (Timeless/1-period) Smooth Ambiguity Model (KMM, 2005)

- The decision maker prefers act \( f \) to act \( g \) if and only if
  \[
  E_\mu \phi (E_\pi u \circ f) \geq E_\mu \phi (E_\pi u \circ g)
  \]

- \( u \) is a vN-M utility function and characterizes attitude towards pure risk, as usual.
- \( \phi \) maps expected utilities to the reals and is shown to capture a notion of ambiguity attitude. \( \phi \) concave = ambiguity averse; \( \phi \) convex = ambiguity loving.
- \( \mu \) is a subjective probability over probability measures (denoted \( \pi \)) on the state space,
- Ambiguity of belief about an event is shown to be characterized in the model by disagreement about the probability of the event across the \( \pi \)'s in the support of \( \mu \).
The decision maker prefers act $f$ to act $g$ if and only if

$$E_\mu \phi (E_\pi u \circ f) \geq E_\mu \phi (E_\pi u \circ g)$$

- $u$ is a vN-M utility function and characterizes attitude towards pure risk, as usual.
- $\phi$ maps expected utilities to the reals and is shown to capture a notion of ambiguity attitude. $\phi$ concave $=$ ambiguity averse; $\phi$ convex $=$ ambiguity loving.
- $\mu$ is a subjective probability over probability measures (denoted $\pi$) on the state space.
- Ambiguity of belief about an event is shown to be characterized in the model by disagreement about the probability of the event across the $\pi$'s in the support of $\mu$.
- The model makes no mention of time or multiple decision points.
The Dynamic Setting

- Discrete time, infinite horizon
The Dynamic Setting

- Discrete time, infinite horizon
- At an initial $t = 0$ and at each subsequent time $t$, the DM chooses a “consumption” plan, detailing current and future consumption (payoffs).
The Dynamic Setting

- Discrete time, infinite horizon
- At an initial $t = 0$ and at each subsequent time $t$, the DM chooses a “consumption” plan, detailing current and future consumption (payoffs).
- Available information at $t$ is given by the realizations of the discrete random variables $X_1, \ldots, X_t$. At $t = 0$, the DM has only prior information.
The Dynamic Setting

- Discrete time, infinite horizon
- At an initial $t = 0$ and at each subsequent time $t$, the DM chooses a “consumption” plan, detailing current and future consumption (payoffs).
- Available information at $t$ is given by the realizations of the discrete random variables $X_1, ..., X_t$. At $t = 0$, the DM has only prior information.
- $x_t \in X_t$ is a realization of the r.v. $X_t$, observed just before time $t$. 
Discrete time, infinite horizon

At an initial $t = 0$ and at each subsequent time $t$, the DM chooses a “consumption” plan, detailing current and future consumption (payoffs).

Available information at $t$ is given by the realizations of the discrete random variables $X_1, ..., X_t$. At $t = 0$, the DM has only prior information.

$x_t \in X_t$ is a realization of the r.v. $X_t$, observed just before time $t$

$S$ is the set of all possible observation paths $s = (x_1, ..., x_t, ...)$
The Dynamic Setting

- Discrete time, infinite horizon
- At an initial $t = 0$ and at each subsequent time $t$, the DM chooses a "consumption" plan, detailing current and future consumption (payoffs).
- Available information at $t$ is given by the realizations of the discrete random variables $X_1, \ldots, X_t$. At $t = 0$, the DM has only prior information.
- $x_t \in X_t$ is a realization of the r.v. $X_t$, observed just before time $t$
- $S$ is the set of all possible observation paths $s = (x_1, \ldots, x_t, \ldots)$
- $S^t$ is the collection of all paths $s^t = (x_1, \ldots, x_t)$ - the histories of observations up to $t$
Realizations map out an event tree; an $s^t$ identifies a time $t$ node in the tree.
The Dynamic Setting

Realizations map out an event tree; an $s^t$ identifies a time $t$ node in the tree.

