Context-Dependent
Forward Induction Reasoning

Pierpaolo Battigalli
Amanda Friedenberg
Forward Induction:

Players think that past behavior of the opponents was "strategically rational" (if possible) and so play a best response.
Battle of the Sexes

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Battle of the Sexes

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\text{A} & 0 & 3 \\
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Battle of the Sexes

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1 & 0 \\
3 & 0 \\
0 & 3 \\
0 & 1 \\
\end{array} \\
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Battle of the Sexes

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2 & 1 & 0 \\
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Battle of the Sexes

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2 & * & 1 & 0 \\
3 & 3 & 0 & 0 \\
3 & 0 & 3 & 1 \\
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\]
Battle of the Sexes

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    In  Out
  ---  ---
B    3   0
L    1   0
U    3   0
D    0   3
A    0   1
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Kohlberg and Mertens introduce forward induction as a criterion for selecting among the Nash equilibria of a game. They do not provide an explicit definition, instead relying on motivating examples and the cryptic label to their theorem that:

(Forward Induction) A stable set contains a stable set of any game obtained by deletion of a strategy with is an inferior response in all the equilibria of the set.

--- from “On Forward Induction”

Govindan and Wilson (Econometrica, 2009)
Language to Formalize FI

The Epistemic Game:

1. Rules of the Game (e.g., Extensive Form)
2. Payoff Functions
3. Beliefs (Modeled as a type structure)
   - About “strategies played”
   - Each type has a belief for each information set
   - Conditional Probability System (CPS)
Back to the Example

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  \hline
  U & 1 & 0 \\
  D & 3 & 0 \\
  O & 0 & 1 \\
  L & 0 & 1 \\
  R & [1] & \\
\end{array}
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  \beta^b(t^b) & t^a & u^a \\
  U & 1 & \\
  D & \\
  O & 1 & \\
  L & \\
  R & [1] & \\
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**Payoff Matrices:**

- **Player A (Actions):** U, D
- **Player B (Actions):** L, R

**Payoffs:**

- **Player A:**
  - U: 1, 0
  - D: 3, 0

- **Player B:**
  - L: 1, 1
  - R: 0, 3

**Equilibrium Strategies:**

- **Player B:**
  - \( \beta^b(t^b) \): U → t^b, D → u^b
  - \( \beta^b(u^b) \): U → t^a, D → u^a

- **Player A:**
  - \( \beta^a(t^a) \): U → t^b, D → u^b
  - \( \beta^a(u^a) \): U → t^a, D → u^a
Back to the Example

\[ \beta^b(t^b) \]

\[ \beta^b(u^b) \]

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Ingredients of Forward Induction

Rationality: strategy-type pair

Forward Induction:

- A *type* of Ann rationalizes Bob’s past behavior, when possible
- A type *strong believes* an event---i.e., assigns probability one to the event, when possible. [Battigalli-Siniscalchi, 2002]

Strong Belief of Rationality
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A & \text{Out} & B \\
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U & 1 & 0 \\
D & 0 & 3 \\
\end{array} \]

\[ \begin{array}{c|cc}
B & t^a & u^a \\
\hline
U & 1 & \beta^b(t^b) \\
D & \beta^b(u^b) \\
\end{array} \]

\[ \begin{array}{c|cc}
A & \text{In} & B \\
\hline
L & 0 & 1 \\
R & \beta^a(t^a) \\
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\[ \begin{array}{c|cc}
D & t^a & u^a \\
\hline
O & 1 & \beta^a(u^a) \\
L & \beta^a(t^a) \\
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A Recap

Fix a Type Structure:

Rationality and Common Strong Belief of Rationality:

• Rational strategy-type pair
• Each type strongly believes “the other player is rational”
• Each type strong believes:
  “the other player is rational and strongly believes I am rational”
Type Structure

**Type Structure:** (Implicit Description of Hierarchies of Conditional Beliefs)

- Type Sets: $T^a$ and $T^b$.
- Each type $t^a$ associated with a CPS on $S^b \times T^b$.

**Question:** Why the particular type structure used?
We think of a particular … structure as giving the “context” in which the game is played. In line with Savage’s Small-Worlds idea in decision theory (Savage (1954, pp. 82–91)), who the players are in the given game can be seen as a shorthand for their experiences before the game. The players’ possible characteristics—including their possible types—then reflect the prior history or context.

--- Brandenburger, Friedenberg, Keisler (2008)
The Context

Premise:

• We look at a snapshot of the strategic situation.
• The context may cause certain beliefs to be “transparent.”

Implications for Forward Induction:

• Certain beliefs are “ruled out.”
• So, players may be limited in their ability to rationalize the past.
The Context

Premise:

- We look at a snapshot of the strategic situation.
- The context may cause certain beliefs to be “transparent.”

Implications for Forward Induction:

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- So, players may be limited in their ability to rationalize the past.

Missing Piece of the Puzzle!
Lady’s Choice Convention

It is “transparent” that, if the Lady gets to move in a BoS-like game, she tries to get her “best outcome.”
Lady’s Choice Convention

It is “transparent” that, if the Lady gets to move in a BoS-like game, she tries to get her “best outcome.”

