Forecasting with the Term Structure
The Role of No-Arbitrage

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Principle of no-arbitrage is powerful tool for cross-sectional asset pricing

Can it be useful in the time series?

- Term structure contains information helpful for forecasting
  - Future term structures
  - Macroeconomic conditions
- Does imposing no-arbitrage restrictions on the term structure improve forecast accuracy?
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The standard view

- Imposing no-arb restrictions improves efficiency if true; can lower quality of forecasts if misspecified
- In practice, appears to help
  - Forecasting future yields (Duffee 2002)
  - Joint macro and TS forecasting (Ang and Piazzesi 2003)
My claim: reverse the standard view

If confident in the no-arb restrictions, do not bother imposing them; only useful if not confident about the model

- Relative to VAR, affine no-arb models offer
  1. Dimension reduction + factor dynamics

  But affine no-arb not unique in this

- Duffie-Kan restrictions
  1. Result 1: if no-arb model is true, only (1) is relevant empirically
  2. Result 2: If no-arb model is false, empirical magnitude of rejection of Duffie-Kan restrictions can help detect misspecification in (1)
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The (ir)relevance of no-arbitrage: theory

- The standard no-arb model
  1. State vector $x_t$
  2. Short rate is affine function of state vector
  3. Equivalent martingale dynamics of state in affine class (drift, conditional covariance both affine in $x_t$)
  4. Result: $y_t^{(m)} = A(m; \text{params}) + B(m; \text{params})'x_t$

- “Unrestricted” model

  $y_t^{(m)} = A_m + B_m'x_t$, $A_m, B_m$ unrestricted
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- “Unrestricted” model
  $$y^{(m)}_t = A_m + B'_m x_t, \quad A_m, B_m \text{ unrestricted}$$
Why no-arbitrage is unimportant

1. Both restricted and unrestricted models imply length-$n$ state vector can be rotated into yields on $n$ bonds

$$y_t^{(m)} = a_m + b'_m \begin{pmatrix} y_t^{(\tau_1)} \\ \vdots \\ y_t^{(\tau_n)} \end{pmatrix}$$

2. Duffie-Kan imposes restrictions on $a_m$, $b_m$; unrestricted model does not

3. But taking model literally, OLS uncovers $a_m$, $b_m$ without estimation error ($R^2 = 1$)

4. “Measurement error” breaks perfect link, but in practice for $n = 3$, $R^2$s are so high (0.998-0.999) that estimation error is too small to matter
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Overview of details in paper

- Previous argument made more precise using Gaussian model
  1. 3-factor essentially affine Gaussian model
  2. 3-factor “unrestricted” Gaussian model
- Monte Carlo simulations to evaluate effect of no-arbitrage restriction
  - Use estimated model of (1) as truth
  - Estimate versions (1) and (2) on simulated 88 quarters of yields, compare RMSE of “out-of-sample” forecasts
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Quick summary of empirical and simulation results

• With actual data, models estimated with and without no-arb restrictions are economically indistinguishable

• In simulations, RMSE’s of out-of-sample forecasts are almost identical for two types of models

Differences between RMSE’s are less than 1/2 basis point for forecasts up to a year ahead
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A formal framework

- Length $n$ state vector $x_t$ with dynamics
  \[ x_{t+1} = \mu + Kx_t + \Sigma\epsilon_{t+1}, \quad \epsilon_{t+1} \sim MVN(0, I) \]
- Relation between yields and state vector
  \[ y^{(m)}_t = A_m + B'_m x_t + \eta_{m,t}, \quad \eta_{m,t} \sim N(0, \sigma^2_\eta) \]
- $A_m, B_m$ unrestricted if not imposing Duffie-Kan
- $\eta_{m,t}$ breaks stochastic singularity of $n$ factors and more than $n$ bond yields
Imposing no-arbitrage restrictions

• Duffie-Kan restrictions replace $A_m, B_m$ with functions that depend on equiv-martingale params

$$A_m = \mathbb{A}(m; \delta_0, \delta_1, \mu^q, K^q, \Sigma)$$

$$B'_m = \mathbb{B}(m; \delta_1, K^q)'$$

• Restrictions assume bond prices are only source of cost, payoff to bondholders

• Alternative assumption: no-arb holds, but value of bonds to investors not necessarily just resale price

These effects on yields are unobserved, affine in $x_t$
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Econometric interpretation of no-arb restrictions

- In $n$-factor no-arb version of model, $A_m$ and $B_m$ for $n + 1$ bonds are unrestricted. They pin down short rate and properties of $n$ prices of risk.
- Each additional bond included in estimation adds $1 + n$ overidentifying restrictions (mean, factor loadings).
- Restrictions are those of law of one price + linear setting.
- Not studied here is effect of additional restrictions on prices of risk (e.g., risk-neutrality, consumption-based asset pricing, constant prices of risk).

Restrict relation of initial $n + 1$ bond yields to state vector.
Estimation and hypothesis tests

Two methods

1. Estimate both models with ML (Kalman filter)

measurement eq. \[ y_t = A + Bx_t + \eta_t, \quad \eta_t \sim MVN(0, \sigma^2_\eta I). \]

transition eq. \[ x_{t+1} = Dx_t + \Sigma \epsilon_{t+1}, \quad \epsilon_{t+1} \sim MVN(0, I), \quad D \text{ diagonal} \]

Either impose restrictions on \( A, B \) or not; use LR test of null hypothesis that restrictions are true.

In practice, this is how I estimate both models.
2. Rewrite unrestricted model to express overidentifying restrictions as specific parameters

- Pick $n + 1$ of the bonds (exact), use their unrestricted $A_m, B_m$ to infer equiv-martingale parameters
- Write unrestricted $A_m, B_m$ for remaining bonds (overident) as sum of piece consistent with above $n + 1$ bonds and a “no-arb error” piece

$$\begin{pmatrix} y_t^x \\ y_t^y \end{pmatrix} = \begin{pmatrix} A(\ldots) \\ A(\ldots) + c_0 \end{pmatrix} + \begin{pmatrix} B(\ldots) \\ B(\ldots) + C_1 \end{pmatrix} x_t + \eta_t$$

Vector $c_0$, matrix $C_1$ are fixed at zero with Duffie-Kan; free params in unrestricted model

- Estimates of “no-arb errors” params imply fitted yields deviate from no-arb by only a few basis points
- Duffie-Kan restrictions strongly rejected statistically (red lines are +/- 2 std errors)
Is it ever useful to consider no-arbitrage?

1. If we want to construct forecasts with a parsimonious model of risk compensation (e.g., consumption betas), need no-arb model to do this
   - Resulting model could produce more accurate forecasts; helps pin down physical dynamics of state vector with information in cross-section of \( n + 1 \) bond yields

2. Can help detect misspecification of unrestricted model
   Differences between restricted, unrestricted model must be due to convenience yields—are magnitudes economically sensible?
A two-factor example

- Model fit to same data as previous models
- Estimates of “no-arb errors” params imply fitted yields deviate from no-arb by up to 27 basis points
  Note – this is different from measurement error
- “Convenience yield” in 1, 2 yr bonds rises with level
Where to go from here?

- Simulation evidence of unimportance of no-arb restriction is only from Gaussian case
  - Logic seems to apply to more general affine settings, but am I missing something?
- Extension to more general case also can help us understand why TS models with time-varying volatility work poorly
  - Models with time-varying volatility have restrictions placed on factor dynamics, both physical and equivalent-martingale
  - Unrestricted model here removes latter restrictions but not former – is that enough to allow the models to forecast successfully?