The DM is subjectively uncertain about which stochastic process gives an appropriate description of probabilities on the event tree. The domain of this uncertainty is a finite parameter space, $\Theta$. 

\textcopyright{} Klibanoff, Marinacci, Mukerji (Northwestern I Recursive Smooth Ambiguity May 2009 6 / 24
Realizations map out an event tree; an $s^t$ identifies a time $t$ node in the tree.

The DM is subjectively uncertain about which stochastic process gives an appropriate description of probabilities on the event tree. The domain of this uncertainty is a finite parameter space, $\Theta$.

$\{\pi_\theta\}_{\theta \in \Theta}$ is a family of probability distributions over paths.
Realizations map out an event tree; an $s^t$ identifies a time $t$ node in the tree.

The DM is subjectively uncertain about which stochastic process gives an appropriate description of probabilities on the event tree. The domain of this uncertainty is a finite parameter space, $\Theta$.

$\{\pi_\theta\}_{\theta \in \Theta}$ is a family of probability distributions over paths.

$\pi_\theta(x_{t+1}; s^t)$ is the probability under distribution $\pi_\theta$ that the next observation will be $x_{t+1}$, given we have reached node $s^t$. 
A plan $f$ is a function mapping pairs $(t, s)$ to payoffs that respects the information structure.
A plan $f$ is a function mapping pairs $(t, s)$ to payoffs that respects the information structure.

Can equivalently think of a plan as mapping nodes to payoffs – $f(s^t)$ is the time $t$ payoff under plan $f$ given that node $s^t$ is reached.
A plan $f$ is a function mapping pairs $(t, s)$ to payoffs that respects the information structure.

Can equivalently think of a plan as mapping nodes to payoffs — $f(s^t)$ is the time $t$ payoff under plan $f$ given that node $s^t$ is reached.

Continuation plans at node $s^t$ are the restrictions of plans to successors of $s^t$ (including $s^t$ itself).
A *plan* \( f \) is a function mapping pairs \((t, s)\) to payoffs that respects the information structure.

Can equivalently think of a plan as mapping nodes to payoffs – \( f(s^t) \) is the time \( t \) payoff under plan \( f \) given that node \( s^t \) is reached.

*Continuation plans* at node \( s^t \) are the restrictions of plans to successors of \( s^t \) (including \( s^t \) itself).

An \( s^t \)-mixed deterministic continuation plan is a Borel probability distribution over payoff streams from \( t \) onwards.
A plan $f$ is a function mapping pairs $(t, s)$ to payoffs that respects the information structure.

Can equivalently think of a plan as mapping nodes to payoffs – $f(s^t)$ is the time $t$ payoff under plan $f$ given that node $s^t$ is reached.

Continuation plans at node $s^t$ are the restrictions of plans to successors of $s^t$ (including $s^t$ itself).

An $s^t$-mixed deterministic continuation plan is a Borel probability distribution over payoff streams from $t$ onwards.

A second order act is a function from $\Theta$ to payoffs.
At each node $s^t$, we consider two preference relations:
At each node $s^t$, we consider two preference relations:

- $\succeq_{s^t}$, over plans and $s^t$-mixed deterministic continuation plans; and
At each node $s^t$, we consider two preference relations:

- $\succeq_{s^t}$, over plans and $s^t$-mixed deterministic continuation plans; and
- $\succsim_{s^t}^2$, over second order acts.
Assumptions on Preferences

