Dynamic Asset Allocation with Ambiguous Return Predictability

Hui Chen, MIT

Nengjiu Ju, HKUST

Jianjun Miao, BU

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Motivation

- do decision makers really have full confidence in their models?
- departure from rational expectation hypothesis
- many different interpretations:
  - ambiguity, misspecification, Knightian uncertainty, robust control, source-dependent risk attitudes
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1. how to invest when misspecification of the dynamics of stock returns (predictability) is a major concern
2. welfare implications of different strategies
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- a long time debate
- evidence of stock-return predictability is controversial
ARE RETURNS PREDICTABLE?

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- evidence of stock-return predictability is controversial
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    ⇒ aggressive market-timing strategies
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⇒ practical usefulness of return forecasts in real-time
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  - economic significance: implies large variations in expected returns
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- limited out-of-sample explanatory power ⇒ practical usefulness of return forecasts in real-time market-timing portfolios in question
Table 1, Cochrane (2007)

Table 1
Forecasting regressions

<table>
<thead>
<tr>
<th>Regression</th>
<th>$b$</th>
<th>$t$</th>
<th>$R^2(%)$</th>
<th>$\sigma(bx)(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{t+1} = a + b(D_t/P_t) + \varepsilon_{t+1}$</td>
<td>3.39</td>
<td>2.28</td>
<td>5.8</td>
<td>4.9</td>
</tr>
<tr>
<td>$R_{t+1} - R_t^f = a + b(D_t/P_t) + \varepsilon_{t+1}$</td>
<td>3.83</td>
<td>2.61</td>
<td>7.4</td>
<td>5.6</td>
</tr>
<tr>
<td>$D_{t+1}/D_t = a + b(D_t/P_t) + \varepsilon_{t+1}$</td>
<td>0.07</td>
<td>0.06</td>
<td>0.0001</td>
<td>0.001</td>
</tr>
<tr>
<td>$r_{t+1} = a_r + b_r(d_t - p_t) + \varepsilon_{t+1}^r$</td>
<td>0.097</td>
<td>1.92</td>
<td>4.0</td>
<td>4.0</td>
</tr>
<tr>
<td>$\Delta d_{t+1} = a_d + b_d(d_t - p_t) + \varepsilon_{t+1}^{dp}$</td>
<td>0.008</td>
<td>0.18</td>
<td>0.00</td>
<td>0.003</td>
</tr>
</tbody>
</table>

$R_{t+1}$ is the real return, deflated by the CPI, $D_{t+1}/D_t$ is real dividend growth, and $D_t/P_t$ is the dividend-price ratio of the CRSP value-weighted portfolio. $R_t^f$ is the real return on 3-month Treasury-Bills. Small letters are logs of corresponding capital letters. Annual data, 1926–2004. $\sigma(bx)$ gives the standard deviation of the fitted value of the regression.
Main Results

1 difference from a Bayesian investor:
   - irreducible compound lotteries
   - endogenous “distorted” beliefs
   - more conservative investment strategies

2 hedging demand of robust strategy:
   - hedge against investment opportunities (market timing)
   - hedge against model uncertainty

3 robustness can lead to significant difference in asset allocation even when there is very little model uncertainty

4 for ambiguity-averse investors, stocks are not that safe in the long run

5 welfare: the costs of adopting suboptimal investment strategies (ignoring model uncertainty) are large
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**Related Literature**


**Preferences**

- homothetic recursive ambiguity utility function:

\[
V_t(C) = \left[ C_t^{1-\gamma} + \beta \left\{ \mathbb{E}_{\mu_t} \left( \mathbb{E}_{\pi_{z,t}} \left[ V_{t+1}^{1-\gamma}(C_{t+1}) \right] \right) \right\}^{\frac{1-\eta}{1-\gamma}} \right]^{\frac{1}{1-\gamma}}
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  - \( \eta > \gamma \Rightarrow \) ambiguity aversion
  - \( \eta \to +\infty \Rightarrow \) recursive multiple prior model (max-min)
Investment Opportunities

- two tradeable assets: a riskfree bond \((r_f)\) and a risky (log return \(r_{t+1}\) from \(t\) to \(t + 1\))
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- Model 1 (IID):

\[
 r_{t+1} - r_f = m + \varepsilon_{1,t+1}
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\(\varepsilon_{1,t+1} \sim i.i.d.\) normal with mean zero and variance \(\sigma_1^2\)
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\(\varepsilon_{1,t+1}\) - i.i.d. normal with mean zero and variance \(\sigma_1^2\)
- Model 2 (VAR):

\[
r_{t+1} - r_f = m + bx_t + \varepsilon_{2,t+1}
\]

\[x_{t+1} = \rho x_t + \varepsilon_{3,t+1}\]

\(x_t\) - demeaned predictive variable
\([\varepsilon_{2,t+1}, \varepsilon_{3,t+1}]'\) - i.i.d. normal with mean zero and covariance matrix \(\Omega\); independent of \(\varepsilon_{1,t+1}\)
Belief dynamics

- $\mu_t = \Pr (z = 1|s^t)$: posterior probability that Model 1 is the true model (given history of data $s^t = \{(r_0, x_0), (r_1, x_1), \ldots, (r_t, x_t)\}$)
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- Bayes’ Rule

\[
\mu_{t+1} = \frac{\mu_t f_1(r_{t+1}, r_f + m)}{\mu_t f_1(r_{t+1}, r_f + m) + (1 - \mu_t) f_2(r_{t+1}, r_f + m + bx_t)}
\]

where

\[
f_z(y, a) = \frac{1}{\sigma_z \sqrt{2\pi}} \exp \left[ -\frac{(y - a)^2}{2\sigma_z^2} \right], \quad z = 1, 2.
\]
Optimal consumption and portfolio choice

