Why Risk Sharing?

- A fundamental question in macroeconomics and finance
  - value created by financial markets
- Welfare theorems
  - better understanding of competitive equilibrium
  - aggregation and asset-pricing models
- Similar structure in other settings
  - dynamic contracts
  - endogenous incomplete markets
- Micro consumption/income empirical puzzles
Recursive Utility

Agent 1:

\[ W_t = \left[ (1 - \beta)(c_t^1)^{\rho_1} + \beta^1 \mu_t^1(W_{t+1})^{\rho_1} \right]^{\frac{1}{\rho_1}} \]

\[ \mu_t^1(W_{t+1}) = \left[ E_t W_{t+1}^{\alpha_1} \right]^{\frac{1}{\alpha_1}} \]

Agent 2:

\[ V_t = \left[ (1 - \beta)(c_t^2)^{\rho_2} + \beta^2 \mu_t^2(W_{t+1})^{\rho_2} \right]^{\frac{1}{\rho_2}} \]

\[ \mu_t^2(V_{t+1}) = \left[ E_t V_{t+1}^{\alpha_2} \right]^{\frac{1}{\alpha_2}} \]
Standard Pareto Problem

\[
\max \{c^1_t, c^2_t\} \lambda W_t + (1 - \lambda)V_t
\]

s.t. \( c^1_t + c^2_t = y_t \) at all dates and states

- Note that the social planner’s objective function is **not** recursive
  - Expected Utility is an exception
- Can’t use Dynamic Programming or Bellman’s equation
Recursive Pareto Problem

\[ J(y, V) = \max_{c, \{V'\}} \left[ (1 - \beta)(y - c)^{\rho_1} + \beta \mu_1 (J')^{\rho_1} \right]^{\frac{1}{\rho_1}} \]

s.t. \[ V \geq \left[ (1 - \beta)c^{\rho_2} + \beta \mu_2 (V')^{\rho_2} \right]^{\frac{1}{\rho_2}} \]

▶ FOC:

\[ J^{1 - \rho_1} (y - c)^{\rho_1 - 1} = \lambda V^{1 - \rho_2} c^{\rho_2 - 1} \]

\[ J^{1 - \rho_1} (\mu_1^{\rho_1 - \alpha_1} (J')^{\alpha_1 - 1} J'_V = \lambda V^{1 - \rho_2} (\mu_2^{\rho_2 - \alpha_2} (V')^{\alpha_2 - 1}) \]

\[ J_V = -\lambda \]
Consider agent with extreme differences in $\alpha$ and $\rho$

- since that is what differentiates these preferences from Expected Utility

Consider two agents with extreme heterogeneity in these differences

- since that maximizes the scope for Pareto improvements

Agent 1: $\alpha_1 = 1$ (risk neutral) and $\rho_1 = -\infty$ (perfectly inelastic deterministic substitution)

\[ W = \min\{(y - c), EW'\} \]

Agent 2: $\alpha_2 = -\infty$ (extreme risk aversion) and $\rho_2 = 1$ (perfectly elastic deterministic substitution)

\[ V = (1 - \beta)c + \beta \min\{V'\} \]
Recursive Pareto problem

\[ J(y, V) = \max_{c, V'} \min \{ y - c, EJ(y', V') \} \]

s.t. \[ V = (1 - \beta)c + \beta \min \{ V' \} \]

After you work out a few recursions, you realize that the value function is linear, so we guess

\[ J(y, V) = p_0 + p_y y - p_v V \]

for parameters \((p_0, p_y, p_v)\) to be determined
Since agent 2 has infinite risk aversion, \( V' = \bar{V}' \) (constant)

The promise constraint then gives us \( c \) and \( y - c \):

\[
c = \left( \frac{1}{1 - \beta} \right) V - \left( \frac{\beta}{1 - \beta} \right) \bar{V}'
\]

\[
y - c = y - \left( \frac{1}{1 - \beta} \right) V + \left( \frac{\beta}{1 - \beta} \right) \bar{V}'
\]

