Cake Eating, Exhaustible Resource Extraction, Life-cycle Saving, and Non-atomic Games: Existence Theorems for a Class of Optimal Allocation Problems

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I. Introduction

- This paper studies the problem concerning the existence of a solution to a popular and diverse class of optimal allocation models, which have the following common underlying structure:

\[
\max_{c(t) \in \Phi} \int_{0}^{1} \alpha(t) f(t) g(c(t)) \, dt \tag{1}
\]

subject to

\[
c(t) \geq 0, \tag{2}
\]

\[
S(t) \geq 0, \tag{3}
\]

\[
S'(t) = j(t) S(t) + m(t) - f(t) c(t), \tag{4}
\]

and

\[
S(0) = S_0, \tag{5}
\]
where $c, S, \alpha, j, m, f : [0, 1] \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}, S_0 \in [0, \infty)$, $\Phi$ denotes the space of piecewise continuous functions, and $\mathbb{R}$ denotes the real line.

- The model (1) - (5) is an optimal control problem with a control constraint and a state constraint (or state variable inequality constraint).

- The range of $t$ is taken to be $[0, 1]$, but it can be any bounded subset of the nonnegative real line.

- A wide variety of economic models can be formulated in the form of (1) - (5), e.g., exhaustible resource extraction (Hotelling (1931)), cake-eating (Gale (1967)), life-cycle saving (Yaari (1965)), and non-atomic games (Aumann and Shapley (1974)).

- The formulation (1) - (5) provides a new and convenient way to encompass all these diverse models.

- Karlin (1959, pp.210-214) was the first to study the existence problem for a special case of (1) - (5) in which $j(t) = 0$, $m(t) = 0$, and $f(t) = 1$ are assumed. In this case, the optimization problem (1) - (5) becomes a calculus of variations problem.
• Yaari (1964) advances Karlin’s (1959) analysis and provides an interesting example to show that the variational problem may not have a solution even when \( g(.) \) is strictly concave and the other functions are smooth and well defined.

• Yaari’s (1964) example is counter-intuitive because it means that there is no optimal way to allocate a given amount of resources to maximize a well-defined objective in a simple and reasonable setting.

• Perhaps even more puzzling is that a solution to the example will exist if \( S_0 \) is sufficiently small. In other words, there is no optimal way to allocate the endowment \( S_0 \) if it is sufficiently large.

• Since Yaari’s (1964) work, a number of follow-up and refinement studies have appeared in both the economics and mathematics literatures, e.g., Aumann and Perles (1965), Kumar (1969), Abrham (1970), Artstein (1974, 1980), and Ioffe (2006). These studies offer a variety of sufficient conditions to guarantee the existence of a solution to the variational problem.
• While previous studies have focused on the special case where \( j(t) = 0, m(t) = 0, \) and \( f(t) = 1, \) the existence problem for the general model (1) - (5) has not yet been studied in the literature.

• The objective of this paper is to develop sufficient conditions for the existence of a solution to the general model (1) - (5) under the assumption that \( \alpha(1) = 0. \) As will be explained later, \( \alpha(1) = 0 \) is a special but not arduous restriction.

• When the condition holds, a precise existence result can be obtained. The solution to (1) - (5), if it exists, possesses a distinctive feature which can be utilized to generate a simple sufficient condition that guarantees the existence of a solution to the optimal allocation problem.

• When applied to the special case where \( j(t) = 0, m(t) = 0, \) and \( f(t) = 1, \) the sufficient condition is substantially simpler than the existing ones in the literature.

• The existence results reveal why (1) - (5) may not have a solution and whether the existence problem depends on the presence of \( j(t), m(t), \) and \( f(t). \)
• In addition, the analysis provides a complete solution to the puzzle raised by Yaari’s (1964) counter-intuitive example.

• While there are many general existence theorems for optimal control problems in the literature (e.g., Cesari (1983)), the relatively simple and explicit structure of this class of optimal allocation problems commensurately deserves a simple and direct existence result.

• The usefulness of the existence theorems will be illustrated by means of a number of different examples.

II. Assumptions and Examples

• A1. \( \alpha(t), m(t), j(t), \) and \( f(t) \) are continuously differentiable, \( m(t) \geq 0, j(t) \geq 0, f(t) > 0, \) \( \alpha(t) \geq 0, \alpha'(t) \leq 0, \alpha(0) = 1, \) and \( \alpha(1) = 0. \)

• A2. \( g(c) \) is twice continuously differentiable, \( g'(c) > 0, \) and \( g''(c) < 0. \)

• A3. \( S(t) \) is piecewise continuously differentiable.
• Let \((c^*(t), S^*(t))\) denote the optimal solution to (1) - (5). Solving \(S^*(t)\) from (4) and (5),

\[
S^*(t) = e^{\int_0^t j(x)dx} \left\{ S_0 + \int_0^t e^{-\int_0^z j(x)dx} [m(z) - f(z)c^*(z)] \, dz \right\}, \quad t \in [0, 1].
\] (6)

• At \(t = 1\),

\[
S^*(1) = e^{\int_0^1 j(x)dx} \left\{ S_0 + \int_0^1 e^{-\int_0^z j(x)dx} [m(z) - f(z)c^*(z)] \, dz \right\}.
\] (7)

**LEMMA 1.** *Optimality requires that \(S^*(t)\) must satisfy*

\[
S^*(1) = 0.
\] (8)

