The Forward Solution for Linear Rational Expectations Model

(Solution Refinement based on The Forward Method)

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Class of Linear Rational Expectations (RE) Models

\[ x_t = a E_t x_{t+1} + b x_{t-1} + c z_t \]

Classes of solutions (REE)

- **Fundamental solutions**
  - Depend on Minimal set of State Variables (MSV)

- **Non-fundamental** (Bubble or Sunspot) solutions
  - Typically depend on additional lagged state variables +Bubble

We discuss the fundamental solutions.
Issue

Standard Solution Methods for RE models
- Blanchard-Kahn(80),
- QZ method (Sims, McCallum, Klein, Uhlig..)

Problem:
1. Potentially multiple stationary fundamental REEs.
   - Examples: New-Keynesian models, Increasing Returns models
2. These methods are silent about which one is the sensible solution.
Previous Works and the Goal of this paper

1. MSV Criterion (McCallum (1983))
   - Unique by construction
   - May not coincide with the “unique solution” (McCallum (04))

2. E-Stability Criterion (Evans and Honkapojha (2001))
   - May be not unique.
   - Depends on specific learning process

➢ We propose a Solution Refinement based on the
   Forward Method of Recursive Substitution
In univariate models without predetermined variable, the forward solution is unique if it exists.

In multivariate models with predetermined variables,

(Q1) Can we apply the forward method?

(Q2) Is the forward solution unique if it exists?

The answers are “Yes”
Traditional Forward Method (FM)

Model:  
\[ x_t = aE_t x_{t+1} + z_t, \quad z_t = \rho z_{t-1} + \varepsilon_t, \quad E_{t-1} \varepsilon_t = 0 \]

Forward Representation “implied by” the model
\[ x_t = a^k E_t x_{t+k} + \gamma_k z_t, \quad \gamma_k = \sum_{i=0}^{k-1} (a \rho)^i \]

Two key concepts in FM
1. **FCC** (Forward Convergence Condition) holds if the coefficients of the state variables converge:
   \[ \gamma^* = \lim_{k \to \infty} \gamma_k \]: FCC is a Model Property

2. **NBC** (No bubble condition) (TVC or boundary condition) holds if expected endogenous variable converges to 0:
   \[ \lim_{k \to \infty} a^k E_t x_{t+k} = 0: \text{NBC is Solution-dependent.} \]
Definition of the Forward Solution (FS)

Model: \[ x_t = aE_t x_{t+1} + z_t, \quad z_t = \rho z_{t-1} + \varepsilon_t, \quad E_{t-1} \varepsilon_t = 0 \]

\[ x_t = a^k E_t x_{t+k} + \gamma_k z_t, \quad \gamma_k = \sum_{i=0}^{k-1} (a \rho)^i \]

Forward Solution: \[ x_t = \gamma^* z_t, \quad \gamma^* = \lim_{k \to \infty} \gamma_k \]

The FS exists if the model satisfies FCC (Here \(|a \rho| < 1\))

- Claim: The FS satisfies NBC, and it is the only one satisfying NBC, among ALL solutions

→ NBC as a Solution Refinement
**Standard Solution Method**

- **Model (Ex: Cagan)**
  \[ x_t = aE_t x_{t+1} + z_t, \quad z_t = \rho z_{t-1} + \varepsilon_t, \quad E_{t-1} \varepsilon_t = 0 \]

- **Guess and verify**:
  \[ x_t = \gamma z_t, \quad \gamma = 1 / (1 - a \rho) \]

  - This is a stationary solution even when \(|a \rho| > 1\)

- **This solution with** \(|a \rho| > 1\) **is dismissed as a legitimate solution**
  - because it violates bounded-ness of exogenous variables in forward representation (Blanchard(79), Woodford(2001))
  - In our terminology, the model violates FCC, and the proposed solution violates NBC

  \[ FCC: \quad \gamma_k = \sum_{i=0}^{k-1} (a \rho)^i, \quad NBC: \quad a^k E_t x_{t+k} = a^k \gamma \rho^k z_t \]

- **Guess-Verify method may lead to a solution that violates NBC.**
(Multivariate) Model with Predetermined Variables

- Model: \( x_t = aE_t x_{t+1} + bx_{t-1} + z_t, \quad z_t = \rho z_{t-1} + \varepsilon_t, \quad E_t^{-1} \varepsilon_t = 0 \)

- Fundamental Class of Solutions

\[
x_t = \omega x_{t-1} + \gamma z_t,
\]

where \((\omega, \gamma)\) must satisfy

\[
\omega = (I - a\omega)^{-1} b,
\]

\[
\gamma = (I - a\omega)^{-1} (I + a\gamma \rho)
\]
Forward Method in General Models

