Credit risk, credit crunch and capital structure

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Motivation

- Predicting corporate default probabilities is instrumental to correctly pricing corporate debt.
- Traditional structural models of default underestimate credit spreads and short-term default probabilities.
- Little attention was paid to the effects of macroeconomic conditions on credit risk and capital structure choices.
- Empirical evidence: macroeconomic conditions influence default probability.
- Market sentiment is “the large missing explanatory variable” in risk management.
- Current crisis: firms may not always be able to borrow.
Objective

Develop a framework for analyzing the impact of

- macroeconomic conditions
- market sentiment
- and credit crunch

on credit risk and optimal capital structure choice.
Structural models of debt

Novelty of the project

Three stochastic factors

- firm specific productivity shocks
- aggregate shocks reflecting the state of the business cycle
- aggregate shocks reflecting the state of the market sentiment

Credit crunch state

Lenders refuse to buy any new debt
Capital structure and coupon payments do not remain constant

- debt principal is chosen optimally at the moment of debt inception
- coupon payment is chosen so that at the moment of debt inception the debt is issued at par
- firm refinances (at a cost) its debt each time macro-economic conditions improve
- if the macro state changes for a worse one, debt holders re-adjust the coupon payments so that the face value equals the debt value; debt profile may also change.
Benchmark model: two stochastic factors

Aggregate uncertainty

- The aggregate shocks to the economy are modeled as the Markov chain $Z = (Z_t)$ with 2 states and the infinitesimal generator

$$
\begin{pmatrix}
-\lambda_{12} & \lambda_{12} \\
\lambda_{21} & -\lambda_{21}
\end{pmatrix},
$$

where state 1 is a boom, and state 2 is a recession; $\lambda_{12} < \lambda_{21}$

- Lenders increase the discount rate $r$ by a margin, $\lambda_j$, $\lambda_1 < \lambda_2$; $q_j = r + \lambda_j$ – lenders’ discount rate in state $j$
Benchmark model

Firm specific uncertainty

- Firm’s after tax profit flow is $\delta_t = G_j e^{X_t^{(j)}}$. For each $j$, $G_j$ is a positive constant and $X^{(j)}$ is a Brownian motion:

$$X_t = X_{t_0}^j = x + \int_0^t \mu(Z_t) dt + \int_0^t \sigma(Z_t) dW_s,$$

where $x$ is the spot value of $X$, $W$ is the standard Wiener process; $\mu(j) \in \mathbb{R}$, $\sigma(j) > 0$ are the drift and volatility of the Brownian motion in state $j$.

- $G_1 > G_2$, $\Psi_1(1) > \Psi_2(1)$, $\Psi_1(1) - \lambda_1 > \Psi_2(1) - \lambda_2$

$$\Psi_j(z) = \frac{1}{2} \sigma(j)^2 z^2 + \mu(j) z \quad j = 1, 2$$
Debt structure

- $P$ – the debt of principal $P$ issued in state $j$; remains constant as long as the firm is solvent and the state does not change
- $C_j = \rho_j P$ – coupon payment in state $j$;
- Firm continuously rolls over a fraction $m_j$ of the debt and replaces it with the new debt of equal coupon, principal and seniority
- If the macro-state improves, the firm refinances all the debt outstanding
- If the macro-state worsens, competitive lenders request debt-renegotiation.
Lenders’ valuation

Value of unlevered firm

\[ v_j^{/0}(x) = c_j^{/0} e^x, \text{ where} \]

\[ c^{/0} = \frac{1}{\det A_l} \left[ \begin{array}{cc} q_2 + \lambda_{21} - \psi_2(1) & \lambda_{12} \\ \lambda_{21} & q_1 + \lambda_{12} - \psi_1(1) \end{array} \right] \cdot \left[ \begin{array}{c} G_1 \\ G_2 \end{array} \right], \]

\[ \det A_l = (q_1 + \lambda_{12} - \psi_1(1))(q_2 + \lambda_{21} - \psi_2(1)) - \lambda_{12} \lambda_{21} \]
Lenders’ valuation