• Lemma 1 means that all the resources must be used up by the terminal time (because there is no salvage value).
EXAMPLE A: Cake-eating or exhaustible resource extraction

• cake-eating (Yaari (1964) and Gale (1967))

• pure depletion of an exhaustible resource (Hotelling (1931), Karlin (1959), and Heal (1993))

\[
\max_{c(t)} \int_0^1 \alpha(t)g(c(t))dt
\]  \hspace{1cm} (9)

subject to

\[
c(t) \geq 0
\]  \hspace{1cm} (10)

and

\[
\int_0^1 c(t)dt = \omega.
\]  \hspace{1cm} (11)

• Cake-eating: \( c(t) \) is consumption at time \( t \), \( g(c) \) is the utility of consumption, \( \alpha(t) \) denotes a discount function, and \( \omega \) is the initial size of the cake.

• Pure depletion of an exhaustible resource: \( c(t) \) is the rate of resource extraction and \( \omega \) is the initial stock of the exhaustible resource.
• The planning horizon is $[0, 1]$. The consumer’s or resource owner’s problem is to allocate the endowment $\omega$ over time to maximize discounted utility.

• This model can be obtained from (1) - (5) by setting $j(t) = 0$, $m(t) = 0$, $f(t) = 1$, and $S_0 = \omega$. In this case, (4) becomes $S'(t) = -c(t)$. Integrating this equation yields

$$S(t) = S(0) - \int_0^t c(\tau)d\tau$$

$$= \omega - \int_0^t c(\tau)d\tau. \quad (12)$$

• Thus, (9) - (11) is a special case of (1) - (5).

• The assumption $\alpha(1) = 0$ means that the discount rate at the terminal date $t = 1$ is zero.
EXAMPLE B: Life-cycle saving under uncertain lifetime

- Life-cycle saving problem by Yaari (1965, p.143):

\[
\max_{c(t)} \int_0^1 \Omega(t)\tilde{\alpha}(t)\gamma(c(t))dt
\]  

subject to (2), (3), (5), and

\[
S'(t) = j(t)S(t) + m(t) - c(t).
\]

- This model can be obtained from (1) - (5) by setting \(\alpha(t) = \Omega(t)\tilde{\alpha}(t)\) and \(f(t) = 1\).

- \(\Omega(t)\) is the consumer’s survival probability at time \(t\), \(\tilde{\alpha}(t)\) is discount function, \(c(t)\) is consumption, \(\gamma(c)\) is the utility of consumption, \(S(t)\) is wealth (accumulated savings), \(m(t)\) is non-interest income (e.g., job earnings, social security benefits), \(j(t)\) is interest rate, and \(S_0\) is initial wealth.

- The consumer faces an uncertain lifetime \(T\), which is assumed to be a continuous random variable distributed on \([0, 1]\). Thus, \(\Omega(t) = \Pr(T > t)\). Clearly, \(\Omega'(t) \leq 0\) because the
survival probability diminishes over time.

- If $t = 1$ denotes the maximum possible lifetime, then the survival probability at $t = 1$ must be zero, i.e., $\Omega(1) = 0$. The probability of living beyond $t = 1$ is zero. Thus, this model, by construction, satisfies the assumption $\alpha(1) = 0$.

- There is a large literature on this Yaari (1965) model. It has been applied to study a wide variety of problems in both microeconomics and macroeconomics, e.g., Blanchard (1985) embeds Yaari’s (1965) model in an overlapping-generation model and the resulting Blanchard-Yaari model has been regarded as one of the workhorse models for modern macroeconomic research (Heijdra and van der Ploeg (2002, p.540)).
EXAMPLE C: Value of an Non-Atomic Game

• In a treatise on non-atomic games, Aumann and Shapley (1974, p.168 and p.179) define the value $\nu$ of a coalition $Q$ as

$$
\nu(Q) = \left\{ \max_{y(t)} \int_Q u(y(t), t) dF(t) : \int_Q y(t) dF(t) = \int_Q a(t) dF(t) \right\},
$$

(15)

where $u$ denotes production function, $y$ denotes raw material, and $a$ denotes endowed raw material.

• Aumann and Shapley (1974): this non-atomic game can be interpreted as a production or an exchange economy.

• Interpreted as a production economy, the model assumes that there is a continuum of producers indexed by $t$ and $F(t)$ is the measure of $t$. The quantity $u(y(t), t) dF(t)$ is the amount of finished good that producer $t$ can produce from an amount $y(t) dF(t)$ of raw material and $a(t) dF(t)$ is the amount of raw material initially available to producer $t$. The total amount of raw material initially available to coalition $Q$ is given by $\int_Q a(t) dF(t)$. 

• The coalition’s objective is to reallocate the endowed raw material \( \int_Q a(t) dF(t) \) among its members to produce the maximum amount of finished product \( \int_Q u(y(t), t) dF(t) \).

• Assuming \( u(y(t), t) = \alpha(t) g(y(t)) \), \( dF(t) = f(t) dt \), \( \int_Q a(t) dF(t) = \omega \), and \( Q = [0, 1] \), then (15) can be expressed as

\[
\max_{y(t)} \int_0^1 \alpha(t) f(t) g(y(t)) dt
\]

subject to

\[
y(t) \geq 0
\]

and

\[
\int_0^1 y(t) f(t) dt = \omega.
\]

• The functions \( F(t) \) and \( f(t) \) are the cumulative distribution function and probability density function of \( t \), respectively. Let

\[
S'(t) = -f(t)y(t)
\]
\[ S(0) = \omega, \quad (20) \]

then (18) can be replaced by (8), (19), and (20). Thus, (16) - (18) is a special case of (1) - (5).