- finite horizon $T$ ($C_T = W_T$)
- at time $t$, choose consumption $C_t$ and portfolio weight $\psi_t \in [0, 1]$
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where

\[ R_{p,t+1} = R_{t+1} \psi_t + R_f (1 - \psi_t) \]
**Optimal Consumption and Portfolio Choice**

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  where
  
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- Bellman equation (subject to budget constraint and belief dynamics)
  
  $$J_t(W_t, x_t, \mu_t) = \max_{C_t, \psi_t} \left[ C_t^{1-\gamma} + \beta \left\{ \mu_t \left( E_t^1 \left[ J_{t+1}^{1-\gamma} (W_{t+1}, x_{t+1}, \mu_{t+1}) \right] \right) \right\}^{\frac{1-\eta}{1-\gamma}}$$

  $$+ (1 - \mu_t) \left( E_t^2 \left[ J_{t+1}^{1-\gamma} (W_{t+1}, x_{t+1}, \mu_{t+1}) \right] \right) \right\}^{\frac{1-\gamma}{1-\eta}} \right\}^{\frac{1}{1-\gamma}}$$
Euler Equation

- when optimal portfolio weight $\psi_t^*$ is an interior solution in $(0, 1)$,

$$\mathbb{E}_t [M_{z,t+1} (R_{t+1} - R_f)] = 0, \quad t = 0, 1, \cdots , T - 1$$

$M_{z,t+1}$ – pricing kernel for the recursive smooth ambiguity utility model ($z = 1, 2$)

$$M_{z,t+1} = \left( \mathbb{E}_t^z \left[ R_{p,t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] \right)^{-\frac{\eta - \gamma}{1 - \gamma}} \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma}$$
“DISTORTED BELIEFS”

Euler equation can be rewritten as:

\[ 0 = \hat{\mu}_t \mathbb{E}_t^1 \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} (R_{t+1} - R_f) \right] \]

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\( \hat{\mu}_t \) is the distorted belief about the IID model:

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\hat{\mu}_t = \frac{\mu_t \left( \mathbb{E}_t^1 \left[ J_{t+1}^{1-\gamma} \right] \right)^{-\frac{\eta-\gamma}{1-\gamma}}}{\mu_t \left( \mathbb{E}_t^1 \left[ J_{t+1}^{1-\gamma} \right] \right)^{-\frac{\eta-\gamma}{1-\gamma}} + (1 - \mu_t) \left( \mathbb{E}_t^2 \left[ J_{t+1}^{1-\gamma} \right] \right)^{-\frac{\eta-\gamma}{1-\gamma}}}
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\]

- ambiguity-averse investors always slant their beliefs towards the “worse” model
- \( \hat{\mu}_t \) is endogenous (preference dependent) and cannot be generated with Bayes Rule using any prior \( \mu_0 \)
**How to calibrate \( \eta \)?**

- a static thought experiment: Ellsberg Paradox
- two urns: one contains 50 black balls and 50 white balls; the other contains 100 balls of the same color (black or white)
- payoff \( d \) if guess the right color; assume prior on urn 2 color is 50/50
- define the ambiguity premium as
  
  \[
  CE_1 = u^{-1} (0.5u(W + d) + 0.5u(W)) \\
  CE_2 = u^{-1} (\phi^{-1} (0.5\phi (u(W + d)) + 0.5\phi(u(W))))
  \]

  Ambiguity premium: \( A = \frac{CE_1 - CE_2}{d^2} \)

- Camerer (1999) and Halevy (2007) report ambiguity premium in the order of 10-20 percent (small gambles)
## Ambiguity Premium

<table>
<thead>
<tr>
<th>$\gamma / \eta$</th>
<th>40.0</th>
<th>60.0</th>
<th>80.0</th>
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<tbody>
<tr>
<td>Prize-wealth ratio</td>
<td>1.0%</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>0.5</td>
<td>9.8</td>
<td>14.6</td>
<td>19.3</td>
<td>23.8</td>
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<td>18.9</td>
<td>23.4</td>
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<td><strong>13.5</strong></td>
<td>18.2</td>
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<tr>
<td>15.0</td>
<td>6.2</td>
<td>11.0</td>
<td>15.7</td>
<td>20.2</td>
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</tbody>
</table>
MARKET-TIMING EFFECT

A. $\mu_0=0.2$

B. $\mu_0=0.4$

C. $\mu_0=0.6$

D. $\mu_0=0.8$
UNCERTAINTY EFFECT

A. $x_0 = -1.12$

B. $x_0 = -0.37$

C. $x_0 = 0.37$

D. $x_0 = 1.12$
HORIZON EFFECT

A. $x_0 = -0.4677$

B. $x_0 = 0$

C. $x_0 = 0.4677$
## Welfare Costs of Suboptimal Strategies

\[ J_{0}^{sub} (W_0 (1 + m), x_0, \mu_0) = J_0 (W_0, x_0, \mu_0) \]

<table>
<thead>
<tr>
<th>( \gamma/\eta )</th>
<th>60</th>
<th>100</th>
<th>60</th>
<th>100</th>
<th>( \mu_0 = 0.2 )</th>
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<th>100</th>
<th>60</th>
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<th>( \mu_0 = 0.4 )</th>
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<td>14.5</td>
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we use uncertainty about return predictability to motivate robust portfolio rules
robust strategy differs from Bayesian strategy: “distorted beliefs”
large effects in some cases even when there is little uncertainty
sizable welfare costs for ignoring model uncertainty
misspecification not limited to predictability: covariance structure, “imperfect” predictors, etc.