\( \succsim_{st} \) on plans satisfies

1. Weak Order
2. Monotonicity
3. Archimedean Continuity
4. Dynamic Consistency
Assumptions on Preferences

\( \succsim_{st} \) on plans satisfies

1. Weak Order
2. Monotonicity
3. Archimedean Continuity
4. Dynamic Consistency

\( \succsim_{st} \) on \( st \)-mixed deterministic continuation plans satisfies

5. Discounting
6. Invariance
Assumptions on Preferences

\( \succsim_{st} \) on plans satisfies

1. Weak Order
2. Monotonicity
3. Archimedean Continuity
4. Dynamic Consistency

\( \succsim_{st} \) on \( st \)-mixed deterministic continuation plans satisfies

5. Discounting
6. Invariance

\( \succsim^{2}_{st} \) on second order acts satisfies

7. SEU on second order acts
Assumptions on Preferences

\(\succeq_{st} \) on plans satisfies

1. Weak Order
2. Monotonicity
3. Archimedean Continuity
4. Dynamic Consistency

\(\succeq_{st} \) on \(s^t\)-mixed deterministic continuation plans satisfies

5. Discounting
6. Invariance

\(\succ_{st}^2 \) on second order acts satisfies

7. SEU on second order acts

\(\succeq_{st} \) and \(\succ_{st}^2 \) jointly satisfy

8. Consistency with associated second order acts.
**Theorem:** Our Assumptions hold if and only if there exists an additive probability \( \mu_{st} : 2^\Theta \rightarrow [0, 1] \) for each \( st \) and continuous and strictly increasing functions \( u : C \rightarrow \mathbb{R} \) and \( \phi : \mathcal{U} \rightarrow \mathbb{R} \), such that
**Theorem:** Our Assumptions hold if and only if there exists an additive probability $\mu_{s^t} : 2^\Theta \rightarrow [0, 1]$ for each $s^t$ and continuous and strictly increasing functions $u : C \rightarrow \mathbb{R}$ and $\phi : U \rightarrow \mathbb{R}$, such that

(i) On plans, each $\sim_{s^t}$ is represented by the monotonic recursive functional $V_{s^t}$ given by

$$V_{s^t} (f) = u (f (s^t)) + \beta \phi^{-1} \left[ \int_{\Theta} \phi \left( \int_{X_{t+1}} V_{(s^t, x_{t+1})} (f) \, d\pi_{\Theta} (\cdot; s^t) \right) \, d\mu_{s^t} \right];$$

(ii) On $s^t$-mixed deterministic continuation plans, each $\sim_{s^t}$ is represented by the expected discounted utility functional given by

$$EU_{s^t} (p) = \int D_{s^t} \sum_{\tau = t}^{\infty} \beta^\tau u (d (\tau)) \, dp;$$
Theorem: Our Assumptions hold if and only if there exists an additive probability $\mu_{st} : 2^\Theta \to [0, 1]$ for each $s^t$ and continuous and strictly increasing functions $u : \mathcal{C} \to \mathbb{R}$ and $\phi : \mathcal{U} \to \mathbb{R}$, such that

(i) On plans, each $\sim_{st}$ is represented by the monotonic recursive functional $V_{st}$ given by

$$V_{st}(f) = u(f(s^t)) + \beta \phi^{-1} \left[ \int_\Theta \phi \left( \int_{\chi_{t+1}} V_{(s^t, x_{t+1})}(f) \, d\pi_{\theta}(\cdot; s^t) \right) \, d\mu_{st} \right];$$

(ii) On $s^t$-mixed deterministic continuation plans, each $\sim_{st}$ is represented by the expected discounted utility functional given by

$$EU_{st}(p) = \int_{D_{st}} \left[ \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(d(\tau)) \right] \, dp;$$
(iii) On second order acts, each \( \succsim^2_{st} \) is represented by the subjective expected utility functional \( V^2_{st} \) given by

\[
V^2_{st}(f) = \int_\Theta \phi \left( \frac{u(f(\theta))}{1 - \beta} \right) d\mu_{st}.
\]
Is there a sense in which Bayesian updating of a prior is entailed in our model?
Is there a sense in which Bayesian updating of a prior is entailed in our model?

Yes, and this prior is related to the *predictive distributions* (i.e., the $\mu_{st}$-average of the $\pi_\theta$’s).