- Except the extra term \( f(t) \), (16) - (18) is similar to (9) - (11). If \( f(t) = 1 \) for all \( t \), i.e., \( F(t) \) is a uniform measure, then (16) - (18) is mathematically equivalent to (9) - (11).

- The function \( \alpha(t) \) can be interpreted as the productivity of producer \( t \). The assumption \( \alpha'(t) \leq 0 \) in A1 can easily be satisfied here because, without loss of generality, the producers can be arranged in descending order of productivity. If the productivity of the least productive producer is zero, then the assumption \( \alpha(1) = 0 \) is satisfied.

- The above three examples are all about optimal resource allocation. However, there is an important distinction between Example C and the other two examples. In contrast to Examples A and B where \( t \) denotes time, the \( t \) in Example C denotes an index for a continuum of producers or consumers.
While there is a large number of studies on the existence problem of Examples A and C (e.g., Yaari (1964), Aumann and Perles (1965), Kumar (1969), Abrham (1970), Artstein (1974, 1980), Aumann and Shapley (1974), and Ioffe (2006)), the existence problem of Example B has never been investigated in the literature.
III. Existence Results

To solve (1) - (5), define the Hamiltonian $H$ and Lagrangian $L$ as

$$H(c(t), S(t), t) = \alpha(t)f(t)g(c(t)) + \lambda(t) [j(t)S(t) + m(t) - f(t)c(t)]$$  \hspace{1cm} (21)

and

$$L(c(t), S(t), t) = \alpha(t)f(t)g(c(t)) + \lambda(t) [j(t)S(t) + m(t) - f(t)c(t)] + \eta(t)c(t) + \mu(t)S(t),$$ \hspace{1cm} (22)

respectively, where $\lambda(t)$, $\eta(t)$, and $\mu(t)$ are multipliers. By the state-constrained maximum principle (Hartl, Sethi, and Vickson (1995)), $c^*(t)$ and $S^*(t)$ must satisfy the following optimality conditions:

$$\frac{\partial L(c^*(t), S^*(t), t)}{\partial c(t)} = \alpha(t)f(t)g'(c^*(t)) - \lambda(t)f(t) + \eta(t) = 0,$$ \hspace{1cm} (23)

$$\frac{\partial L(c^*(t), S^*(t), t)}{\partial S(t)} = -\lambda'(t) = j(t)\lambda(t) + \mu(t),$$ \hspace{1cm} (24)
\[ \eta(t) \geq 0, \eta(t)c^*(t) = 0, \quad (25) \]

\[ \mu(t) \geq 0, \mu(t)S^*(t) = 0, \quad (26) \]

as well as the transversality condition

\[ \lambda(1)S^*(1) = 0. \quad (27) \]

The following theorem shows that the solution \((c^*(t), S^*(t))\), if it exists, contains a distinctive feature.

**Lemma 2.** Assume \(\alpha(1) = 0\). Let \(g'(0^+) = \lim_{c \to 0^+} g'(c)\). If either \(g'(0^+) < \infty\) or \(m(1) > 0\), then there exists a \(t^* \in [0, 1]\) such that \(S^*(t) = 0\) and \(c^*(t) = \frac{m(t)}{f(t)}\) for all \(t \in [t^*, 1]\).
Figure 0
The paths of $c^*(t)$ and $S^*(t)$

$S^*(t)$

$S_0$

0 $t^*$ 1 $t$
• For the cake-eating example, the theorem means that the cake will be completely consumed before the terminal time $t = 1$.

• For the exhaustible resource example, the entire stock of natural resource will be exhausted before the terminal time.

• For Yaari’s (1965) life-cycle model, the consumer’s wealth will be depleted before the
terminal time.

- The driving force behind the theorem is the assumption that $\alpha(1) = 0$.
- How is the $t^*$ in Lemma 2 determined?
- Assume that $\eta(t) = 0$ and $\mu(t) = 0$ for $t \in [0, t^*]$. Combining (23) and (24),

$$\alpha(t)g'(c^*(t)) = \lambda(t) = \lambda(0)e^{-\int_0^t j(x)dx} \tag{28}$$

for $t \in [0, t^*]$. From Lemma 2, $c^*(t^*) = m(t^*)/f(t^*)$, hence (28) implies that

$$\lambda(0) = g' \left( \frac{m(t^*)}{f(t^*)} \right) \alpha(t^*)e^{\int_0^{t^*} j(x)dx}. \tag{29}$$