Value of a bond with face value 1 and maturity $t$

Bond was issued in state $j$ at $X^j_{in} = x$; $y = X^j_0$ is the spot value

$$d^1_j(x, y, t) = \mathbb{E}^y \left[ \int_0^{\tau_j \wedge t} (\rho_j + \pi_j) e^{-(q_j + \Lambda_j)s} ds \right] + \mathbb{E}^y \left[ e^{-(q_j + \Lambda_j)t} \mathbf{1}_{t < \tau_j} \right]$$

$$+ \frac{\alpha_j}{P} \mathbb{E}^y \left[ e^{-(q_j + \Lambda_j)\tau_j} v^j_{\tau_j} (X^{(j)}_{\tau_j}) \mathbf{1}_{\tau_j < t} \right]$$

$\Lambda_1 = \lambda_{12}, \Lambda_2 = \lambda_{21}$

$\tau_j = \inf\{ t' \geq 0 | X^{j}_{t'} \leq h_j \}$

$h_j = h_j(x)$ – default threshold

$\alpha_j$ – recovery rate

$\pi_j = \pi^0_j \Lambda_j$, $\pi^0_1 = 0$, $\pi^0_2 = \pi \geq 0$ – prepayment penalty
Lenders’ valuation

Value of currently issued debt

\[ \phi_j(t) = m_j e^{-m_j t} \quad \text{maturity profile in state } j; \ 1/m_j \quad \text{average maturity} \]

\[ d_j(x, y) = \int_0^\infty \phi_j(t) d^1_j(x, h_j; y, t) dt \]

\[ = p_j \mathbb{E}^{y} \left[ \int_0^{\tau_j} e^{-(q_j + \Lambda_j + m_j) t} (\rho_j + m_j + \pi_j) dt \right] \]

\[ + m_j \alpha_j \mathbb{E}^{y} \left[ e^{-(q_j + \Lambda_j + m_j) \tau_j} v_j^{l0}(X_{\tau_j}^{(j)}) \right]. \]

Value of total debt outstanding

\[ \nu_j^d(x, y) = \int_{-\infty}^{0} e^{m_j t} d_j(x, h_j; y) dt = \frac{1}{m_j} d_j(x, y) \]

\[ \nu_j^d(x, x) = P = d_j(x, x)/m_j \]
\( \beta_{l,j}^{\pm} \) – positive and negative roots of the characteristic equation

\[
q_j + \Lambda_j + m_j - \Psi_j(\beta) = 0,
\]

\[
E_{l,j}^{+,u}(y) = \beta_{l,j}^{+} \int_{0}^{+\infty} e^{-\beta_{l,j}^{+}z} u(y + z) dz,
\]

\[
E_{l,j}^{-,u}(y) = -\beta_{l,j}^{-} \int_{-\infty}^{0} e^{-\beta_{l,j}^{-}z} u(y + z) dz
\]

\[
E_{l,j}^{\pm} e^{zy} = \kappa_{l,j}^{\pm}(z) e^{zy} \quad \kappa_{l,j}^{\pm} = \frac{\beta_{l,j}^{\pm}}{\beta_{l,j}^{\pm} - z}
\]
Debt value and endogenous interest rate

Debt value

\[ v^d_j(x, y) = \frac{(\rho_j + m_j + \pi_j)m_j P}{q_j + \Lambda_j + m_j} \mathcal{E}_{l,j}^-(1_{(h_j, +\infty)}(y)) + \frac{\alpha_j c_j^l 0}{\kappa_{l,j}^-}(1) \mathcal{E}_{l,j}^-(1_{(-\infty, h_j)} e^y) \]

\[ v^d_j(x, y) = \frac{(\rho_j + m_j + \pi_j)m_j P}{q_j + \Lambda_j + m_j} (1 - e^{\beta_{l,j}^- (y-h_j)}) + \alpha_j c_j^l 0 e^{h_j} e^{\beta_{l,j}^- (y-h_j)} \]

No-arbitrage condition \( P = v^d_j(x, y) \) implies

\[ \rho_j = \rho_j(x) = -\pi_j - m_j + \frac{q_j + \Lambda_j + m_j}{P (1 - e^{\beta_{l,j}^- (x-h_j)})} \left( P - \alpha_j c_j^l 0 e^{h_j} e^{\beta_{l,j}^- (x-h_j)} \right) \]
Equity value and endogenous default threshold