**Definition**

Given $\mu_{st}$, the *predictive distribution* $P_{st}$ is:

$$P_{st}(B) = \int_{\Theta} \pi_\theta (B \mid s^t) \, d\mu_{st}(\theta), \quad \forall B \in \Sigma.$$
Proposition: Given, for each $s^t$, a $V_{st}$ on plans of the form above, there exists a unique countably additive probability $\eta$ such that,

(i) If $\phi$ is affine, then

$$V_{st}(f) = \int_S \left[ \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(f(s^\tau)) \right] d\eta_{st}$$

where the measure $\eta$ is related to the $\pi_\theta$'s and the $\mu_{st}$'s (via the predictive distributions) through the formula

$$\eta(\{x_1, \ldots, x_m\}) = P_{s^0}(x_1) P_{(x_1)}(x_2) \cdots P_{(x_1,\ldots,x_{m-1})}(x_m),$$

for each $m \geq 1$ and $x_1, \ldots, x_m$, and $\eta_{st}$ is the Bayesian update of $\eta$;
(ii) if $\phi$ is differentiable, then

$$\frac{\partial V_{s^t}(f)}{\partial [\beta^n u(f((s^t, x_{t+1}, \ldots, x_{t+n})))]} \bigg|_{f=d} \frac{\partial V_{s^t}(f)}{\partial [u(f(s^t))]}$$

$$= \eta_{s^t} \left( \{s^t, x_{t+1}, \ldots, x_{t+n}\} \right)$$

for each $n \geq 1, t, s^t, x_{t+1}, \ldots, x_{t+n}$ and deterministic plan $d$. 
Updating

(ii) if $\phi$ is differentiable, then

$$
\left. \frac{\partial V_{st} (f) / \partial [\beta^n u(f((s^t, x_{t+1}, \ldots, x_{t+n})))]}{\partial V_{st} (f) / \partial [u(f(s^t))]} \right|_{f=d} = \eta_{st} (\{s^t, x_{t+1}, \ldots, x_{t+n}\})
$$

for each $n \geq 1, t, s^t, x_{t+1}, \ldots, x_{t+n}$ and deterministic plan $d$.

Result (ii) shows that that $\eta$ can be meaningfully characterized solely in terms of behavior toward plans even when preferences are ambiguity sensitive, as long as $\phi$ is differentiable (i.e., preferences are smooth).
Updating

- Under expected utility, subjective probability measures the trade-offs at the margin, evaluated at a riskless position, between sure payoffs and payoffs contingent on a specific event.
Updating

- Under expected utility, subjective probability measures the trade-offs at the margin, evaluated at a riskless position, between sure payoffs and payoffs contingent on a specific event.

- Analogously, the result shows, \( \eta \) measures the trade-offs at the margin between utility at \( s^0 \) and (discounted) utility at successor nodes, when these trade-offs are evaluated at a deterministic plan.
Updating

- Under expected utility, subjective probability measures the trade-offs at the margin, evaluated at a riskless position, between sure payoffs and payoffs contingent on a specific event.
- Analogously, the result shows, $\eta$ measures the trade-offs at the margin between utility at $s^0$ and (discounted) utility at successor nodes, when these trade-offs are evaluated at a deterministic plan.
- So,
  - $\eta$ measures these marginal trade-offs evaluated at any deterministic position;
Under expected utility, subjective probability measures the trade-offs at the margin, evaluated at a riskless position, between sure payoffs and payoffs contingent on a specific event.

Analogously, the result shows, $\eta$ measures the trade-offs at the margin between utility at $s^0$ and (discounted) utility at successor nodes, when these trade-offs are evaluated at a deterministic plan.

So,

- $\eta$ measures these marginal trade-offs evaluated at any deterministic position;
- $\eta$ is updated according to Bayes’ rule;
Updating

- Under expected utility, subjective probability measures the trade-offs at the margin, evaluated at a riskless position, between sure payoffs and payoffs contingent on a specific event.