**Type Structure for Lady’s Choice Convention:**

- Each type of Bob mapped into a CPS that “believes” \( \{Up\} \times T^a \).
- For any CPS that “believes” \( \{Up\} \times T^a \), there is a type \( t^b \) associated with that CPS.
- For any CPS on \( S^b \times T^b \), there is a type \( t^a \) associated with that CPS.
Epistemic Type Structure:

- For any CPS that believes \( \{Up\} \times T^a \), there is a type \( t^b \) associated with that CPS. (And nothing else!)
- For any CPS on \( S^b \times T^b \), there is a type \( t^a \) associated with that CPS.
### Back to the Example

#### Epistemic Type Structure:
- For any CPS that believes \{Up\}×\(T^a\), there is a type \(t^b\) associated with that CPS. (And nothing else!)
- For any CPS on \(S^b×T^b\), there is a type \(t^a\) associated with that CPS.

#### Rational Strategy-Type Pairs:
- \{Out\} × \(T^b\)
- Pairs (Up, \(t^a\)) and (Down, \(u^a\))
Back to the Example

Rational Strategy-Type Pairs:
- \{Out\} × \(T^b\)
- Pairs \((Up, t^a)\) and \((Down, u^a)\)
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#### Rational Strategy-Type Pairs:
- $\{\text{Out}\} \times T^b$
- Pairs ($\text{Up}, t^a$) and ($\text{Down}, u^a$)

#### Rationality and Strong Belief of Rationality:
- ($\text{Out}, t^b$)
- Pairs ($\text{Up}, t^a$) and ($\text{Down}, u^a$)
Rational Strategy-Type Pairs:
- $\{Out\} \times T^b$
- Pairs $(Up, t^a)$ and $(Down, u^a)$

Rationality and Strong Belief of Rationality:
- $(Out, t^b)$
- Pairs $(Up, t^a)$ and $(Down, u^a)$
Context-Dependent Forward Induction Reasoning

Fix a Type Structure: Context

Rationality and Common Strong Belief of Rationality:

- Rational strategy-type pair
- Each type strongly believes “the other player is rational”
- Each type strongly believes:
  “the other player is rational and strongly believes I am rational”

...
Can we characterize sets of the form $Q^a \times Q^b$?
Call $Q^a \times Q^b$ an **Extensive-Form Best Response Set** (EFBRS) if: for each $s^a$ in $Q^a$, there exists a CPS $\mu(s^a)$ satisfying

i. $s^a$ is sequentially optimal under $\mu(s^a)$;

ii. $\mu(s^a)$ strongly believes $Q^b$;

iii. if $r^a$ is sequentially optimal under $\mu(s^a)$, then $r^a$ is also contained in $Q^a$.

And, likewise, with $a$ and $b$ reversed.
A Characterization Theorem

**Theorem:** Fix an extensive-form game.

1. For any given type structure, the set of strategies consistent with RCSBR form one EFBRS.

2. Fix a given EFBRS, viz. $Q^a \times Q^b$. Then there exists a type structure so that the set of strategies consistent with RCSBR is exactly $Q^a \times Q^b$. 
Back to Battle of the Sexes

Three EFBRS’s:
• \{Out\} × \{Up\}
• \{Out\} × \{Up, Down\}
• \{Right\} × \{Down\}

Result: Extensive-Form Rationalizable Strategies form one EFBRS.
Questions

1. What is the relationship to other solution concepts?
   Is it “new”?

2. What does EFBRS give in games of applied interest?
   Does it rule out anything?
Close Relatives

1. Extensive-Form Rationalizability (Pearce, 1984)
   • Battigalli-Siniscalchi (2002)
2. F-Rationalizability (Battigalli-Siniscalchi, 2003 and 2007)
3. Self-Admissible Sets (Brandenburger-Friedenberg-Keisler, 2008)
   • Brandenburger-Friedenberg (2004)
Applications

**Question:** What does an EFBRS give in games of interest?

- Some sharp answers
- Use properties (i)-(ii)-(iii) of an EFBRS to get answers
- Relative of a self-admissible set (SAS)
  
  Brandenburger-Friedenberg-Keisler, 2008
- Similar output to SAS
  
  Borrow arguments from Brandenburger-Friedenberg, 2004
Perfect Information Games

**Result:** Fix a perfect information game satisfying “no ties.”

1. Fix an EFBRS, viz. $Q^a \times Q^b$. Each strategy profile in $Q^a \times Q^b$ is outcome equivalent to some pure strategy Nash equilibrium.

2. Fix a Nash Equilibrium in sequentially justifiable strategies. Then there exists an EFBRS, viz. $Q^a \times Q^b$, that contains the Nash equilibrium.
Centipede

Claim: In any EFBRS, Ann plays Out immediately.
A Coordination Example

Three EFBRS's:

- \{Out\} × \{Out\}
- \{Out\} × \{Out, In\}
- \{In-Across\} × \{In\}
A Coordination Example

\[ \beta^a(t^a) \quad \boxed{Out \quad In \quad 1 \quad [1]} \]

\[ \beta^b(t^b) \quad \boxed{Out \quad In-Across \quad In-Down \quad 1 \quad [1]} \]

The diagram shows a coordination example with labeled transitions and states.
The EFBRS Properties

Tell us Something About Applications:

- Finitely repeated prisoner’s dilemma;
- Finitely repeated chain store game;
- …