Higher Order Beliefs in Dynamic Environments

Antonio Penta

University of Pennsylvania
Department of Economics

June 22, 2008
Introduction: Higher Order Beliefs

- **Global Games (Carlsson and Van Damme, 1993):**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0, 0</td>
<td>0, $\theta - 2$</td>
</tr>
<tr>
<td>B</td>
<td>$\theta - 2, 0$</td>
<td>$\theta, \theta$</td>
</tr>
</tbody>
</table>

**Dominance Regions:**
- A if $\theta < 0$;
- B if $\theta > 2$

**Multiplicity:** $\theta \in (0, 2)$
Global Games (Carlsson and Van Damme, 1993):

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0, 0</td>
<td>0, $\theta - 2$</td>
</tr>
<tr>
<td>$B$</td>
<td>$\theta - 2, 0$</td>
<td>$\theta, \theta$</td>
</tr>
</tbody>
</table>

**Dominance Regions:**
- $A$ if $\theta < 0$
- $B$ if $\theta > 2$

**Multiplicity:** $\theta \in (0, 2)$

$\theta^1 = 3$  $\theta^2 = 1.5$  $\theta^3 = 1.5$

Player 1: $\bigcirc$ $\bigcirc$ $\bigcirc$

Player 2: $\bigcirc$ $\bigcirc$ $\bigcirc$
Introduction: Higher Order Beliefs

- Global Games (Carlsson and Van Damme, 1993):

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0, 0</td>
<td>0, θ − 2</td>
</tr>
<tr>
<td>B</td>
<td>θ − 2, 0</td>
<td>θ, θ</td>
</tr>
</tbody>
</table>

**Dominance Regions:**
- A if θ < 0; B if θ > 2

**Multiplicity:** θ ∈ (0, 2)

\[ \theta^1 = 3 \quad \theta^2 = 1.5 \quad \theta^3 = 1.5 \]

Player 1:
- \( \bullet \) \( \bullet \) \( \bullet \)

Player 2:
- \( \bullet \) \( \bullet \) \( \bullet \) \( \bullet \)

- Discontinuity between Common Knowledge and k-Mutual Belief (k finite) (cf. Rubinstein, 1989):

The fact that **Dominance Regions** are **NOT** ruled out by **Higher Order Beliefs** determines the result.
**Static Games:**
multiplicity of rationalizable outcomes is not robust to perturbation of *higher order beliefs*
Introduction: Higher Order Beliefs

Static Games:
multiplicity of rationalizable outcomes is not robust to perturbation of higher order beliefs

Weinstein and Yildiz (2007, WY): If the underlying space of uncertainty is rich enough, whenever a game has multiple Rationalizable outcomes, any of these is uniquely rationalizable in an arbitrarily close game (perturbing higher order beliefs only).
Introduction: Higher Order Beliefs

**Static Games:**
multiplicity of rationalizable outcomes is not robust to perturbation of *higher order beliefs*

- **Weinstein and Yildiz (2007, WY):** If the underlying space of uncertainty is *rich enough*, whenever a game has multiple Rationalizable outcomes, any of these is uniquely rationalizable in an arbitrarily close game (perturbing *higher order beliefs* only).

- **(Non-)Robustness of Refinements:** refinements of rationalizability (e.g. *Nash Eq.*) are *not* robust to perturbations of *higher order beliefs* if and only if they agree with those of rationalizability.
  (Robustness = upper hemicontinuity)
Introduction: What is the Role of Higher Order Beliefs in Dynamic Games?

Dynamic Environments: little is known
Introduction: What is the Role of Higher Order Beliefs in Dynamic Games?

Dynamic Environments: little is known

- Dynamic Global Games: no systematic analysis – contrasting results (in terms of impact of perturbations of higher order beliefs on multiplicity)
**Dynamic Environments:** little is known

- WY’s analysis cannot be applied to normal forms of dynamic games:
  
  - **WY’s Richness Condition:** the underlying space of uncertainty $\Theta$ must be rich enough that for each player strategy $s_i$ there is a payoff state $\theta^{s_i}$ that makes $s_i$ strictly dominant.
**Introduction:** What is the Role of Higher Order Beliefs in Dynamic Games?