- Analogously, the result shows, $\eta$ measures the trade-offs at the margin between utility at $s^0$ and (discounted) utility at successor nodes, when these trade-offs are evaluated at a deterministic plan.

- So,
  - $\eta$ measures these marginal trade-offs evaluated at any deterministic position;
  - $\eta$ is updated according to Bayes’ rule;
  - the one-step-ahead marginals for each $\eta_{st}$ agree with the one-step-ahead marginals for the corresponding predictive distribution, $P_{st}$. 
Can we go further and say that the $\mu_{st}$ are the Bayesian updates of $\mu (= \mu_{s0})$?
Can we go further and say that the $\mu_{st}$ are the Bayesian updates of $\mu (=\mu_{s0})$?

It turns out that this question is tightly linked to the question of whether the $\eta_{st}$ and the predictive distributions $P_{st}$ agree everywhere.
Consider a DM's marginal trade-off between utility at $s^t$ and (discounted) utility at a successor node $(s^t, x_{t+1}, \ldots, x_{t+n})$, when evaluated at a *deterministic* plan.
Consider a DM’s marginal trade-off between utility at $s^t$ and (discounted) utility at a successor node $(s^t, x_{t+1}, \ldots, x_{t+n})$, when evaluated at a deterministic plan.

One could imagine, because this trade-off is being made given only information available at $s^t$, the trade-off could be recoverable from preferences over second order acts at $s^t$ alone, in particular from $\mu_{s^t}$.
Consider a DM’s marginal trade-off between utility at $s^t$ and (discounted) utility at a successor node $(s^t, x_{t+1}, \ldots, x_{t+n})$, when evaluated at a deterministic plan.

One could imagine, because this trade-off is being made given only information available at $s^t$, the trade-off could be recoverable from preferences over second order acts at $s^t$ alone, in particular from $\mu_{s^t}$.

If the DM were expected utility over plans with beliefs given by the $\mu_{s^t}$-average of the $\pi_\theta$, the trade-off would indeed be identified by the “$n$-step-ahead” predictive distribution $P_{s^t} (x_{t+1}, \ldots, x_{t+n})$. 
Consider a DM’s marginal trade-off between utility at $s^t$ and (discounted) utility at a successor node $(s^t, x_{t+1}, \ldots, x_{t+n})$, when evaluated at a deterministic plan.

One could imagine, because this trade-off is being made given only information available at $s^t$, the trade-off could be recoverable from preferences over second order acts at $s^t$ alone, in particular from $\mu_{s^t}$.

If the DM were expected utility over plans with beliefs given by the $\mu_{s^t}$-average of the $\pi_\theta$, the trade-off would indeed be identified by the “$n$-step-ahead” predictive distribution $P_{s^t} (x_{t+1}, \ldots, x_{t+n})$.

Assuming global ambiguity neutrality is very strong, and would defeat the whole purpose of our modeling exercise. All that is needed for $P_{s^t} (x_{t+1}, \ldots, x_{t+n})$ to identify these trade-offs is that the DM is ambiguity neutral “locally around determinism” with beliefs given there by the $\mu_{s^t}$-average of the $\pi_\theta$. 

Klibanoff, Marinacci, Mukerji (Northwestern University, Collegio Carlo Alberto, University of Oxford)
If \( \phi \) differentiable, such local ambiguity neutrality is perfectly compatible with overall sensitivity to ambiguity, just as local risk-neutrality is compatible with global risk aversion or love in the standard expected utility model.
Updating

- If $\phi$ differentiable, such local ambiguity neutrality is perfectly compatible with overall sensitivity to ambiguity, just as local risk-neutrality is compatible with global risk aversion or love in the standard expected utility model.