The optimization is therefore

\[
\max_{\bar{V}'} \min \{ y - \left( \frac{1}{1 - \beta} \right) V + \left( \frac{\beta}{1 - \beta} \right) \bar{V}', p_0 + p_y Ey - p_v \bar{V}' \}
\]

The max comes where the two terms are equal:

\[
\bar{V}' \left[ p_v + \frac{\beta}{1 - \beta} \right] = p_0 + p_y Ey - y + \frac{V}{1 - \beta}
\]
Substitute into the Bellman equation and equate terms:

\[ p_0 = \beta E_y, \quad p_y = 1 - \beta, \quad p_v = 1 \]

The Pareto frontier is

\[ W = J(y, V) = (1 - \beta)E_y + \beta y - V \]

The controlled decision rules and laws of motion include

\[ y - c = Ey - V + (1 - \beta)(y - Ey) \]
\[ W' = (y - c) + (1 - \beta)(y' - Ey) \]
\[ c = V + \beta(y - Ey) \]
\[ \bar{V}' = c - (y - Ey) \]
What’s going on?

- If there’s a shock to $y$, agent 1 consumes a fraction $(1 - \beta)$ of it and agent 2 consumes a fraction $\beta$.
- In agent 1’s case, current consumption and expected future utility go up in by the same amount (Leontief over time).
- In agent 2’s case, an increase in $y$ leads to an increase in current consumption and a fall in future utility with the same discounted value (Linear over time).
- Who bears (most of) the risk? The infinitely risk averse agent!!!
Why?

- Distinction between consumption risk and utility risk
- With recursive utility, “risk aversion” (ie, $\alpha$) is a statement about one-step-ahead utility lotteries
- “Resolution of uncertainty” (ie, $\alpha - \rho$) is a statement about consumption lotteries
- In the LL example, agent 2 wants his utility next period to be perfectly predictable, but doesn’t care how much consumption has to fluctuate over time to deliver that
- Agent 1 doesn’t care at all about how much utility fluctuates period to period, provided future consumption is perfectly predictable
Exact Aggregation

- What do prices look like in this economy? Is there a representative agent?

- Answer: the pricing kernel is a constant, so the economy looks like one with a representative agent with linear utility: 
  \[ \rho = \alpha = 1 \]

- Pricing kernel is

  \[
  m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho - 1} \left( \frac{U_{t+1}}{\mu_t(U_{t+1})} \right)^{\alpha - \rho}
  \]

Note: at an optimum, both agents have the same pricing kernel

- Consider agent 2:
  - The first term in brackets is 1 because \( \rho - 1 = 0 \)
  - The second term is 1 because \( V' = \bar{V}' \), so that \( \mu(V') = V' \)
  - So the pricing kernel is simply \( \beta \)
Now turn to agent 1: With $\alpha = 1$, we can combine terms

$$m_{t+1} = \beta \left( \frac{(y_{t+1} - c_{t+1})E_t W_{t+1}}{(y_t - c_t) W_{t+1}} \right)^{\rho^{-1}}.$$

The min time aggregator sets $y_t - c_t = E_t W_{t+1}$ so those two terms cancel.

The Bellman equation sets $W_{t+1} = y_{t+1} - c_{t+1}$ so those two terms cancel, too.

We’re left with

$$m_{t+1} = \beta.$$
Decentralized Equilibrium (Lucas)

- Equity and 1-Period Bond Prices:

\[ P_t^s = E_t \sum_{j=1}^{\infty} \left[ \prod_{i=1}^{j} m_{t+i} \right] y_{t+j} \]

\[ = \sum_{j=1}^{\infty} \beta^j E_t y_{t+j} \]

\[ = \left( \frac{\beta}{1 - \beta} \right) \bar{y} \]

\[ P_t^b = E_t m_{t+1} \]

\[ = \beta \]
Portfolio Choice

- Date-\(t\) budget constraint:

\[
\theta_{t-1}^s(P_t^s + y_t) + \theta_{t-1}^b = c_t + \theta_t^s P_t^s + \theta_t^b P_t^b,
\]

where \(\theta_t^s\) and \(\theta_t^b\) investments in equity and one-period bonds respectively.

