Random Matching and Aggregate Uncertainty

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Random matching models

What is random matching?

Explore foundations and economic implications
Existence results

Nonuniqueness problem

Growth model example

- Two approaches to nonuniqueness:
  - No aggregate uncertainty
  - Selection criterion (entropy)
Hardy-Weinberg (Matching and Types in Genetics) (1908)

Doob, Judd (Non-measurability) (1953, 1985)

Duffie-Sun (Solve Measurability Problem) (2007)

Boylan (Deterministic Models of Countable Sets of Agents) (1991)

Alós-Ferrer (Dynamical System with a Continuum of Agents) (1999)

Aliprantis-Camera-Puzzello (2007)
\[ A = \{a_1, a_2, b_1, b_2\} \]

Consider matches

<table>
<thead>
<tr>
<th>Match</th>
<th>Representation</th>
<th>Probability</th>
<th>( Q_1 )</th>
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<tr>
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Both \( Q_1 \) and \( Q_2 \) satisfy proportionality

Convex combinations of distributions generate more
SOME NOTATION

- Measure spaces:
  Agents: \((A, \mathcal{L}, \mu)\)
  Matches: \((\Phi, \mathcal{F}, Q)\)

- Types and type function:
  \(S = \{\tau_1, \ldots, \tau_L\}\)
  \(\tau: A \to S\) (measurable)

- Proportion of type \(k\) agents:
  \[P_k = \lim_{n \to \infty} \left[ \frac{\mu\{g \in A_n : \tau(g) = \tau_k\}}{\mu\{g \in A_n\}} \right]\]
  where \(A = \bigcup A_n\) with \(A_n \subset A_{n+1}\)
  (e.g., \(A = \mathbb{N}, A_n = \{1, 2, \ldots, n\}\), \(\mu =\) counting measure)
Random Matching Scheme satisfies Proportionality Laws if:

**M1 Measure Preserving** \( \mu(E) = \mu(\varphi(E)) \) for all measurable \( E \)

**M2 Proportional Law** \( Q \{ \varphi \mid \tau(\varphi(g)) = \tau_I \} = P_I \) for a.e. \( g \in A \)

**M3 Quasi-independence** \( \lim_{n \to \infty} \frac{\mu\{g \in A_n | \tau(g) = \tau_i \text{ and } \tau(\varphi(g)) = \tau_j \}}{\mu(A_n)} = P_i P_j \) for a.e. \( \varphi \in \Phi \)
Existence and nonuniqueness:

- Finite Populations
- Countable Populations
- Continuum Populations
\begin{itemize}
  \item $A = \{a_1, a_2, b_1, b_2\}$
  \item Consider matches

\begin{tabular}{|c|c|c|c|c|}
  \hline
  Match & Representation & Probability & $Q_1$ & $Q_2$ \\
  \hline
  $\varphi_1$ & $(a_1, a_2)(b_1, b_2)$ & $Q(\varphi_1)$ & $\frac{1}{2}$ & $\frac{1}{2}$ \\
  $\varphi_2$ & $(a_1, b_1)(a_2, b_2)$ & $Q(\varphi_2)$ & $\frac{1}{4}$ & $\frac{1}{2}$ \\
  $\varphi_3$ & $(a_1, b_2)(a_2, b_1)$ & $Q(\varphi_3)$ & $\frac{1}{4}$ & 0 \\
  \hline
\end{tabular}

  \item Both $Q_1$ and $Q_2$ satisfy proportionality
  \item Convex combinations of distributions generate more
  \item Observe: uniform does not satisfy proportionality
\end{itemize}
WHY DO WE CARE ABOUT NONUNIQUENESS?

- Growth model
  - $k(t)$ capital available at start of period $t$
  - $c(t)$ consumption at the end of period $t$ (fraction with $0 \leq c(t) \leq 1$)
  - $z(t)$ random shock (i.i.d. random variables with $0 \leq z(t) \leq 1$)
  - $F(k(t))$ production function

- Updating rule
  \[
  k(t + 1) = k(t) + z(t)F(k(t)) - c(t)z(t)F(k(t)).
  \]
Shock comes from a random matching of agents with complementary resources