**Assumption MRS:** For each deterministic plan $d$,

$$
\frac{\partial V_{s^t}(f)}{\partial \beta^n u(f((s^t, x_{t+1}, \ldots, x_{t+n})))} = P_{s^t}(x_{t+1}, \ldots, x_{t+n})
$$

for each $n \geq 1, t, s^t, x_{t+1}, \ldots, x_{t+n}$. 
If $\phi$ differentiable, such local ambiguity neutrality is perfectly compatible with overall sensitivity to ambiguity, just as local risk-neutrality is compatible with global risk aversion or love in the standard expected utility model.

**Assumption MRS:** For each deterministic plan $d$,

$$
\frac{\partial V_{s^t}(f)}{\partial \left[ \beta^n u(f((s^t, x_{t+1}, \ldots, x_{t+n}))) \right]} = P_{s^t}(x_{t+1}, \ldots, x_{t+n})
$$

for each $n \geq 1, t, s^t, x_{t+1}, \ldots, x_{t+n}$.

Recall that $P_{s^t}(x_{t+1}, \ldots, x_{t+n})$ is completely determined by $x_{t+1}^2$ through $\mu_{s^t}$.  

Klibanoff, Marinacci, Mukerji (Northwestern University, Collegio Carlo Alberto, University of Oxford)  
Recursive Smooth Ambiguity  
May 2009 18 / 24
Theorem: Given our previous assumptions and $\phi$ differentiable, Assumption MRS is equivalent to the predictive distributions being related by Bayes’ rule (i.e., $P_{s^t}(B) = P_{s^0}(B \cap s^t) / P_{s^0}(s^t)$ for each $t, s^t, B \in \Sigma$. )
Theorem: Given our previous assumptions and $\phi$ differentiable, Assumption MRS is equivalent to the predictive distributions being related by Bayes’ rule (i.e., $P_{s^t}(B) = P_{s^0}(B \cap s^t) / P_{s^0}(s^t)$ for each $t, s^t, B \in \Sigma$).

- It is easy to show that if $\mu_{s^t}$ are the Bayesian updates of $\mu (= \mu_{s^0})$ then the predictive distributions are related by Bayes’ rule.
**Theorem:** Given our previous assumptions and $\phi$ differentiable, Assumption MRS is equivalent to the predictive distributions being related by Bayes’ rule (i.e., $P_{st}(B) = P_{s0}(B \cap s^t) / P_{s0}(s^t)$ for each $t, s^t, B \in \Sigma$).

- It is easy to show that if $\mu_{st}$ are the Bayesian updates of $\mu (= \mu_{s0})$ then the predictive distributions are related by Bayes’ rule.
- What about the other direction?
**Full Rank Condition**

**Definition:** The *full rank condition* holds if, for each node $s^t$, there exist $m - 1$ elementary cylinder sets $B^s_1, \ldots, B^s_{m-1}$ such that the $m \times m$ matrix $A^s_t \equiv \begin{bmatrix} 1 & \cdots & 1 \\ \pi_{\theta_1} \left( B^s_1 \mid s^t \right) & \cdots & \pi_{\theta_m} \left( B^s_1 \mid s^t \right) \\ \vdots & \vdots & \vdots \\ \pi_{\theta_1} \left( B^s_{m-1} \mid s^t \right) & \cdots & \pi_{\theta_m} \left( B^s_{m-1} \mid s^t \right) \end{bmatrix}$ is of full rank, where $\Theta = \{\theta_1, \ldots, \theta_m\}$.
**Full Rank Condition**

**Definition:** The *full rank condition* holds if, for each node \( s^t \), there exist \( m - 1 \) elementary cylinder sets \( B_1^{s^t}, \ldots, B_{m-1}^{s^t} \) such that the \( m \times m \) matrix

\[
A^{s^t} \equiv \begin{bmatrix}
1 & \cdots & 1 \\
\pi_{\theta_1} \left( B_1^{s^t} \mid s^t \right) & \cdots & \pi_{\theta_m} \left( B_1^{s^t} \mid s^t \right) \\
\vdots & \ddots & \vdots \\
\pi_{\theta_1} \left( B_{m-1}^{s^t} \mid s^t \right) & \cdots & \pi_{\theta_m} \left( B_{m-1}^{s^t} \mid s^t \right)
\end{bmatrix}
\]

is of full rank, where \( \Theta = \{\theta_1, \ldots, \theta_m\} \).