- The budget constraint with equilibrium prices:

\[
\theta_{t-1}^s \left[ \left( \frac{1}{1-\beta} \right) \bar{y} + \varepsilon_t \right] + \theta_{t-1}^b = c_t + \theta_t^s \left( \frac{\beta}{1-\beta} \right) \bar{y} + \theta_t^b \beta.
\]
Agent 2’s Optimal Portfolio

- Conjecture a solution for equity holdings of $\theta_s^t = \beta$ constant for all $t$
- Optimal consumption given by

$$c_t = V_0 + (\beta - 1) \sum_{j=1}^{t} \varepsilon_{t-j} + \beta \varepsilon_t,$$

- Implies bond holdings solve the difference equation:

$$\theta_t^b = \kappa_0 + \kappa_1 \theta_{t-1}^b + \eta_t,$$

where $\kappa_0 = \bar{y} - V_0 / \beta$, $\kappa_1 = 1 / \beta$ and $\eta_t - \eta_{t-1} = \left(\frac{1-\beta}{\beta}\right) \varepsilon_{t-1}$

- Agent 1 holds $-\theta_t^b$ bonds and $(1 - \beta)$ stocks
- Infinitely risk averse agent holds (almost) all the risky asset!
L-L with a persistent endowment

- Suppose $y$ is AR(1) with parameter $\varphi$

\[ Ey' = (1 - \varphi)\bar{y} + \varphi y \]

where $E$ is the one-period conditional expectation and $\bar{y}$ is the unconditional mean

- The structure of the problem implies that no other features of the process matter (the conditional variance, for example)

- Follow the same steps, which leads to value function parameters

\[ p_0 = \frac{\beta(1 - \varphi)}{1 - \beta \varphi} \bar{y}, \quad p_y = \frac{(1 - \beta)}{1 - \beta \varphi}, \quad p_v = 1 \]
The controlled decision rules and laws of motion include

\[ c = V + \left( \frac{\beta(1 - \varphi)}{1 - \beta \varphi} \right) (y - \bar{y}) \]

\[ \bar{V}' = V + \left( \frac{1 - \beta}{1 - \beta \varphi} \right) [(1 - \varphi)\bar{y} + \varphi y] \]

\[ y - c = \bar{y} - V + \left( \frac{1 - \beta}{1 - \beta \varphi} \right) (y - \bar{y}) \]

\[ W' = \left( \frac{(1 - \beta)\varphi}{1 - \beta \varphi} \right) [(y' - \bar{y}) - \varphi(y - \bar{y})] - V \]

Note the impact of persistence: If we increase \( \varphi \), then an increase in \( y \) leads to a smaller increase in \( c \) and a larger increase in \( y - c \) than we had with \( \varphi = 0 \).
Robust/Risk-Sensitive Control Example

- Modify agent 1’s preferences

\[
J(y, V) = \max_{c, V'} \min \{y - c, A^{-1} \log(Ee^{AJ(y', V')})\}
\]

s.t. \[V = (1 - \beta)c + \beta \min \{V'\}\]
\[y \sim N(Ey, \sigma^2)\]

where \(A < 0\)

- Follow the same steps to obtain the solution

\[
p_v = 1, \quad p_y = 1 - \beta, \quad p_0 = \beta Ey + A\beta(1 - \beta)\sigma^2/2
\]

- Note: only adjusts the intercept

- Persistence in \(y\) will work the same way as before

- Square-root volatility in \(y\) will also change \(p_y\) (and make it \(\alpha\) sensitive), but will leave the \(p_v = 1\) result unchanged
Conclusions?