Substituting (29) into (28),

$$\alpha(t)g'(c^*(t))e^{\int_0^t j(x)dx} = \alpha(t^*)g' \left( \frac{m(t^*)}{f(t^*)} \right) e^{\int_0^{t^*} j(x)dx}, \quad t \in [0, t^*]. \tag{30}$$
Thus, the solution $(c^*(t), S^*(t))$ can be characterized as follows:

$$
c^*(t) = \begin{cases} 
(g')^{-1}(\gamma(t,t^*)) & \text{if } t \in [0, t^*] \\
m(t) & \text{if } t \in [t^*, 1] 
\end{cases}, \quad (31)
$$

$$
S^*(t) = \begin{cases} 
\int_0^t j(x)dx \left\{ S_0 + \int_0^t e^{-\int_0^z j(x)dx} [m(z) - f(z) c^*(z)] dz \right\} & \text{if } t \in [0, t^*] \\
0 & \text{if } t \in [t^*, 1] 
\end{cases}, \quad (32)
$$

where $(g')^{-1}$ denotes the inverse function of $g'$ and

$$
\gamma(t,t^*) = \frac{g' \left( \frac{m(t^*)}{f(t^*)} \right) \alpha(t^*) e^{\int_0^{t^*} j(x)dx}}{\alpha(t) e^{\int_0^t j(x)dx}}, \quad 0 \leq t \leq t^* < 1. \quad (33)
$$

From (4) and (5),

$$
S_0 = \int_0^{t^*} e^{-\int_0^t j(x)dx} [f(t)c^*(t) - m(t)] dt. \quad (34)
$$
Substituting (31) into (34), the terminal wealth depletion time $t^*$ is determined by the equation

$$S_0 = \int_0^{t^*} e^{-\int_0^t j(x)dx} \left[ f(t) (g')^{-1} (\gamma(t, t^*)) - m(t) \right] dt. \quad (35)$$

To study whether such a $t^*$ exists, replace every $t^*$ in (35) by $t$ and define the function

$$\varphi(t) = S_0 - \int_0^t e^{-\int_0^t j(x)dx} \left[ f(z) (g')^{-1} (\gamma(z, t)) - m(z) \right] dz, \quad t \in [0, 1). \quad (36)$$

- Clearly, (34) implies that $\varphi(t^*) = 0$. Hence, solving for $t^*$ from (35) is equivalent to locating the root of $\varphi(t)$.

- Therefore, if $\varphi(t)$ does not have a root, then the optimal allocation problem (1) - (5) does not have a solution.

- If $\varphi(t)$ has a root, then (1) - (5) has a solution.

- Hence, Lemma 2 serves to convert the original existence problem of (1) - (5) into a root existence problem for $\varphi(t)$. 
• It is possible for $\varphi(t)$ to have multiple roots (Leung (2007)).

The following theorem follows readily from the property of $\varphi(t)$.

**THEOREM 1.** Assume that (i) $\eta(t) = 0$ and $\mu(t) = 0$ for $t \in [0, t^*]$, (ii) $g'(0^+) < \infty$ or $m(1) > 0$, and (iii) $S_0 > 0$. Let $g'(\infty) = \lim_{c \to \infty} g'(c)$. If $g'(\infty) = 0$, then the optimal allocation problem (1) - (5) has a solution.

• Theorem 1 shows that the optimal allocation problem has a solution if the marginal utility of infinite consumption is zero. Many popular utility functions satisfy this condition, e.g., logarithmic ($\log c$), CRRA ($c^{1-\delta}/(1 - \delta)$), and exponential ($-e^{-c}$).

**EXAMPLE 1.** Let $g(c) = 1 - e^{-c}$, $\alpha(t) = 1 - t$, $j(t) = 0$, $m(t) = 0$, $f(t) = 1$, and $S_0 > 0$. In this case, $g'(c) = e^{-c}$, hence $g'(0^+) = 1$ and $g'(\infty) = 0$. The conditions in Theorem 1 are
satisfied, hence $\varphi(t)$ has a solution. It is straightforward to verify that (31) and (36) become
\[
c^*(t) = \begin{cases} 
\ln \left( \frac{1-t}{1-t^*} \right) & \text{if } t \in [0,t^*] \\
0 & \text{if } t \in [t^*,1]
\end{cases}
\] (37)
and
\[
\varphi(t) = S_0 - \int_0^t \ln \left( \frac{1-z}{1-t} \right) \, dz \\
= S_0 + t + \ln(1-t), \quad t \in [0,1),
\] (38)
respectively. As $\varphi(0) = S_0 > 0$, $\varphi(1^-) = -\infty$, and $\varphi'(t) = -\frac{1}{1-t} < 0$ for all $t \in [0,1)$, $\varphi(t)$ must have a root in $(0,1)$. \hfill \blacktriangleright

- Some utility functions do not satisfy the condition $g'(\infty) = 0$.
- The following example demonstrates what can happen when $g'(\infty) > 0$. 

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EXAMPLE 2. Let $g(c) = c - e^{-c}$, $\alpha(t) = 1 - t$, $j(t) = 0$, $m(t) = 0$, $f(t) = 1$, and $S_0 > 0$. In this case, $g'(c) = 1 + e^{-c}$, hence $g'(0^+) = 2$, $g'(\infty) = 1$, and $1 < g'(c) < 2$ for $c \in (0, \infty)$. Hence, $g'(c)$ is bounded below by one. Since $g'(0^+) < \infty$, Lemma 2 is applicable. It is straightforward to verify that (31) becomes

$$c^*(t) = \begin{cases} \ln \left( \frac{1-t}{1-2t^*+t} \right) & \text{if } t \in [0, t^*] \\ 0 & \text{if } t \in [t^*, 1] \end{cases}.$$  \hspace{1cm} (39)$$