- For each \( s^t \), there is an infinite number of ways to select \( m - 1 \) elementary cylinder sets. The full rank condition requires simply that *one* such selection yield a non-singular matrix.
**Full Rank Condition**

**Definition:** The *full rank condition* holds if, for each node $s^t$, there exist $m - 1$ elementary cylinder sets $B_1^{s^t}, \ldots, B_{m-1}^{s^t}$ such that the $m \times m$ matrix

$$A^{s^t} \equiv \begin{bmatrix}
1 & \cdots & 1 \\
p_{\theta_1} \left(B_1^{s^t} \mid s^t\right) & \cdots & p_{\theta_m} \left(B_1^{s^t} \mid s^t\right) \\
\vdots & \ddots & \vdots \\
p_{\theta_1} \left(B_{m-1}^{s^t} \mid s^t\right) & \cdots & p_{\theta_m} \left(B_{m-1}^{s^t} \mid s^t\right)
\end{bmatrix}$$

is of full rank, where $\Theta = \{\theta_1, \ldots, \theta_m\}$.

- For each $s^t$, there is an infinite number of ways to select $m - 1$ elementary cylinder sets. The full rank condition requires simply that *one* such selection yield a non-singular matrix.
- Satisfied, for example, when each $p_\theta$ makes the sequence $\{X_t\}_{t \in \mathcal{T}}$ a homogeneous Markov chain (see Proposition 4).
Corollary: Given our previous assumptions, $\phi$ differentiable, and the full rank condition, Assumption MRS is equivalent to Bayesian updating of the $\mu_{st}$ (i.e., $\mu_{st}(\theta) = \frac{\mu(\theta)\pi_{\theta}(s^t)}{\int_{\Theta} \pi_{\theta}(s^t) d\mu}$ for all $t, s^t, \theta$).
Assumption MRS as Limited Closure

- Assumption MRS is implied by a reduced-form dynamic smooth ambiguity model with discounted utility when $\phi$ is differentiable:

$$\hat{V}_{s^t}(f) = u(f(s^t)) + \beta \phi^{-1} \left[ \int_{\Theta} \phi \left( \sum_{s \in S} \pi_\theta(s | s^t) \sum_{\tau \geq t+1} \beta^{\tau-(t+1)} u(f(s)(\tau)) \right) d\mu_{s^t} \right]$$
Assumption MRS as Limited Closure

- Assumption MRS is implied by a reduced-form dynamic smooth ambiguity model with discounted utility when \( \phi \) is differentiable:
  \[
  \hat{V}_{s^t}(f) = u(f(s^t)) + \beta \phi^{-1} \left[ \int_{\Theta} \phi \left( \sum_{s \in S \setminus \{s^t\}} \pi_\theta(s | s^t) \sum_{\tau \geq t+1} \beta^{\tau-(t+1)} u(f(s)(\tau)) \right) d\mu_{s^t} \right]
  \]

- Any such model is locally ambiguity neutral (i.e., locally expected utility) around any deterministic plan. Under ambiguity neutrality, \( \mu_{s^t} \)-average probabilities precisely reveal utility trade-offs at the margin.
Assumption MRS as Limited Closure

- Assumption MRS is implied by a reduced-form dynamic smooth ambiguity model with discounted utility when $\phi$ is differentiable:

$$
\hat{V}_{st}(f) = u(f(s^t)) + 
\beta \phi^{-1} \left[ \int_{\Theta} \phi \left( \sum_{s \in S \cap \{s^t\}} \pi_\theta (s \mid s^t) \sum_{\tau \geq t+1} \beta^{\tau-(t+1)} u(f(s)(\tau)) \right) d\mu_{st} \right]
$$

- Any such model is locally ambiguity neutral (i.e., locally expected utility) around any deterministic plan. Under ambiguity neutrality, $\mu_{st}$-average probabilities precisely reveal utility trade-offs at the margin.