**Equity value**

\[
V_1(x, y) = \mathbb{E}^y \left[ \int_0^{\tau_1(x)} e^{-(r+\Lambda_1)t} \left( G_1 e^{X_t^{(1)}} - (\rho_1(x)(1 - \tau) + m_1)P 
+ d_1(x, y) + \lambda_{12} V_2(X_t^{(2)}, X_t^{(2)}, x) \right) dt \right]
\]

\[
V_2(x, y, z) = \mathbb{E}^y \left[ \int_0^{\tau_2(x)} e^{-(r+\Lambda_2)t} \left( G_2 e^{X_t^{(2)}} - (\rho_2(x)(1 - \tau) + m_2 + \pi_2)P(z) + d_2(x, y) + \lambda_{21} V_1(X_t^{(2)}, X_t^{(2)}) \right) dt \right]
\]

\( \tau \) – tax advantage to debt
Equity value and endogenous default threshold

\( \beta_{f,j}^\pm \) – positive and negative roots of the characteristic equation

\[ r + \Lambda_j - \Psi_j(\beta) = 0, \]

\[ E^+_{f,j} u(y) = \beta_{f,j}^+ \int_0^{+\infty} e^{-\beta_{f,j}^+ z} u(y + z) dz, \]

\[ E^-_{l,j} u(y) = -\beta_{f,j}^- \int_{-\infty}^0 e^{-\beta_{f,j}^- z} u(y + z) dz \]

\[ E^\pm_{f,j} e^{zy} = \kappa_{f,j}^\pm(z) e^{zy} \quad \kappa_{f,j}^\pm = \frac{\beta_{f,j}^\pm}{\beta_{f,j}^\pm - z} \]
Equity value and endogenous default threshold

\[ g_1(x, y) = G_1 e^y - (\rho_1(x)(1 - \tau) + m_1)P + d_1(x, y) + \lambda_{12} V_2(y, y, x) \]

\[ g_2(x, y, z) = G_2 e^y - (\rho_2(x)(1 - \tau) + m_2 + \pi_2)P(z) + d_2(x, y) + \lambda_{21} V_1(y, y) \]

\[ V_1(x, y) = (r + \Lambda_1)^{-1} \left( \mathcal{E}_{\mathcal{F},1}^-(h_1(x), +\infty)(\cdot)w_1(x, \cdot) \right)(y) \]

\[ V_2(x, y, z) = (r + \Lambda_2)^{-1} \left( \mathcal{E}_{\mathcal{F},2}^-(h_2(x, z), +\infty)(\cdot)w_2(x, \cdot, z) \right)(y) \]

\[ w_1(x, y) = \mathcal{E}_{\mathcal{F}_1}^+ g_1(x, y) \quad w_2(x, y, z) = \mathcal{E}_{\mathcal{F}_2}^+ g_2(x, y, z) \]

\[ w_1(x, h_1) = 0 \quad w_2(x, h_2, z) = 0 \]
Optimal capital structure

The optimal principal $P$ is chosen so as to maximize

$$V_1(x, x) + P$$
Iteration procedure

Step 1

Find $V_1^0(x, y), h_1^0(x), P^0(x)$ s.t.

- $w_1^0(x, h_1^0) = 0, \ w_1^0(x, y) = E_{f_1}^+ g_1^0(x, y)$

$$g_1^0(x, y) = G_1 e^y - (\rho_1^0(x)(1 - \tau) + m_1)P^0 + d_1^0(x, y)$$

- $V_1^0(x, y) = (r + \Lambda_1)^{-1} \left( E_{f_1}^- 1_{(h_1^0(x), +\infty)}(\cdot) w_1^0(x, \cdot) \right)(y)$

- $P^0 = \text{argmax } V_1^0(x, x) + P^0$

- $\rho_1^0 = \rho_1^0(x)$ is defined from $P^0 = d_1^0(x, x)/m_1$

**Important observation** $h_1^0(x) = x + h_1^0(0), \ P^0(x) = P^0(0)e^x, \ V_1^0(x, x) = V_1^0(0, 0)e^x, \ \rho_1^0$ is independent of $x.$
Iteration procedure

Step 2

Find $V^0_2(x, y, z)$, $h^0_2(x, z)$ s.t.