For $c^*(t)$ to be well defined, $t^*$ must be strictly smaller than $\frac{1}{2}$. If $t^* = \frac{1}{2}$, then $c^*(0) = \infty$. If $t^* > \frac{1}{2}$, then $1 - 2t^* + t < 0$ for $t < 2t^* - 1$, hence $c^*(t)$ is undefined for $t \in [0, 2t^* - 1]$. Therefore, $t^* \in [0, \frac{1}{2})$ and (36) becomes

$$\varphi(t) = S_0 + 2(1-t) \ln(1-t) - (1-2t) \ln(1-2t), \ t \in \left[0, \frac{1}{2}\right).$$ \hspace{1cm} (40)$$

As $\varphi\left(\frac{1}{2}\right) = S_0 - \ln 2$ and $\varphi'(t) = -1 + 2 \ln[(1-2t)/(1-t)] < 0$, $\varphi(t)$ does not have a root if $S_0 > \ln 2$. Clearly, $\varphi(t)$ has a root if and only if $S_0 < \ln 2$. This optimal allocation problem
does not have a solution if $S_0 \geq \ln 2$. A solution exists if and only if $S_0 < \ln 2$. ▲

- Example 2 first appeared in Yaari (1964, pp.586-7). This interesting example demonstrates that the optimal allocation problem (1) - (5) may not have a solution. T

- The result is counter-intuitive because it means that there is no optimal way to allocate the endowed wealth $S_0$ to maximize a well-defined objective (utility) function in a relatively simple setup.

- Perhaps even more puzzling is that there exists a solution if $S_0$ is sufficiently small (smaller than $\ln 2$), but there is no solution if $S_0$ is sufficiently large (greater than or equal to $\ln 2$). Thus, an infinitesimally small decrease in $S_0$ can change the optimal allocation problem from one without a solution to one with a solution.

- A major factor that renders $\varphi(t)$ rootless in Example 2 is the property of the utility function, namely $g'(c) \geq 1$, which prevents $\varphi(t)$ from diminishing to $-\infty$. 

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In Example 2, it is assumed that $m(t) = 0$. What will happen if $m(t) > 0$? Will the addition of a positive $m(t)$ guarantee the existence of a solution? Example 3 sheds light on these questions.

**EXAMPLE 3.** Let $g(c) = c - e^{-c}$, $\alpha(t) = 1 - t$, $j(t) = 0$, $m(t) = M > 0$, $f(t) = 1$, and $S_0 \geq 0$. It is straightforward to verify that (31) becomes

$$c^*(t) = \begin{cases} \ln \left[ \frac{1-t}{e^{-M(1-t^*)-t^*+t}} \right] & \text{if } t \in [0, t^*] \\ 0 & \text{if } t \in [t^*, 1] \end{cases}. \quad (41)$$

For $c^*(t)$ to be well defined, $t^*$ must be strictly smaller than $\frac{1}{1+e^M}$. If $t^* = \frac{1}{1+e^M}$, then $e^{-M(1-t^*)} - t^* = 0$ and (41) implies that $c^*(0) = \infty$. If $t^* > \frac{1}{1+e^M}$, then $e^{-M(1-t^*)} - t^* < 0$ and (41) implies that $c^*(0)$ is undefined. Therefore,

$$t^* \in \left[0, \frac{1}{1 + e^M} \right). \quad (42)$$
In addition, for $c^*(t)$ to be well defined, it is necessary that $(1 - t) \geq e^{-M}(1 - t^*) - t^* + t$ for $t \in [0, t^*]$, i.e., $t \leq \frac{1}{2}[1 + t^* - e^{-M}(1 - t^*)]$. The last inequality is satisfied because $e^{-M} \leq 1$ implies that $2t^* \leq 1 + t^* - e^{-M}(1 - t^*)$, i.e., $t^* \leq \frac{1}{2}[1 + t^* - e^{-M}(1 - t^*)]$. Given $t \in [0, t^*]$, the inequalities $t \leq t^* \leq \frac{1}{2}[1 + t^* - e^{-M}(1 - t^*)]$ are satisfied.

It is straightforward to verify that (36) becomes

\[
\varphi(t) = S_0 + Mt + (1 - t) \ln(1 - t) + e^{-M}(1 - t) \ln[e^{-M}(1 - t)] \\
- [e^{-M}(1 - t) - t] \ln[e^{-M}(1 - t) - t], \quad t \in \left[0, \frac{1}{1 + e^M}\right],
\]

and $\varphi'(t) < 0$. Evaluating the left-hand limit of $\varphi(t)$ at $\frac{1}{1 + e^M}$ and using the fact that $0 \ln 0 = 0$,

\[
\varphi\left(\left(\frac{1}{1 + e^M}\right)^-\right) = S_0 + M - \ln\left(1 + e^M\right).
\]

Thus, if

\[
S_0 + M \geq \ln\left(1 + e^M\right),
\]

then $\varphi(t)$ does not have a root and the optimal allocation problem does not have a solu-
tion. Although $M < \ln (1 + e^M)$ (as $e^M < 1 + e^M$), (45) will hold if $S_0$ is sufficiently large. Furthermore,

$$d \left[ M - \ln (1 + e^M) \right] = \frac{1}{1 + e^M} > 0$$

(46)

and

$$\lim_{M \to \infty} \left[ M - \ln (1 + e^M) \right] = \lim_{M \to \infty} \left[ \ln \frac{e^M}{1 + e^M} \right] = 0.$$  

