A Consistent Nonparametric Test of Affiliation in Auction Models

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What is this paper about?

- We propose a new test of affiliation.
- The test is nonparametric; it does not need any ex ante distributional assumptions.
- The test is consistent against all alternatives to affiliation.
Affiliation is the most important assumption in structural auction models.

- Milgrom and Weber (1982) showed the existence of a pure strategy monotone equilibrium in a variety of auction settings.
- One of their main assumptions is that signals/values are affiliated.
- However, de Castro (2007) showed that a pure strategy monotone equilibrium may not exist if the affiliation assumption is violated.
Affiliation is a strong form of positive dependence.

For its importance, affiliation is not routinely tested in empirical work.

It is useful to have a diagnostic test that can be applied on the raw data before any modeling assumptions are made.
Testing affiliation has many potential uses.

- Testing affiliation of signals/values: If the bid function is monotonic, then affiliation of signals/values implies that of bids.

- Testing IPV if $n$ is exogenous: If $n$ is exogenous in the IPV setup, then bids will be affiliated with $n$.

- Testing the presence of a certain type of collusion: In the IPV setup, if ring members other than the designated winner do not submit bids, then bids will not affiliated with $n$. 
What is affiliation?

- The figure below illustrates the notion of affiliation.
  
  For any boxes, $P(A)P(B) - P(C)P(D) \leq 0$. 

It is strictly stronger than many other forms of positive dependence (e.g. de Castro (2007)).

An equivalent definition can also be found in Milgrom and Weber (1982).
A nonparametric test of affiliation

For any \( a, b, \) and a positive \( \delta, \)

\[
Q(a, b, \delta) = P(\xi \in B(a, \delta))P(\xi \in B(b, \delta)) - P(\xi \in B(a \land b, \delta))P(\xi \in B(a \lor b, \delta)),
\]

where \( B(a, \delta) \) is a box with volume \( \delta \) and centroid \( a. \) \( a \land b \) (\( a \lor b \)) denotes the element-wise minimum (maximum).
Then, under the null of affiliation, $Q(a, b, \delta)$ is non–positive for any $a, b,$ and $\delta$.

Therefore, we consider

$$T(Q) = \int \max(Q(a, b, \delta), 0) dW(a, b, \delta),$$

where $W(a, b, \delta)$ is some continuous nonnegative weighting function.

Note that $T(Q) = 0$ under the null and that $T(Q) > 0$ under the alternative.
max(·, 0) has a kink, which causes some difficulty.

- A nonstandard, non-pivotal distribution of $T(\hat{Q})$.

- Instead, we introduce a sample size dependent sequence $\beta_n \prec 1$ and use

$$T_n(Q) = \int_{Q(a, b, \delta) > -\beta_n} Q(a, b, \delta) dW(a, b, \delta).$$
The test statistic

Suppose that we have a set of iid data, \( \{\xi_1, \cdots, \xi_n\} \).

Let

\[
h_{ij}^*(a, b, \delta) = I(\xi_i \in B(a, \delta))I(\xi_j \in B(b, \delta)) - I(\xi_j \in B(a \land b, \delta))I(\xi_i \in B(a \lor b, \delta)),
\]

where \( I(\cdot) \) is the standard indicator function.
We then estimate $Q(a, b, \delta)$ by

$$\hat{Q}(a, b, \delta) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} h_{ij}^*(a, b, \delta).$$

Our statistic is defined by

$$\hat{\tau} = \frac{\sqrt{n} T_n(\hat{Q})}{2\hat{\nu}},$$

where $\hat{\nu}^2$ is an estimator for $\nu^2 = \text{var} \left( E(h_{12}|\xi_1) \right)$ with

$$h_{12} = \int_{Q=0} \frac{(h_{12}^* + h_{21}^*)}{2} dW.$$
Asymptotic validity

- Using U-procесс theory (e.g. Nolan and Pollard (1987)), we show the following theorem:

**Theorem:** If $n^{-1/(2+2\gamma)} < \beta_n < 1$ for some $0 < \gamma < \infty$ that satisfies a technical condition, we have

$$\limsup_{n \to \infty} P(\hat{\tau} > C_\alpha) \leq \alpha$$

under the null of affiliation, where $C_\alpha$ is the $1 - \alpha$ quantile from the standard normal distribution.
Some comments on the validity

- In the case of independence, the rejection probability is equal to $\alpha$.
- Otherwise, the test is conservative.
- Being conservative is natural, though.
Consistency

**Theorem:** For any $C < \infty$, we have

$$P(\hat{r} > C) \rightarrow 1$$

under any alternative to affiliation.
Some comments on the consistency

- If the elements of $\xi_i$ are strictly affiliated, then the test will be conservative.
- Therefore, if $\xi_i$ is strictly affiliated on large parts of the support while affiliation is violated only in a small area, the test is likely to have little power in samples of moderate size, although the null of affiliation will be rejected in the limit.
Simulations: Size and the choice of $\beta_n$

- Under independence and under positive dependence:

![Graph showing the size of the test statistic under different conditions]
Simulations: Power and the choice of $\beta_n$

- Under negative dependence:
Simulations: Further power experiments

- DGP1: $\xi_{i1}$ and $\xi_{i2}$ are generated by

\[
\Phi(u_{i1})/2, \quad \Phi(u_{i2})/2 \quad \text{w.p. 0.25}
\]

independent $U(0, 0.5), \ U(0, 0.5) \quad \text{w.p. 0.25}$

independent $U(0.5, 1), \ U(0, 0.5) \quad \text{w.p. 0.25}$

independent $U(0.5, 1), \ U(0.5, 1) \quad \text{w.p. 0.25},$

where $u_{i1}$ and $u_{i2}$ are drawn from a joint normal with mean zero, unit variance with correlation $\rho$.

- $\rho = -0.8$ corresponds to a correlation of $-0.04$ between $\xi_{i1}$ and $\xi_{i2}$. 
With $\beta_n = 0.02/\sqrt[3]{n}$,

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$n = 300$</th>
<th>$n = 500$</th>
<th>$n = 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.6$</td>
<td>0.015</td>
<td>0.129</td>
<td>0.210</td>
</tr>
<tr>
<td>$-0.7$</td>
<td>0.133</td>
<td>0.184</td>
<td>0.353</td>
</tr>
<tr>
<td>$-0.8$</td>
<td>0.167</td>
<td>0.254</td>
<td>0.684</td>
</tr>
</tbody>
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We conducted similar experiments with $d = 3$, but all the results were similar.
Applications

- OCS (Offshore Continental Shelf) and RFSRMA (Russian Federal Suboil Resources Management Agency): the right to drill for oil and gas.
- Caltrans (CA Department of Transportation):Procurements on construction projects.
Overall results

- We could not reject affiliation in most of the cases. (The test statistics were largely negative.)
  - Affiliation between two randomly chosen bids with and without conditioning on other variables such as ex post revenue: Not rejected the null.
  - Affiliation between a randomly chosen bid and the number of bidders: Not rejected the null.
  - Affiliation between the lowest bid and the number of bidders: Rejected the null in the OCS data. (Not surprising.)
Discussions

- It is surprising that affiliation could not be rejected in most of the cases.

- In the framework of CV or APV, bids are not affiliated with the number of bidders, if the number of bidders is exogenous. The examples of OCS and RFSRMA indicates that the number of bidders is likely to be endogenous.

- It seems that modelling the entry decision is quite important.