Model: \( x_t = aE_tx_{t+1} + bx_{t-1} + z_t, \quad z_t = \rho z_{t-1} + \varepsilon_t, \quad E_{t-1}\varepsilon_t = 0 \)

Forward Representation, \((m_k, \omega_k, \gamma_k)\) for \(k \geq 1\) such that

\[
(2) \quad x_t = m_k E_t x_{t+k} + \omega_k x_{t-1} + \gamma_k z_t
\]

Where \((m_1, \omega_1, \gamma_1) = (a, b, I)\), and for \(k \geq 2\)

\[
m_k = (I - a\omega_{k-1})^{-1} am_{k-1}
\]

\[
\omega_k = (I - a\omega_{k-1})^{-1} b
\]

\[
\gamma_k = (I - a\omega_{k-1})^{-1}(I + a\gamma_{k-1}\rho)
\]

FCC: \(\exists \omega^* = \lim_{k \to \infty} \omega_k, \text{ and } \gamma^* = \lim_{k \to \infty} \gamma_k\)

NBC: \(\lim_{k \to \infty} m_k E_t x_{t+k} = 0\)
Forward Solution

Forward Solution under FCC

\[ x_t = \omega^* x_{t-1} + \gamma^* z_t \]

**Proposition**  Consider a Model (1),

1. The FS exists if the model satisfies FCC.
2. The FS is the unique fundamental solution that satisfies NBC.
3. For any other solution, either fundamental, or non-fundamental, the NBC is violated.
Corollary

1. Under FCC, the expectational term is finite but not equal to zero for any other solution

2. When FCC does not hold, the expectational term either explodes or oscillates for any solution.
NBC, a Solution Refinement

(1) When the FCC holds

- Any other fundamental REE, \( x_t = \omega' x_{t-1} + \gamma' z_t \) must solve
  \[
  x_t = \lim_{k \to \infty} m_k E_t x_{t+k} + \omega^* x_{t-1} + \gamma^* z_t
  \]

- NBC: \( \lim_{k \to \infty} m_k E_t x_{t+k} = C x_t + D z_t \neq 0 \)

where C and D are complicated constants.

\( \Rightarrow \) NBC is a natural refinement scheme for fundamental REEs

- For any REE, the terminal condition is a very complicated function of current variables.
- Difficult to justify this particular terminal condition economically
NBC, a Solution Refinement

(2) When the FCC does not hold, the FS does not exist.

Even in this case, one may obtain an REE. (e.g., Cagan)

But, for any REE, the expectational term does not converge.

\[ x_t = \lim_{k \to \infty} m_k E_t x_{t+k} + \lim_{k \to \infty} \omega_k x_{t-1} + \lim_{k \to \infty} \gamma_k z_t \]

- Difficult to justify non-convergent terminal condition
- This solution can be eliminated à la Blanchard and Woodford
- To do so, one must apply FM and examine NBC.

NBC is a natural refinement scheme for fundamental REEs
## Summary

<table>
<thead>
<tr>
<th>Model’s FCC</th>
<th>Existence of Fundamental Solutions</th>
<th>NBC</th>
</tr>
</thead>
<tbody>
<tr>
<td>YES</td>
<td>Forward</td>
<td>YES</td>
</tr>
<tr>
<td></td>
<td>Others</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Possible</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ex 1: Evans-Honkaphoja(2001)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ex 2: McCallum(2004)</td>
<td>NO</td>
</tr>
<tr>
<td>NO</td>
<td>Others</td>
<td>NO</td>
</tr>
<tr>
<td></td>
<td>Possible</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ex 3: Cagan, Our NK Model</td>
<td></td>
</tr>
</tbody>
</table>
To Conclude,

1. The FS exists under the FCC.
2. The FS is unique by construction if it exists.
3. The FS is a fundamental solution.
4. The FM is a solution method itself.
5. The FCC and NBC are economically sensible, therefore, NBC can serve as a solution refinement.

Remark: Non-fundamental solutions are designed Not to satisfy the NBC.
EX 1. Multiple solutions can pass E-stability. Only one of them is the FS.

**Dornbusch model of Evans and Honkapohja(2001)**

\[ p_t = p_{t-1} + \pi d_t \]
\[ d_t = -\gamma (r_t - E_t p_{t+1} + p_t) + \eta (e_t - p_t) \]
\[ r_t = \lambda^{-1} (p_t - \delta p_{t-1}) \]
\[ e_t = E_t e_{t+1} - r_t \]
\[ \Rightarrow p_t = \beta_1 E_t p_{t+1} + \beta_2 E_{t+2} + \delta p_{t-1} \]

- The model can have three stable solutions. \( p_t = \omega p_{t-1} \)
- The first and third solutions pass their E-stability conditions.
- The FS exists and it coincides with the first solution.
- The third solution violates NBC.