(47)

If $S_0 \geq \ln 2$, then (46) and (47) imply that (45) will hold for any $M \geq 0$. In this case, $\varphi(t)$ has no root and the addition of $M$ does not affect the non-existence result. If $0 < S_0 < \ln 2$, then (46) and (47) imply that (45) will hold for any $M \geq -\ln(e^{S_0} - 1)$. In other words, $\varphi(t)$ has a root if $M < -\ln(e^{S_0} - 1)$ and $\varphi(t)$ has no root if $M \geq -\ln(e^{S_0} - 1)$. The larger the value of $M$, the more likely that $\varphi(t)$ will not have a root. Thus, the addition of $M$ exacerbates the existence problem.\hfill

- Example 3 reveals that the addition of $m(t)$ increases the likelihood that the optimal allocation problem will not have a solution.
- Examples 2 and 3 suggest that the existence problem arises because \( g'(\infty) \) is bounded away from zero. The following example examines whether \( g'(\infty) > 0 \) is the source of the problem.

**Example 4.** Let \( g(c) = c + \ln c \), \( \alpha(t) = 1 - t \), \( j(t) = 0 \), \( m(t) = M \geq 0 \), \( f(t) = 1 \), and \( S_0 > 0 \). In this case, \( g'(c) = 1 + 1/c \), hence \( g'(0^+) = \infty \) and \( g'(\infty) = 1 \). Thus, \( g'(c) \) is bounded below by one. It is straightforward to verify that (31) becomes

\[
 c^*(t) = \begin{cases} 
 \frac{M(1-t)}{(1+M)(1-t^*)-M(1-t)} & \text{if } t \in [0, t^*] \\
 0 & \text{if } t \in [t^*, 1] 
\end{cases}.
\]

(48)

For \( c^*(t) \) to be well defined, \( t^* \) must be strictly less than \( \frac{1}{1+M} \). If \( t^* = \frac{1}{1+M} \), then (48) implies that \( c^*(0) = \infty \). If \( t^* > \frac{1}{1+M} \), then \( (1+M)(1-t^*) - M < 0 \) and (48) implies that \( c^*(0) < 0 \). Therefore,

\[
 t^* \in \left[ 0, \frac{1}{1 + M} \right). 
\]

(49)
In addition, for \( c^*(t) \) to be well defined, it is necessary that \((1 + M)(1 - t^*) - M > 0\), i.e., \( t^* < \frac{1}{1+M} \), which is satisfied by virtue of (49). It is straightforward to verify that (36) becomes

\[
\varphi(t) = S_0 + (1 + M)t + \frac{(1 + M)(1 - t)}{M} \ln \left[ \frac{1 - (1 + M)t}{1 - t} \right], \quad t \in \left[ 0, \frac{1}{1+M} \right].
\]  

(50)

and \( \varphi'(t) < 0 \). Evaluating the left-hand limit of \( \varphi(t) \) at \( \frac{1}{1+M} \),

\[
\varphi \left( \left( \frac{1}{1+M} \right)^- \right) = -\infty.
\]

Thus, \( \varphi(t) \) has a root regardless of the magnitude of \( S_0 \) and \( M \). As long as \( S_0 > 0 \), the optimal allocation problem has a solution.\( \blacklozenge \)

- Example 4 shows that a solution exists even when \( g'(\infty) > 0 \). Therefore, \( g'(\infty) > 0 \) does not always imply that the optimal allocation problem will not have a solution.

- Examples 2 - 4 reveal that \( g'(\infty) > 0 \) plays an important but not decisive role on the non-existence result.
The following theorem provides a simple sufficient condition to guarantee the existence of a solution for (1) - (5) when $g'(\infty) = k > 0$.

**THEOREM 2.** Assume that (i) $\eta(t) = 0$ and $\mu(t) = 0$ for $t \in [0, t^*]$, (ii) $g'(0^+) < \infty$ or $m(1) > 0$, and (iii) $S_0 > 0$. Suppose $g'(\infty) = k > 0$. Without loss of generality, assume that $k = 1$. Then there exists a $\tau \in (0, 1)$ such that

$$
\alpha(\tau)g'\left(\frac{m(\tau)}{f(\tau)}\right) e^{\int_0^\tau j(x)dx} = 1. \quad (51)
$$

If there is more than one $\tau$ that satisfies (51), assume that

$$
\tau = \max\{\theta \in (0, 1) \mid \alpha(\theta)g'\left(\frac{m(\theta)}{f(\theta)}\right) e^{\int_0^\theta j(x)dx} = 1\} \quad (52)
$$

exists. Then the optimal allocation problem (1) - (5) has a solution if

$$
\int_0^\tau e^{-\int_0^t j(x)dx} f(t)(g')^{-1}\left(\frac{1}{\alpha(t)e^{\int_0^t j(x)dx}}\right) dt > S_0 + \int_0^\tau e^{-\int_0^t j(x)dx} m(t) dt. \quad (53)
$$
In particular, if
\[
\int_0^\tau e^{-\int_0^t j(x)dx}f(t)(g')^{-1}\left(\frac{1}{\alpha(t)e^{\int_0^t j(x)dx}}\right)dt = \infty,
\]
then (53) is satisfied and (1) - (5) has a solution.