**Dynamic Environments:** little is known

- WY’s analysis cannot be applied to normal forms of dynamic games:
  - **WY’s Richness Condition:** the underlying space of uncertainty $\Theta$ must be rich enough that for each player strategy $s_i$ there is a payoff state $\theta^{s_i}$ that makes $s_i$ strictly dominant.

```
1
  a1
   1
( 1  1 )
  a2
   2
 b1
( 2 -1 )
 b2
( -2 0 )
 b1
( 0 -2 )
 b2
( 0 -1 )
```
Dynamic Environments

Do WY’s results (uniqueness /(non-)robustness) extend to dynamic environments? How?
Do WY’s results (uniqueness /(non-)robustness) extend to dynamic environments? How?

one possibility is to maintain the same richness condition for the (reduced) normal form of the game and modify (normal fom)-rationalizability introducing trembles (see WY)
Do WY’s results (uniqueness / (non-)robustness) extend to dynamic environments? How?

- one possibility is to maintain the same richness condition for the (reduced) normal form of the game and modify (normal form) rationalizability introducing trembles (see WY)
- PROBLEM: are tremble-based solution concepts still refinements of such modified rationalizability?
Do WY’s results (uniqueness /(non-)robustness) extend to dynamic environments? How?

- one possibility is to maintain the same richness condition for the (reduced) normal form of the game and modify (normal form)-rationalizability introducing trembles (see WY)
  - PROBLEM: are tremble-based solution concepts still refinements of such modified rationalizability?

- alternatively: weaken the richness condition, and exploit the extensive form:
Do WY’s results (uniqueness / (non-)robustness) extend to dynamic environments? How?

- one possibility is to maintain the same richness condition for the (reduced) normal form of the game and modify (normal form)-rationalizability introducing trembles (see WY)
  - PROBLEM: are tremble-based solution concepts still refinements of such modified rationalizability?

- alternatively: weaken the richness condition, and exploit the extensive form: **Sequential Rationalizability** ($\text{SR}$) (similar to (but weaker than) Pearce’s (1984) **Extensive Form Rationalizability**)
Do WY’s results (uniqueness /(non-)robustness) extend to dynamic environments? How?

- one possibility is to maintain the same *richness condition* for the (reduced) normal form of the game and modify (normal fom)-rationalizability introducing *trembles* (see WY)
  - PROBLEM: are tremble-based solution concepts still refinements of such modified rationalizability?

- alternatively: weaken the *richness condition*, and exploit the extensive form: **Sequential Rationalizability** (*SR*) (similar to (but weaker than) Pearce’s (1984) *Extensive Form Rationalizability*)

- **MAIN RESULT**: *SR* is the *strongest* solution concept that is *robust* to perturbations of higher order beliefs.
The Model: Finite Multistage Games with Observable Actions

Dynamic Bayesian Games

\[ \Gamma^T = \langle N, \mathcal{H}, \mathcal{Z}, \Theta^*, (A_i, T_i, u_i)_{i \in N} \rangle \]

- **Type Space:**
  \[ \mathcal{T} = \langle (T_i, \tau_i)_{i \in N} \rangle \]
  s.t. \( \tau_i : T_i \rightarrow \Delta (\Theta^* \times T_{-i}) \) (each type in a type space represents an infinite hierarchy of beliefs)

- **Universal Type Space** (Mertens and Zamir, 1985):
  \[ \mathcal{T}^* = \langle (T_i^*, \tau_i^*)_{i \in N} \rangle \]
  where \( T^* \) is the set of all (collectively coherent) profiles of hierarchies generated by \( \Theta^* \)

- **payoffs:** \( u_i : \mathcal{Z} \times \Theta^* \times T \rightarrow \mathbb{R} \)

- **interim (reduced form) strategies:** \( s_i \in S_i \) s.t. \( s_i : \mathcal{H} \rightarrow A_i \) and \( s_i(h) \in A_i(h) \)

- **outcome function:** \( O : S \rightarrow \mathcal{Z} \)
Beliefs: Conditional Probability Systems [CPS] (Myerson, 1986)

\[ \mu^i \in \Delta^H (\Theta^* \times T_{-i} \times S_{-i}) \] such that:

1) for each \( h \in H \), \( \mu^i (\cdot | h) \in \Delta (\Theta^* \times T_{-i} \times S_{-i}) \)

2) \( \int_{\Theta^* \times T_{-i} \times S_{-i}(h)} d\mu^i (\cdot | h) = 1 \)

3) Consistent with Bayesian Updating whenever possible
Beliefs: Conditional Probability Systems [CPS] (Myerson, 1986)
\(\mu^i \in \Delta^\mathcal{H} (\Theta^* \times T_{-i} \times S_{-i})\) such that:
1) for each \(h \in \mathcal{H}\), \(\mu^i (\cdot | h) \in \Delta (\Theta^* \times T_{-i} \times S_{-i})\)
2) \(\int_{\Theta^* \times T_{-i} \times S_{-i}} d\mu^i (\cdot | h) = 1\)
3) Consistent with Bayesian Updating whenever possible

- **Sequential Best Responses**
  \(s_i \in r_i (\mu^i | t_i)\) if and only if \(\forall h \in \mathcal{H} (s_i)\)
  \(s_i\) maximizes over \(s_i \in S_i (h)\)
  \(\int_{\Theta_0 \times T_{-i} \times S_{-i}} u_i (O (s_i, s_{-i}), \theta^*, t_{-i}, t_i) d\mu^i (\cdot | h)\)
Beliefs: Conditional Probability Systems [CPS] (Myerson, 1986)

\[ \mu^i \in \Delta^H (\Theta^* \times T_{-i} \times S_{-i}) \] such that:

1) for each \( h \in \mathcal{H} \), \( \mu^i (\cdot|h) \in \Delta (\Theta^* \times T_{-i} \times S_{-i}) \)

2) \( \int_{\Theta^* \times T_{-i} \times S_{-i}(h)} d\mu^i (\cdot|h) = 1 \)

3) Consistent with Bayesian Updating whenever possible

- **Sequential Best Responses**
  
  \( s_i \in r_i (\mu^i|t_i) \) if and only if \( \forall h \in \mathcal{H} (s_i) \)

  \( s_i \) is a best response for \( t_i \) to \( \mu^i (\cdot|h) \)
Beliefs: Conditional Probability Systems [CPS] (Myerson, 1986)

\[ \mu^i \in \Delta^\mathcal{H} (\Theta^* \times T_{-i} \times S_{-i}) \] such that:

1) for each \( h \in \mathcal{H} \), \( \mu^i (\cdot | h) \in \Delta (\Theta^* \times T_{-i} \times S_{-i}) \)

2) \( \int_{\Theta^* \times T_{-i} \times S_{-i}} (h) \ d\mu^i (\cdot | h) = 1 \)

3) Consistent with Bayesian Updating whenever possible

- **Sequential Best Responses**
  
  \( s_i \in r_i (\mu^i | t_i) \) if and only if \( \forall h \in \mathcal{H} (s_i) \)

  \( s_i \) is a best response for \( t_i \) to \( \mu^i (\cdot | h) \)

- \( s_i \in S_i \) is **sequentially rational** for \( t_i \), if there exist a conjecture \( \mu^i \in \Delta^\mathcal{H} (\Theta \times T_{-i} \times S_{-i}) \) s.t. \( s_i \in r_i (\mu^i | t_i) \) and \( mrg_{\Theta \times T_{-i}} \mu^i (\cdot | \phi) = \tau_{t_i} \).
Sequential Rationalizability

**Definition (SR)**

For each $i$ and $t_i \in T_i$, let $\mathcal{SR}_i^{T,0}(t_i) = S_i$.

