Evaluating Value-at-Risk models via Quantile Regression

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Recent financial disasters have emphasized the need for accurate risk measures for financial institutions;

Since 1998, regulatory guidelines have required banks to set aside capital to insure against extreme portfolio losses

the size of the set-aside capital requirement is directly related to a measure of portfolio risk.

Currently, portfolio risk is measured in terms of its "value-at-risk"
Value-at-Risk (VaR): a statistical risk measure of potential losses:

\[
\Pr(R_t < \text{VaR}_t \mid \mathcal{F}_{t-1}) = \tau^*
\]

(1)

\[
F(\text{VaR}_t) = \tau^* \\
\text{VaR}_t = F^{-1}(\tau^*) = Q_{R_t}(\tau^* \mid \mathcal{F}_{t-1})
\]

(2)

Under the internal model approach of the Basle Accord on banking, financial institutions have the freedom to specify their own model to compute value at risk!
Value-at-Risk models

Define a "violation" by the hit sequence: 

$$H_t = \begin{cases} 
1 & \text{if } R_t < \text{VaR}_t \\
0 & \text{otherwise}
\end{cases}$$

- Berkowitz et al (2009) extended and unified the existing tests by noting that the de-meaned violations $\text{Hit}_t = H_t - \tau^*$ form a martingale difference sequence (m.d.s.) with respect to $\mathcal{F}_{t-1}$.

- This implies that $\text{Hit}_t = H_t - \tau^*$ is uncorrelated at all leads and lags. In other words, for any vector $X_t$ in $\mathcal{F}_{t-1}$, 

$$E[(H_t - \tau^*) \otimes X_t] = 0$$
Notice that $E[(H_t - \tau^*) \otimes X_t] = 0$ is the basis of GMM estimation.
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Kupiec (1995) sets \( X_t = 1 \) and test \( H_o: p = E(H_t) = \tau^* \).
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This framework offers a natural way to construct a Lagrange Multiplier (LM) test.

Kupiec (1995) sets $X_t = 1$ and test $H_o: E(H_t) = \tau^*$.

Christoffersen (1998) considers $X_t = [1, H_{t-1}]'$ and Engle and Manganelli (2004) set $X_t = [Hit_{t-1}, Hit_{t-2}, ..., VaR_t, VaR_{t-1}...]$.
Some Comments

- The existing Lagrange Multiplier (LM) test has low power in finite samples.
- Berkowitz et al (2009) show that the Dynamic Quantile (DQ) test of Engle and Manganelli (2004) with $X_t = [1, \text{VaR}_t]'$ has the highest power against a variety of misspecified models.
- No guidance to why a given VaR is misspecified.
Main Results

- We show that Quantile regression offers a natural setup for the development of a Wald type of test.
- This Wald test has more power than the DQ test test in finite samples.
- Monte Carlo simulations and an empirical exercise corroborate our findings.
- Our framework is useful to show why a given VaR model is misspecified.
Model

Random coefficient model representation for the time series $R_t$ (Koenker & Xiao, 2002)

\[
R_t = \alpha_0(U_t) + \alpha_1(U_t) \text{VaR}_t
\]

(3)

\[
= x_t' \beta(U_t)
\]

(4)

where $U_t \sim \text{iid } U(0,1)$, $\alpha_i(U_t)$, $i = 0, 1$ are assumed to be comonotonic in $U_t$ and $\beta(U_t) = [\alpha_0(U_t); \alpha_1(U_t)]'$ and $x_t' = [1, \text{VaR}_t]$. 
On Comonotonicity

- Definition: Two random variables $X$, $Y$ are comonotonic if there exists a third random variable $Z$ and increasing functions $f$ and $g$ such that $X = f(Z)$ and $Y = g(Z)$.
- $X$ and $Y$ are driven by the same random (uniform) variable.
- From our point of view the crucial property of comonotonic random variables is the behavior of quantile functions of their sums, $X$, $Y$ comonotonic implies:

$$Q_{X+Y}(\tau) = Q_X(\tau) + Q_Y(\tau).$$
A Wald-type test to evaluate VaRs

**Proposition 1** Given the random coefficient model (3) and the comonotonicity assumption of $\alpha_i(U_t), i = 0, 1$, the $\tau$th conditional quantile of $R_t$ can be written as

$$Q_{R_t}(\tau \mid F_{t-1}) = \alpha_0(\tau) + \alpha_1(\tau) \text{ VaR}_t; \text{ for all } \tau \in (0, 1).$$

- What do we really want to test? $\text{VaR}_t = Q_{R_t}(\tau^* \mid F_{t-1})$
“Mincer-Zarnowitz” (quantile) regression:

\[
Q_{R_t}(\tau^* \mid \mathcal{F}_{t-1}) = \alpha_0(\tau^*) + \alpha_1(\tau^*) \text{VaR}_t
\]  

(2)

\[
H_0 : \begin{cases} 
\alpha_0(\tau^*) = 0 \\
\alpha_1(\tau^*) = 1 
\end{cases}
\]

or \( H_0 : \theta(\tau^*) = 0 \), where \( \theta(\tau^*) = [\alpha_0(\tau^*); (\alpha_1(\tau^*) - 1)]' \)