- Without Theorem 2, it would be natural to tackle the existence problem by checking the value of \(\varphi(1)\). If \(\varphi(1) < 0\), then (1) - (5) will have a solution provided that \(S_0 > 0\).

- However, Theorem 2 shows that \(\varphi(1)\) will not exist if \(g'(\infty) > 0\). If \(g'(\infty) = 0\), the supremum of \(t\) is 1 and the domain of \(\varphi(t)\) is \([0, 1]\). If \(g'(\infty) > 0\), then the domain of \(\varphi(t)\) becomes \([0, \tau]\), where \(\tau < 1\). In other words, when \(g'(\infty)\) has a positive lower bound, it will put an upper bound (which is strictly less than one) on \(t^*\). The supremum \(\tau\) is determined by (52).

- As long as \(m(t)\) is smooth enough (e.g., piecewise smooth), then (52) will exist. With the upper bound \(\tau\), one can check whether \(\varphi(t)\) has a root by ascertaining whether \(\varphi(\tau) \leq 0\).
Therefore, Theorem 2 shows that one can determine whether (1) - (5) has a solution by evaluating the left-hand side of (54).

A salient feature of condition (54) is that the left-hand side does not depend on \( m(t) \) except through the upper limit of the integral, \( \tau \). The integrand depends on \( j(t) \), \( \alpha(t) \), \( f(t) \), and \( g(c) \) but not on \( m(t) \).

Condition (54) is relatively simple and verifiable. It is different from the condition proposed by Yaari (1964).

The following illustrates how Theorem 2 can be applied to examine the existence problem for Examples 3 and 4.

**EXAMPLE 3 (Continued).** Given \( g(c) = c - e^{-c} \), \( \alpha(t) = 1 - t \), \( f(t) = 1 \), and \( j(t) = 0 \), (51) implies that \( (1 - \tau)(1 + e^{-M}) = 1 \), which yields \( \tau = \frac{1}{1+e^M} \). Thus, the left-hand side of (54) becomes

\[
\int_0^{\tau} (g')^{-1} \left( \frac{1}{\alpha(t)} \right) dt = \int_0^{\frac{1}{1+e^M}} \ln \left( \frac{1 - t}{t} \right) dt = \ln (1 + e^M) - \frac{Me^M}{1 + e^M} < \infty. \quad (55)
\]
Hence, \( \phi(t) \) may not have a root. Substituting (55) into (53), \( \phi(t) \) has a root if

\[
\ln \left(1 + e^M\right) - \frac{Me^M}{1 + e^M} > S_0 + \frac{M}{1 + e^M},
\]

i.e., \( \ln \left(1 + e^M\right) > S_0 + M \). If this inequality is not satisfied, then \( \phi(t) \) has no root. This result is the same as (45).

\[\square\]

**EXAMPLE 4 (Continued).** Given \( g(c) = c + \ln c, \alpha(t) = 1 - t, f(t) = 1, \) and \( j(t) = 0, \) (51) implies that \( (1 - \tau)(1 + \frac{1}{M}) = 1, \) which yields \( \tau = \frac{1}{1+M}. \) Hence, the left-hand side of (54) becomes

\[
\int_{0}^{\tau} (g')^{-1} \left( \frac{1}{\alpha(t)} \right) dt = \int_{0}^{\frac{1}{1+M}} \left( \frac{1 - t}{t} \right) dt = \ln t - t \bigg|_{0}^{\frac{1}{1+M}} = \infty.
\]

Condition (54) is satisfied, thus \( \phi(t) \) has a root. \[\square\]

- Examples 3 and 4 expose the source of the existence problem.
- For Theorem 1, when \( g'(\infty) = 0, \) the lowest possible value for the right-hand-side of
(30) is zero, which occurs when \( t^* = 1 \). This will force \( c^*(t) = \infty \) for all \( t \in [0, 1] \), thus guaranteeing a root for \( \varphi(t) \).

- For Theorem 2, when \( g'(\infty) = 1 \), the lowest possible value for the right-hand-side of (30) is one, which occurs when \( t^* = \tau < 1 \). This will force \( c^*(t) = \infty \) at only one point \( (t = 0) \) but not elsewhere, thus \( \varphi(t) \) may not have a root. Nevertheless, although \( c^*(t) \) is unbounded only at one point, it can be sufficient to generate a root for \( \varphi(t) \).

- Examples 3 and 4 reveal that \( c^*(0) \) plays a pivotal role in generating a root for \( \varphi(t) \).

- It is possible to offer an intuitive explanation for the difference in the results in Examples 3 and 4. As shown in (54), the key of the existence problem is the convergence or divergence of the integral \( \int_0^\tau (g')^{-1} (1/\alpha(t)) \, dt \). At \( t = 0 \), the integrand \( (g')^{-1} (1/\alpha(0)) = (g')^{-1} (1) = \infty \). As \( (g')^{-1} (1/\alpha(t)) < \infty \) for \( t \in (0, \tau] \), \( (g')^{-1} (1/\alpha(0)) = \infty \) is the sole source of the divergence of \( \int_0^\tau (g')^{-1} (1/\alpha(t)) \, dt \).