Define recursively, for $k = 1, 2, \ldots$, and $t_i \in T_i$

$$\mathcal{SR}_i^{T,k}(t_i) = \begin{cases} \hat{s}_i \in \mathcal{SR}_i^{T,k-1}(t_i) : & (1) \hat{s}_i \text{ is sequentially rational for } t_i \text{ w.r.t. some CPS } \mu^i \\ & (2) \mu^i(t_{-i}, s_i|\phi) = 0 \text{ if } s_i \notin \mathcal{SR}_{-i}^{T,k-1}(t_{-i}) \end{cases}$$

**Sequential Rationalizability** is defined as the set of pairs $(t, s) \in \mathcal{SR}^T = \bigcap_{k \geq 0} \mathcal{SR}^{T,k}$
Sequential Rationalizability

\[ SR = \{a_1, a_2\} \times \{b_1, b_2\} \]

\[ EFR = \{a_2, b_1\} \]

\[
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 \\
-1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
-2
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
-2
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
-1
\end{pmatrix}
\]
Sensitivity to Higher Order Beliefs: The Richness Condition

\[
\begin{pmatrix}
1 \\
1 \\
\end{pmatrix}
\quad
\begin{pmatrix}
2 \\
-1 \\
\end{pmatrix}
\quad
\begin{pmatrix}
0 \\
-2 \\
\end{pmatrix}
\quad
\begin{pmatrix}
0 \\
-2 \\
\end{pmatrix}
\quad
\begin{pmatrix}
0 \\
-1 \\
\end{pmatrix}
\]
Sensitivity to Higher Order Beliefs: The Richness Condition

\[
\begin{align*}
&\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&\begin{pmatrix} 2 \\ -1+10 \end{pmatrix} \\
&\begin{pmatrix} 0 \\ -2 \end{pmatrix} \\
&\begin{pmatrix} 0 \\ -2+10 \end{pmatrix} \\
&\begin{pmatrix} 0 \\ -1 \end{pmatrix}
\end{align*}
\]
Richness Condition: for each player strategy $s_i$ there exists a payoff state $\theta^{s_i}$ s.t. for all $s_{-i} \in S_{-i}$, $s_i$ is the unique strategy s.t.: for all histories that can be reached by $s_i$, $s_i|_h$ is best response to $s_{-i}|_h$. ($s_i$ is sequentially dominant)
**Richness Condition:** for each player strategy $s_i$ there exists a payoff state $\theta^{s_i}$ s.t. $s_i$ is *sequentially dominant*.

**Theorem**

*Under the richness condition, for any finite types profile $\hat{t} \in T^*$ and any $s \in SR(\hat{t})$, there exists a sequence of finite types profiles $\hat{t}^m \rightarrow \hat{t}$ and s.t. $SR(\hat{t}^m) = \{s\}$ for each $m$ (the convergence $\hat{t}^m \rightarrow \hat{t}$ is in the product topology)*
\[
\theta = (\theta_1, \theta'_1, \theta_2) \in \Theta^* = \{\theta^*, \theta^{a_3}, \theta'^{a_3}, \theta^{a_2b_1}\} \\
\theta^* = (0, 0, 0) \\
\theta^{a_3} = (0, 3, 0) \\
\theta'^{a_3} = (0, 4, 0) \\
\theta^{a_2b_1} = (2, 0, 2) \\
\theta^* = (0, 0, 0) \\
\text{let } t^* \text{ denote CK of } \theta^* \\
S^R(t^*) = \{a_1, a_2\} \times \{b_1, b_2\}
\]
Suppose that 1 observes the true state \( \theta \in \Theta = \{ \theta^*, \theta'^{\alpha_3}, \theta^{\alpha_3} \} \) with prior \( (\frac{1-2\epsilon}{2}, \epsilon, \frac{1}{2}) \). Fix \( \epsilon \in (0, \frac{1}{6}) \).

If 1 observes \( \theta^* \) or \( \theta'^{\alpha_3} \), the email game starts, with messages delivered with prob \( p \in (0, \frac{\epsilon}{1-2\epsilon}) \) at each round.