- In this sense, Assumption MRS is a very limited version of the closure sometimes assumed for recursive models – here we are not demanding that each $\Rightarrow_{st}$ on plans is represented by a reduced-form dynamic smooth ambiguity model, but simply that it shares the relationship between the predictive distributions and marginal rates of substitution around determinacy with that model.
Closure is (in combination with recursion) what delivers anything that dynamic recursive models have to say about updating – e.g., for recursive multiple priors it delivers prior–by-prior Bayesian updating applied to the overall rectangular set of priors, for recursive variational preferences it delivers a condition on how the ambiguity index must be updated.
In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:
In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:

- the model is recursive
Conclusion

- In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:
  - the model is recursive
  - discounted expected utility is a special case

Yet:

- the DM may be non-neutral to ambiguity
- ambiguity attitudes are modelled very much like risk attitudes
- ambiguity modelled through familiar probabilities

Caveat/feature: As with other non-linear recursive models, the preferences are sensitive to the information filtration with respect to which the recursion is defined – different information flow may lead to different preferences over plans.

Much more in the paper, including important results on existence, uniqueness, learning and examples.
In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:

- the model is recursive
- discounted expected utility is a special case
- beliefs take the form of a probability (over parameters)
In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:

- the model is recursive
- discounted expected utility is a special case
- beliefs take the form of a probability (over parameters)
- this belief is updated over time using Bayes’ rule
Conclusion

- In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:
  - the model is recursive
  - discounted expected utility is a special case
  - beliefs take the form of a probability (over parameters)
  - this belief is updated over time using Bayes’ rule

- Yet:
  - the DM may be non-neutral to ambiguity

Much more in the paper, including important results on existence, uniqueness, learning and examples.
Conclusion

- In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:
  - the model is recursive
  - discounted expected utility is a special case
  - beliefs take the form of a probability (over parameters)
  - this belief is updated over time using Bayes’ rule

- Yet:
  - the DM may be non-neutral to ambiguity
  - ambiguity attitudes are modelled very much like risk attitudes
In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:

- the model is recursive
- discounted expected utility is a special case
- beliefs take the form of a probability (over parameters)
- this belief is updated over time using Bayes’ rule

Yet:

- the DM may be non-neutral to ambiguity
- ambiguity attitudes are modelled very much like risk attitudes
- ambiguity modelled through familiar probabilities

Caveat/feature: As with other non-linear recursive models, the preferences are sensitive to the information filtration with respect to which the recursion is defined – different information flow may lead to different preferences over plans.

Much more in the paper, including important results on existence, uniqueness, learning and examples.
Conclusion

- In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:
  - the model is recursive
  - discounted expected utility is a special case
  - beliefs take the form of a probability (over parameters)
  - this belief is updated over time using Bayes’ rule

- Yet:
  - the DM may be non-neutral to ambiguity
  - ambiguity attitudes are modelled very much like risk attitudes
  - ambiguity modelled through familiar probabilities

- Caveat/feature: As with other non-linear recursive models, the preferences are sensitive to the information filtration with respect to which the recursion is defined – different information flow may lead to different preferences over plans.
In the end, we have a dynamic model of preferences under uncertainty where much is preserved from standard analysis:

- the model is recursive
- discounted expected utility is a special case
- beliefs take the form of a probability (over parameters)
- this belief is updated over time using Bayes’ rule

Yet:

- the DM may be non-neutral to ambiguity
- ambiguity attitudes are modelled very much like risk attitudes
- ambiguity modelled through familiar probabilities

Caveat/feature: As with other non-linear recursive models, the preferences are sensitive to the information filtration with respect to which the recursion is defined – different information flow may lead to different preferences over plans.

Much more in the paper, including important results on existence, uniqueness, learning and examples.