- Put differently, if \( (g')^{-1} (1/\alpha(0)) \) were finite, then \( \int_0^\tau (g')^{-1} (1/\alpha(t)) \, dt \) could not diverge because \( (g')^{-1} (1/\alpha(t)) < \infty \) for all \( t \in [0, \tau] \). Since \( g'(\infty) = 1 \), the faster \( g'(c) \) goes to one
(the smaller the value of $c$ required for $g'(c)$ to approach one), the smaller the values of $(g')^{-1}(1/\alpha(t))$ near $t = 0$, hence the smaller will be the magnitude of $\int_0^\tau (g')^{-1}(1/\alpha(t)) \, dt$. Thus, the faster $g'(c)$ approaches one, the less likely that $\int_0^\tau (g')^{-1}(1/\alpha(t)) \, dt$ will diverge.

- The magnitude of $\tau$ does not play a role in the divergence of $\int_0^\tau (g')^{-1}(1/\alpha(t)) \, dt$ because $(g')^{-1}(1/\alpha(0)) = \infty$ is the sole source of the divergence. However, it plays an important role in the existence problem when $\int_0^\tau (g')^{-1}(1/\alpha(t)) \, dt$ converges.
IV. Discussion

(a) Smoothness Assumptions

- What if $m(t)$ is not continuous?

**EXAMPLE 9.** Let $g(c) = \ln c$, $\alpha(t) = 1 - t$, $j(t) = 0$, $f(t) = 1$, $m(t) = E$ if $t < R$, and $m(t) = M$ if $t \geq R$. Assume that $E$, $M$, and $S_0$ are positive constants, and $0 < R < 1$. Hence, $m(t)$ is continuous on $[0, 1]$ except at $t = R$. Under these assumptions, it is straightforward to verify that (36) becomes

$$
\varphi(t) = \begin{cases} 
S_0 - \frac{Et^2}{2(1-t)} & \text{if } t \in [0, R) \\
S_0 + (E - M)R - \frac{Mt^2}{2(1-t)} & \text{if } t \in [R, 1]
\end{cases}
$$

Clearly, $\varphi(t)$ has two special features. First, $\varphi(t)$ is continuous on $[0, 1]$ except at $t = R$. The discontinuity arises because $m(t)$ is discontinuous at $t = R$. Second, $\varphi(t)$ is strictly decreasing and strictly concave on each of the two intervals $[0, R)$ and $[R, 1]$. If $\varphi(R^-) < 0$ or $\varphi(R) > 0$, 
then \( \varphi(t) \) has a root. However, if \( \varphi(R^-) > 0 \) and \( \varphi(R) < 0 \), then \( \varphi(t) \) has no root. For instance, let \( S_0 = 10, E = 10, \) and \( M = 50, \) then \( \varphi(0.65^-) = 3.96 \) and \( \varphi(0.65) = -46.18, \) hence \( \varphi(t) \) has no root. \( \blacklozenge \)

- Example 9 shows that, even if \( g'(\infty) = 0 \) and \( m(t) \) is discontinuous only at one point, the optimal allocation problem may not have a solution. A discontinuous \( m(t) \) generates a discontinuous \( \varphi(t) \). The existence problem caused by a discontinuous \( \varphi(t) \) is substantively different from that caused by a positive \( \varphi(\tau) \). This paper focuses on the latter and develops simple sufficient conditions to guarantee that \( \varphi(\tau) < 0. \)

(b) Interior Solution

The following lemma gives some sufficient conditions for \( \eta(t) = 0 \) and \( \mu(t) = 0 \) for \( t \in [0, t^*]. \)

**Lemma 3.** If \( (C1) \) \( j(t) = 0 \) and \( m(t) = 0, \) or \( (C2) \) \( j(t) = 0, m'(t) \leq 0, \) and \( f(t) = 1, \) then \( \eta(t) = 0 \) and \( \mu(t) = 0 \) for \( t \in [0, t^*]. \)
• If either $(C1)$ or $(C2)$ is satisfied, then the solution to $\text{(1)} - \text{(5)}$, if it exists, must be interior throughout $[0, t^*)$.

• Although these two conditions may appear restrictive, they are satisfied in most of the economic applications of $\text{(1)} - \text{(5)}$. 
V. Conclusion

- This paper investigates the problem concerning the existence of a solution to a diverse class of optimal allocation problems which include models of cake eating, exhaustible resource extraction, life-cycle saving, and non-atomic games.

- A new formulation that encompasses all these diverse models is provided.

- Examples of these models for which a solution does not exist and the causes of the non-existence are studied.

- Two theorems are provided to tackle the existence problem under different conditions.

- Several analytical examples with a closed-form solution are offered to illustrate the usefulness of the existence theorems.

- This paper considers only the case where $\alpha(1) = 0$. The existence theorems are built on this pivotal assumption as it provides a special route by which the existence problem can be resolved. Without this assumption, Lemma 2 will not hold, hence Theorems 1 and 2
cannot be established.

- While the assumption may appear restrictive, it is not contrived because it does include a meaningful class of optimal allocation models such as Yaari’s (1965) life-cycle model of saving and the non-atomic games in Aumann and Shapley (1974).

- For $\alpha(1) > 0$, existence results are available only for the case where $j(t) = 0$ and $m(t) = 0$, see Ioffe (2006) and the references therein.

- The existence problem for the general model (1) - (5) where $\alpha(1) > 0$ has not yet been investigated in the literature.

- Whether the results of this paper can shed light on the existence problem for the general model remains to be studied.