Unit root tests when the data are a trigonometric transformation of an integrated process

Chien-Ho Wang¹, Robert de Jong²

¹Department of Economics
National Taipei University

²Department of Economics
Ohio State University

ESNASM, Pittsburgh, 2008
Outline

1 Motivation
   - Nonlinear time series
   - Previous literature

2 Our contribution
   - The model
   - Central idea
   - Consistency
   - Asymptotic normality
   - Test consistency

3 Further research
   - Further research
Outline

1 Motivation
   • Nonlinear time series
   • Previous literature

2 Our contribution
   • The model
   • Central idea
   • Consistency
   • Asymptotic normality
   • Test consistency

3 Further research
   • Further research
Central motivation

- Nonlinear models are often used in modeling economic relationships like inflation rate, and exchange rate.
- When we use nonlinear transformations for time series variables, the characteristics of original data series may be changed.
- This problems are serious if the original data series are integrated processes.
Therefore we investigate the transformed I(1) series and obtain the asymptotic properties of Dickey-Fuller test under different functional forms.

The basic model:

\[ T(x_t) = \rho T(x_{t-1}) + u_t, \]

where \( x_t \) is I(1) process and \( T(.) \) are periodic transformations.
Outline

1. Motivation
   - Nonlinear time series
   - Previous literature

2. Our contribution
   - The model
   - Central idea
   - Consistency
   - Asymptotic normality
   - Test consistency

3. Further research
   - Further research

Chien-Ho Wang, Robert de Jong
Unit root tests when the data are a trigonometric transformation
2. Franses and McAleer (1998); Kobayashi and McAleer (1999)
4. de Jong (2001)
Some mathematical transformations

The mathematical transformations discussed in early literatures:

1. Integrable functions
2. Asymptotically homogeneous functions: polynomial functions
3. Explosive functions: $\exp(x)$
4. Periodic functions: Fourier transformations
Outline

1 Motivation
   - Nonlinear time series
   - Previous literature

2 Our contribution
   - The model
     - Central idea
     - Consistency
     - Asymptotic normality
     - Test consistency

3 Further research
   - Further research

Chien-Ho Wang, Robert de Jong
Unit root tests when the data are a trigonometric transformation
Our models under consideration are

\begin{equation}
\sin(x_t) = \rho \sin(x_{t-1}) + u_t \tag{1}
\end{equation}

\begin{equation}
\sin(x_t) = \mu + \rho \sin(x_{t-1}) + u_t \tag{2}
\end{equation}

\begin{equation}
\cos(x_t) = \rho \cos(x_{t-1}) + u_t \tag{3}
\end{equation}

\begin{equation}
\cos(x_t) = \mu + \rho \cos(x_{t-1}) + u_t \tag{4}
\end{equation}

where \( x_t = x_{t-1} + \epsilon_t, \ \epsilon_t \sim iid(0, \sigma^2) \)
Outline

1. Motivation
   - Nonlinear time series
   - Previous literature

2. Our contribution
   - The model
   - Central idea
   - Consistency
   - Asymptotic normality
   - Test consistency

3. Further research
   - Further research
From Theorem 2 of de Jong (2001),

\[ n^{-1/2} \sum_{t=1}^{n} T(x_t) \xrightarrow{p} (2\pi)^{-1} \int_{-\pi}^{\pi} T(x) \, dx \]

where \( T(.) \) are periodic transformations, and

\[ n^{-1} \sum_{t=1}^{n} \sin(x_t) \cos(x_t) \xrightarrow{p} 0 \]
Central idea

For cross-product item like

\[ n^{-1/2} \sum_{t=1}^{n} \sin(x_t) \sin(x_{t-1}) \]

Use the addition formulas

\[ \sin(x_{t-1} + \epsilon_t) = \sin(x_{t-1}) \cos(\epsilon_t) + \cos(x_{t-1}) \sin(\epsilon_t) \]
\[ \cos(x_{t-1} + \epsilon_t) = \cos(x_{t-1}) \cos(\epsilon_t) - \sin(x_{t-1}) \sin(\epsilon_t) \]
Outline

1. Motivation
   - Nonlinear time series
   - Previous literature

2. Our contribution
   - The model
   - Central idea
   - Consistency
     - Asymptotic normality
     - Test consistency

3. Further research
   - Further research
The convergence behavior of $\hat{\rho}$

If $y_t = T(x_t)$ and $\bar{y} = 1/T \sum_{t=1}^{T} T(x_t)$, OLS estimators of $\hat{\rho}$ and $\hat{\rho}_{\mu}$ are

$$\hat{\rho} = \frac{n^{-1} \sum_{t=2}^{n} y_t y_{t-1}}{n^{-1} \sum_{t=2}^{n} y_{t-1}^2}$$

and

$$\hat{\rho}_{\mu} = \frac{n^{-1} \sum_{t=2}^{n} (y_t - \bar{y})(y_{t-1} - \bar{y})}{n^{-1} \sum_{t=2}^{n} (y_{t-1} - \bar{y})^2}$$

The asymptotic behaviors of $\hat{\rho}$ and $\hat{\rho}_{\mu}$ are

$$\hat{\rho} \xrightarrow{p} E \cos(\epsilon_t) \quad \text{and} \quad \hat{\rho}_{\mu} \xrightarrow{p} E \cos(\epsilon_t),$$
Motivation

- Nonlinear time series
- Previous literature

Our contribution

- The model
- Central idea
- Consistency
- Asymptotic normality
- Test consistency

Further research

- Further research

Outline

1 Motivation

2 Our contribution

3 Further research

Chien-Ho Wang, Robert de Jong

Unit root tests when the data are a trigonometric transformation
If the $\epsilon_t$ has a symmetric distribution, the OLS regression coefficients $\hat{\rho}$ and $\hat{\rho}_\mu$ satisfy

$$n^{1/2}(\hat{\rho} - E \cos(\epsilon_t)) \xrightarrow{d} N(0, V)$$

$$n^{1/2}(\hat{\rho}_\mu - E \cos(\epsilon_t)) \xrightarrow{d} N(0, V),$$

where

$$V = \left(\frac{3}{8}\right)E(\cos(\epsilon_t) - E \cos(\epsilon_t))^2 + \left(\frac{1}{8}\right)E(\sin(\epsilon_t))^2.$$
Outline

1. Motivation
   - Nonlinear time series
   - Previous literature

2. Our contribution
   - The model
   - Central idea
   - Consistency
   - Asymptotic normality
   - Test consistency

3. Further research
   - Further research
Dickey-Fuller t test is defined as below:

$$\hat{t} = \frac{\hat{\rho} - 1}{\hat{\sigma}_{\rho}}$$

The asymptotic behavior of Dickey-Fuller t-statistic is

$$n^{-1/2}\hat{t} \xrightarrow{p} (E \cos(\epsilon_t) - 1)(1 - (E \cos(\epsilon_t))^2)^{1/2}$$
Outline

1. Motivation
   - Nonlinear time series
   - Previous literature

2. Our contribution
   - The model
   - Central idea
   - Consistency
   - Asymptotic normality
   - Test consistency

3. Further research
   - Further research

Chien-Ho Wang, Robert de Jong
Unit root tests when the data are a trigonometric transformation
Further research

1. Applications in empirical economics: Do not find suitable example for economic applications this time.
2. Periodic transformations for cointegrated variables
3. Different estimation methods for Periodic transformations of I(1) process.