- $w^0_2(x, h^0_2, z) = 0$, $w^0_2(x, y, z) = \mathcal{E}^+_{f_2} g^0_2(x, y, z)$

- $g^0_2(x, y, z) = G_2 e^y - (\rho^0_2(x)(1-\tau)+m_1) P^0_0(z) + d^0_2(x, y) + \lambda_{21} V^0_1(y, y)$

- $V^0_2(x, y, z) = (r + \Lambda_2)^{-1} \left( \mathcal{E}^-_{f,2} \mathbf{1}(h^0_2(x, z), +\infty)(\cdot) w^0_2(x, \cdot, z) \right) (y)$

- $\rho^0_2 = \rho^0_2(x, z)$ is defined from $P^0_0(z) = d^0_2(x, x)/m_2$
Iteration procedure

For \( n=1,2, \ldots \)

Find \( V_1^n(x, y), h_1^n(x), P^n(x) \) s.t.

- \( w_1^n(x, h_1^n) = 0, \ w_1^n(x, y) = \mathcal{E}^+_f g_1^n(x, y) \)
- \( g_1^n(x, y) = G_1 e^y - (\rho_1^n(x)(1-\tau) + m_1)P^n + d_1^n(x, y) + \lambda_12 V_2^{n-1}(y, y, x) \)
- \( V_1^n(x, y) = (r + \Lambda_1)^{-1} \left( \mathcal{E}^- f, 1_{(h_1^n(x), +\infty)}(\cdot)w_1^n(x, \cdot) \right)(y) \)
- \( P^n = \text{argmax } V_1^n(x, x) + P^n \)
- \( \rho_1^n = \rho_1^n(x) \) is defined from \( P^n = d_1^n(x, x)/m_1 \)
Iteration procedure

**For n=1,2,...**

Find $V_n^2(x, y, z), h_n^2(x, z)$ s.t.

- $w_n^2(x, h_n^2, z) = 0, \ w_n^2(x, y, z) = \mathcal{E}_{f2} g_n^2(x, y, z)$

$$g_n^2(x, y, z) = G_2 e^y - (\rho_n^2(x)(1-\tau) + m_1) P_n^2(z) + d_n^2(x, y) + \lambda_2 V_1^{n-1}(y, y)$$

- $V_n^2(x, y, z) = (r + \Lambda_2)^{-1} \left( \mathcal{E}^-_{f,2} 1_{(h_n^2(x, z), +\infty)}(\cdot) w_n^2(x, \cdot, z) \right) (y)$

- $\rho_n^2 = \rho_n^2(x, z)$ is defined from $P_n^2(z) = d_n^2(x, x) / m_2$
Parameter values

- Aggregate uncertainty: $\lambda_{12} = 0.1, \lambda_{21} = 0.33$
- Discount rates: $r = 0.055, \lambda_1 = 0.01, \lambda_2 = 0.015$
- Average debt maturity: 5 years in booms, 3 years in recessions; $m_1 = 1/5, m_2 = 1/3$
- Idiosyncratic uncertainty:
  - $\sigma_1 = 0.25, \sigma_2 = 0.2, \mu_1 = -0.0262, \mu_2 = -0.0170$
  - (this choice of $\mu_{1,2}$ corresponds to cash flows growth rates of 5% in booms and 3% in recessions)
- Firm’s productivity: $G_1 = 2, G_2 = 1$
- Recovery rates: $\alpha_1 = \alpha_2 = 0.5$
- Prepayment penalty: $\pi = 0.001$
Default thresholds in state 1: two-vs. one-regime models
Main conclusion so far

- In a one regime model, a firm of any size issues debt proportional to its size.
- If a firm faces a possibility of adverse changes in debt contract, the firm has to be sufficiently large in order to take on any debt.
Optimal debt: two-vs. one-regime models

![Graph of optimal debt values](image)

- $P_0(x)$
- $P_1(x)$
Optimal coupon in state 1: two-vs. one-regime models

Optimal coupon

$C_0(x)$, $C_1(x)$
Future work

- Calibrate the model
- Models with irrational market sentiment (secondary market trends)
  three possible states: boom, recession, bear rally in recession
- Models with credit crunch state