- Suppose that 1 observes the true state \( \theta \in \Theta = \{ \theta^*, \theta'^{\alpha_3}, \theta^{\alpha_3} \} \) with prior \( (\frac{1-2\epsilon}{2}, \epsilon, \frac{1}{2}) \). Fix \( \epsilon \in (0, \frac{1}{6}) \).
- If 1 observes \( \theta^* \) or \( \theta'^{\alpha_3} \), the email game starts, with messages delivered with prob \( p \in (0, \frac{\epsilon}{1-2\epsilon}) \) at each round.

\[
t^m \rightarrow t^* \text{ s.t. } \mathcal{SR}(t^m) = \{a_1 b_2\} \text{ for each } m
\]
Example

\[ t^m \rightarrow t^* \text{ s.t. } \mathcal{S}(t^m) = \{a_2b_1\} \text{ for each } m \]

**email game:**
suppose that 1 observes the true state 
\( \theta \in \Theta = \{\theta^*, \theta^{a_2b_1}\} \) s.t.
\( \theta^* = (0, 0, 0) \)
and \( \theta^{a_2b_1} = (2, 0, 2) \)
If he observes \( \theta^* \), the email game starts, with messages delivered with prob \( \frac{1}{2} \) at each round.
Theorem (Structure of $\mathcal{SR}$)

Under the richness assumption, the set

$$U = \{ t \in T^* : \#(\mathcal{SR}(t)) = 1 \}$$

is open and dense in $T^*$. Moreover, the unique outcome is locally constant. If $\#(\mathcal{SR}_i(t_i)) = n$, then $t_i$ is on the boundary of $n$ open sets in which each of the $n$ strategies is uniquely $\mathcal{SR}$.

Corollary

Generic Uniqueness of any Perfect-Bayesian Equilibrium outcome.

Corollary

Non-Robustness of PBE-predictions to perturbations of higher order beliefs.
• **Upper Hemicontinuity:** the $SR$-correspondance is u.h.c in $t$. 
Other Results

- **Upper Hemicontinuity**: the $\mathcal{SR}$-correspondance is u.h.c in $t$.
- **Type Space-Invariance**: if two types $t_i$ and $t'_i$ in two (possibly different) type-spaces induce the same hierarchy of beliefs over $\Theta$, $\mathcal{SR}_i(t_i) = \mathcal{SR}_i(t'_i)$ (only hierarchies matter)
Other Results

• **Upper Hemicontinuity:** the $\mathcal{SR}$-correspondance is u.h.c in $t$.

• **Type Space-Invariance:** if two types $t_i$ and $t'_i$ in two (possibly different) type-spaces induce the same hierarchy of beliefs over $\Theta$, $\mathcal{SR}_i(t_i) = \mathcal{SR}_i(t'_i)$ (only hierarchies matter)

• **Alternative Characterization:** (for generic games) $\mathcal{SR}$ is equivalent to the Dekel-Fudenberg (1991) procedure(*) applied to the *interim normal form* of the game. (*): one round of deletion of weakly dominated strategies followed by iterated deletion of (strictly) dominated strategies)
In writing down a game (a model), we generally make simplifying assumptions about higher order beliefs. It is important then that the implications of the analysis are robust to possible misspecifications of these. (Robustness = u.h.c.)
Conclusion

In writing down a game (a model), we generally make simplifying assumptions about higher order beliefs. It is important then that the implications of the analysis are robust to possible misspecifications of these. (*Robustness = u.h.c.*)

- **Robustness**: suppose that uncertainty is *rich* (i.e.: *richness condition*). What solution concept to use, if you’re concerned with this problem of robustness? (ANSWER: **Sequential Rationalizability**: it is the strongest u.h.c. solution concept)
Perturbations of beliefs:

- Do higher order beliefs play the same role in dynamic environments that they play in static ones?
- What kind of perturbations to beliefs deliver the familiar results in dynamic environments?
Perturbations of beliefs:

- Do higher order beliefs play the same role in dynamic environments that they play in static ones?
- What kind of perturbations to beliefs deliver the familiar results in dynamic environments?

In the literature on Dynamic Global Games (DGG), beliefs are perturbed at each stage (more like a repeated global game). Here, perturbations are once and for all (more similar to reputation models).
Perturbations of beliefs:

- Do higher order beliefs play the same role in dynamic environments that they play in static ones?
- What kind of perturbations to beliefs deliver the familiar results in dynamic environments?

The results of this paper suggest that nothing inherent to dynamic environments prevents higher order beliefs from having the familiar impact on multiplicity. The "unusual" results from DGG are rather due to the nature of the perturbations that have been used.