- \( \theta(\tau^*) \) can be consistently estimated by using Koenker and Bassett (1978)
A Wald-type test to evaluate VaRs

- CLT of Koenker (2005):

**Assumption 1:** Let \( x_t = (1, \text{VaR}_t)' \) be measurable with respect to \( \mathcal{F}_{t-1} \) and \( z_t \equiv \{ R_t; x_t \} \) be a strictly stationary process;

**Assumption 2:** (Density) Let \( \{ R_t \} \) have distribution functions \( F_t \), with continuous Lebesgue densities \( f_t \) uniformly bounded away from 0 and \( \infty \) at the points \( Q_{R_t}(\tau \mid x_t) = F_{R_t}^{-1}(\tau \mid x_t) \);

**Assumption 3:** (Design) There exist positive definite matrices \( J \) and \( H_{\tau^*} \), such that

\[
J = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} x_t x_t'
\]

and

\[
H_{\tau^*} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} x_t x_t'[f_t(Q_{R_t}(\tau^* \mid x_t))]
\]

**Assumption 4:** \( \max_{t=1,\ldots,T} \| x_t \| / \sqrt{T} \to 0. \)
A Wald-type test to evaluate VaRs

$$\sqrt{T}(\hat{\theta}(\tau^*) - \theta(\tau^*)) \overset{d}{\rightarrow} N(0, \tau^*(1 - \tau^*)H_{\tau^*}^{-1} JH_{\tau^*}^{-1}) = N(0, \Lambda_{\tau^*})$$

Definition 1:

$$\zeta_{VQR} = T[\hat{\theta}(\tau^*)' (\tau^*(1 - \tau^*)H_{\tau^*}^{-1} JH_{\tau^*}^{-1})^{-1} \hat{\theta}(\tau^*)]$$

Proposition

(VQR test) Consider the quantile regression (2). Under the null hypothesis, if assumptions (1)-(4) hold, then, the test statistic \( \zeta_{VQR} \) is asymptotically \( \chi^2(2) \).
DQ versus VQR test

- The null hypothesis in the DQ test is based on the orthogonality condition

\[ n^{-1} \sum_{t=1}^{n} \text{Hit}_t [1 \text{ VaR}_t] = 0, \]

In quantile regression we minimize the loss function

\[ R(\beta) = \sum_{t=1}^{n} \rho_\tau (R_t - \alpha_0 (\tau) - \alpha_1 (\tau) \text{ VaR}) \]

- If \( \beta = [\alpha_0 (\tau), \alpha_1 (\tau)]' \) minimizes \( R(\beta) \), then the directional derivative \( \nabla R(\beta, w) \geq 0 \) for all \( w \in \mathbb{R}^2 \) with \( \| w \| = 1 \)
under the null hypothesis, $\beta_0 = [0, 1]'$ the directional derivative of $R(\beta_0)$ becomes

$$\nabla R(\beta_0, w) = - \sum_{t=1}^{n} \left\{ \begin{array}{ll}
Hit_t \cdot [1 \ VaR_t] \cdot w & \text{if } u_t \neq 0 \\
Hit^*_t \cdot [1 \ VaR_t] \cdot w, & \text{if } u_t = 0
\end{array} \right.$$

$$Hit_t = I(R_t - VaR_t < 0) - \tau^* \text{ if } u_t \neq 0$$

$$Hit^*_t = I(x_t^l w < 0) - \tau^* \text{ if } u_t = 0$$

for all $w \in \mathbb{R}^2$ with $\|w\| = 1$.

- the DQ and VQR tests are asymptotically equivalent under the null and under local alternatives

- In finite samples, the GMM estimation and the quantile estimation can be quite different $\Rightarrow$ DQ and VQR can yield quite different results
Monte Carlo

- DGP is a zero mean, unit unconditional variance normal innovation-based GARCH model:
- we consider $\tau^* = 0.01$ and $0.05$. $T = \{250, 500, 1000, 2500\}$ (i.e., approximately 1, 2, 4, and 10 years of daily data).
- We computer the size adjusted power: Historical simulation is the misspecified model
- we consider 5% tests: Kupiec (1995), Christoffersen (1998) the DQ test (2004) in which we considered the instruments $X_t = [1 \; \text{VaR}_t]'$.
- We look at the one-day ahead forecast and simulate 5,000 sample paths of length $T + T_e$ observations, with $T_e = 250$. 
- All four tests have small size distortions for $T = 1000$ and 2500.
- The VQR test is oversized for small sample sizes, $T = 250$ and 500.
- The VQR test is more powerful than any other existing test for any sample size and level of significance $\tau^*$. 
Table 2: Size-adjusted Power of 5% tests