**Remark:** The concept of E-stability differs across different representations of a given model and a solution.
EX 2. Unique stationary solution = Forward solution ≠ Solution by MSV

- *McCallum (2004)*’s example that the unique “model” solution is not equal to the solution via MSV.

\[
x_t = \begin{bmatrix} -0.4 & 0.01 \\ 0.02 & -1.5 \end{bmatrix} E_t x_{t+1} + \begin{bmatrix} 1.2 & 0.02 \\ 0.01 & 1.6 \end{bmatrix} x_{t-1} + \epsilon_t
\]

- G-eigenvalues are (-3.4463, 1.0551, -0.8275, 0.1610)

- There is unique stationary solution associated with (-0.8275, 0.1610) : This is the forward solution.
  - The solution via MSV is associated with (1.0551, 0.1610)
Case 3. FCC violated but the MSV solution exists and it violates the NBC.

A standard New-Keynesian macro model with AS, IS and forward-looking Taylor rule.

\[
\pi_t = \delta_1 E_t \pi_{t+1} + \delta_2 \pi_{t-1} + \kappa y_t + v_t
\]
\[
y_t = \mu_1 E_t y_{t+1} + \mu_2 y_{t-1} - (i_t - E_t \pi_{i+1}) + u_t
\]
\[
i_t = (1 + \beta) E_t \pi_{t+1} + \lambda y_t
\]

Substituting the MP rule out yields two equation system.

\[
\pi_t = \delta_1 E_t \pi_{t+1} + \delta_2 \pi_{t-1} + \kappa y_t + v_t
\]
\[
y_t = \mu'_1 E_t y_{t+1} + \mu'_2 y_{t-1} - \beta'E_t \pi_{t+1} + u'_t
\]

Assumption: \( \kappa = 0 \) \( \Rightarrow \) Closed form solution exists.
Case 3 (Continued)

- Under certain parameters, there may be multiple solutions.
- The QZ solutions \((g_1, g_2, g_3, g_4) = (0.9231, 1, 0.7611, 0.8614)\)

\[
\omega(g_1, g_3) = \begin{bmatrix} 0.9231 & 0 \\ 2.2860 & 0.7611 \end{bmatrix}, \quad \omega(g_1, g_4) = \begin{bmatrix} 0.9231 & 0 \\ 0.8712 & 0.8614 \end{bmatrix}
\]

* \(\omega(g_1, g_2)\) violates the rank condition. (So it is not an REE.)
- The Forward solution does not exist. (FCC violated)

<table>
<thead>
<tr>
<th>(k)</th>
<th>(\omega_{11})</th>
<th>(\omega_{12})</th>
<th>(\omega_{21})</th>
<th>(\omega_{22})</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.9161</td>
<td>-6.2866</td>
<td>0</td>
<td>0.7589</td>
</tr>
<tr>
<td>70</td>
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<td>0</td>
<td>0.7611</td>
</tr>
<tr>
<td>100</td>
<td>0.9231</td>
<td>-988.5229</td>
<td>0</td>
<td>0.7611</td>
</tr>
</tbody>
</table>

MSV solution is \(\omega(g_1, g_3)\), violating NBC.
Graphical Representation (Univariate)

Recall 

\[(m_1, \omega_1, \gamma_1) = (a, b, 1)\]

\[m_k = (1 - a \omega_{k-1})^{-1} am_{k-1}\]

\[\omega_k = (1 - a \omega_{k-1})^{-1} b\]

\[\gamma_k = (1 - a \omega_{k-1})^{-1}\]

\[\text{Regularity condition (Part of FCC)}\]

\[v_k = 1 - a \omega_k\]

\[\omega_k = v_{k-1}^{-1} b\]

\[v_{k+1} = 1 - \theta / v_k, \quad v_1 = 1 - \theta, \text{ where } \theta = ab\]

\[\text{Essence: } v_1 \text{ is given by the Model}\]
Figure 1: Existence of the FS

* It can be shown that $v_1 > v(1) > v(2)$
Figure 2: Non-existence of the FS
Panel A: Regularity condition holds
Panel B: Regularity condition is violated.
Practical Summary

1. Given the model, \[ x_t = aE_t x_{t+1} + bx_{t-1} + cz_t \]

2. Construct sequences \((\omega_k, \gamma_k)\)

3. If \((\omega_k, \gamma_k)\) converge (FCC) and \(\omega^*\) is stationary, then, the forward solution, \[ x_t = \omega^* x_{t-1} + \gamma^* z_t \]
is the unique, stationary solution satisfying NBC.