<table>
<thead>
<tr>
<th>Panel A: $\tau^* = 1%$</th>
<th>Sample Size</th>
<th>250</th>
<th>500</th>
<th>1000</th>
<th>2500</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta_{\text{Kupiec}}$</td>
<td>0.059</td>
<td>0.113</td>
<td>0.189</td>
<td>0.322</td>
<td></td>
</tr>
<tr>
<td>$\zeta_{\text{Christ.}}$</td>
<td>0.087</td>
<td>0.137</td>
<td>0.216</td>
<td>0.396</td>
<td></td>
</tr>
<tr>
<td>$\zeta_{DQ}$</td>
<td>0.084</td>
<td>0.171</td>
<td>0.402</td>
<td>0.644</td>
<td></td>
</tr>
<tr>
<td>$\zeta_{VQR}$</td>
<td>0.091</td>
<td>0.174</td>
<td>0.487</td>
<td>0.800</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: $\tau^* = 5%$</th>
<th>Sample Size</th>
<th>250</th>
<th>500</th>
<th>1000</th>
<th>2500</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta_{\text{Kupiec}}$</td>
<td>0.068</td>
<td>0.146</td>
<td>0.267</td>
<td>0.423</td>
<td></td>
</tr>
<tr>
<td>$\zeta_{\text{Christ.}}$</td>
<td>0.142</td>
<td>0.227</td>
<td>0.319</td>
<td>0.509</td>
<td></td>
</tr>
<tr>
<td>$\zeta_{DQ}$</td>
<td>0.179</td>
<td>0.316</td>
<td>0.454</td>
<td>0.769</td>
<td></td>
</tr>
<tr>
<td>$\zeta_{VQR}$</td>
<td>0.215</td>
<td>0.366</td>
<td>0.644</td>
<td>0.883</td>
<td></td>
</tr>
</tbody>
</table>
Figure 4 - S&P500 daily returns

Series: SP500
Sample 1 1000
Observations 1000
Mean 0.000416
Median 0.000788
Maximum 0.028790
Minimum -0.035343
Std. Dev. 0.007140
Skewness -0.253716
Kurtosis 4.553871
Jarque-Bera 111.3334
Probability 0.000000
Local Analysis

Definition 2 \( W_t \equiv \left\{ Q_{U_t}(\tau) = \tau \in [0, 1] \mid \text{VaR}_t = Q_{R_t}(\tau \mid F_{t-1}) \right\} \), representing the empirical quantile of the standard iid uniform random variable, \( U_t \), such that the equality \( \text{VaR}_t = Q_{R_t}(\tau \mid F_{t-1}) \) holds at period \( t \).

- In other words, \( W_t \) is obtained by comparing \( \text{VaR}_t \) with a full range of estimated conditional quantiles evaluated at \( \tau \in [0, 1] \).
- Note that \( W_t \) enables us to conduct a local analysis, whereas the proposed VQR test is designed for a global evaluation based on the whole sample.
- If \( \text{VaR}_t \) is a correctly specified VaR model, then \( W_t \) should be as close as possible to \( \tau^* \) for all \( t \).
- However, if \( \text{VaR}_t \) is misspecified, then it will vague away from \( \tau^* \), suggesting that \( \text{VaR}_t \) does not correctly approximate the \( \tau^* \) th conditional quantile.
### Figure 3 - Local Analysis of VaR Models.

<table>
<thead>
<tr>
<th></th>
<th>HS12m - 1% VaR</th>
<th>GARCH - 1% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>HS12m - 5% VaR</td>
<td><img src="image3" alt="Graph" /></td>
<td><img src="image4" alt="Graph" /></td>
</tr>
<tr>
<td></td>
<td><img src="image5" alt="Graph" /></td>
<td><img src="image6" alt="Graph" /></td>
</tr>
</tbody>
</table>
Table 3: Backtesting Value-at-Risk Models

<table>
<thead>
<tr>
<th>Model</th>
<th>% of Hits</th>
<th>$\zeta_{Kupiec}$</th>
<th>$\zeta_{Christ.}$</th>
<th>$\zeta_{DQ}$</th>
<th>$\zeta_{VQR}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tau^* = 1%$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HS12M</td>
<td>1.6</td>
<td>0.080</td>
<td>0.110</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>GARCH(1,1)</td>
<td>1.1</td>
<td>0.749</td>
<td>0.841</td>
<td>0.185</td>
<td>0.100</td>
</tr>
<tr>
<td>$\tau^* = 5%$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HS12M</td>
<td>5.5</td>
<td>0.466</td>
<td>0.084</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>GARCH(1,1)</td>
<td>4.5</td>
<td>0.470</td>
<td>0.615</td>
<td>0.841</td>
<td>0.042</td>
</tr>
</tbody>
</table>

Notes: P-values are shown in the $\zeta$’s columns
Although the proposed methodology has several appealing properties, it should be viewed as complementary rather than competing with the existing approaches.

Furthermore, several important topics remain for future research, such as:
(i) time aggregation: how to compute and properly evaluate a 10-day regulatory VaR?

(ii) Our randomness approach of VaR also deserves an extended treatment and leaves room for weaker conditions;

(iii) extension of the analysis for the multivariate quantile regression (see Chaudhuri (1996) and Laine (2001));

(iv) inclusion of other variables to increase the power of VQR test in other directions;

(v) improvement of the BIS formula for market required capital;

(vi) nonlinear quantile regressions; among many others.
The End