- Agents of two types, \( a \) and \( b \). \( A = \{a_1, \ldots, a_N, b_1, \ldots, b_N\} \)
- Capital \( k(t) = (k_{a_1}(t), k_{a_2}(t), \ldots, k_{a_N}(t), k_{b_1}(t), \ldots, k_{b_N}(t)) \). Typical component \( k_g(t) \)
- Consumption vector \( c(t) \). Typical component \( c_g(t) \)
- Utility of consumption, \( u(c_g(t)F_g(\cdot)) \)
- Production only when opposite types meet
- Probability of match \( \varphi \in \Phi \) is \( Q(\varphi) \)
- Production function (vector) \( F(k(t), \varphi) \) with production for agent \( g \) given by \( F_g(k(t), \varphi) \)
- Updating rule

\[
k_g(t+1) = k_g(t) + F_g(k(t), \varphi) - c_g(t)F_g(k(t), \varphi)
\]
REWARDS AND TRANSITION

- Production function

\[ F_g(k(t), \varphi) = \begin{cases} 
  f \left( \min [k_g(t), k_{\varphi(g)}(t)] \right) & \text{if } g \text{ and } \varphi(g) \text{ have different types} \\
  0 & \text{if } g \text{ and } \varphi(g) \text{ have the same type}
\end{cases} \]

with \( f(0) = 0 \)
Maximize over all policies, $\pi$, total expected (discounted) utility of consumption over an infinite time horizon.

$$V_\pi = E_\pi \left[ \sum_t \beta^t \left( \sum_g u(c_g(t)F_g(k(t),*)) \right) \right]$$

Policy, $\pi$, will simply select a fraction, $c$, of production for consumption.

Bellman Equation

$$V(k) = \max_c \{ R(k, c) + \beta \sum P_{k\tilde{k}}(c)V(\tilde{k}) \}$$

Assume matching satisfies proportionality laws.

Assume distribution of capital. Both types hold 0 or 1 with equal probability.
Given a probability distribution $Q$ on the set of possible matches

Expected reward depends explicitly on matching distribution

$$R(k, c) = \sum_{g} \sum_{\varphi} Q(\varphi) u(c_g(t) F_g(k, \varphi))$$

The transition probabilities are computed in a similar way

Dynamic behavior of the process depends on the matching rule

Expected rewards and transition probabilities cannot be computed using proportionality conditions alone.
Take $N = 4$ so we have 8 agents; 4 of type $a$ and 4 of type $b$.

Two distinct probability distributions satisfying the same proportionality laws

**Distribution I**

- $(a_1, a_2)(a_3, b_3)(a_4, b_4)(b_1, b_2)$ - .5
- $(a_1, b_1)(a_2, b_2)(a_3, a_4)(b_3, b_4)$ - .5

**Distribution II**

- $(a_1, a_2)(a_3, b_1)(a_4, b_2)(b_3, b_4)$ - .25
- $(a_1, b_1)(a_2, b_2)(a_3, a_4)(b_3, b_4)$ - .25
- $(a_1, b_3)(a_2, b_4)(a_3, a_4)(b_1, b_2)$ - .25

**Distribution of Capital**

<table>
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<tr>
<th>Agent</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$b_1$</th>
<th>$b_2$</th>
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<tr>
<td>Capital</td>
<td>1</td>
<td>1</td>
<td>0</td>
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Distribution I: There will be no consumption since there can be no production.

Distribution II: \((a_1, b_3)\) and \((a_2, b_4)\) production occurs. Utility of consumption will be positive.

Both cases proportionality laws are satisfied.

Dynamics are different.

Example can be generalized.
HOW TO DEAL WITH NONUNIQUENESS?

Two approaches:

1. Formulate the model so that nonuniqueness is not an issue
   Sufficient Condition for no aggregate uncertainty

2. Provide a selection criterion for matching distribution
   Natural candidate: maximal entropy
AGGREGATE UNCERTAINTY

- Def. The reward structure of the model contains *no aggregate uncertainty* if

\[ E_Q[R(s, u)] = E_{\tilde{Q}}[R(s, u)] \]

whenever \( Q \) and \( \tilde{Q} \) satisfy the same Proportionality Laws. Similarly, the transition structure of the model contains *no aggregate uncertainty* if

\[ \text{Prob}_Q(s(t + 1, *) | s(t, *), u) = \text{Prob}_{\tilde{Q}}(s(t + 1, *) | s(t, *), u) \]

whenever \( Q \) and \( \tilde{Q} \) satisfy the same Proportionality Laws. (If both, the model contains no aggregate uncertainty.)

- Sufficient conditions: reward function and transition probability become deterministic

- Nonuniqueness not an issue
Rewards depends on factors other than types

Use entropy in some cases

The entropy for a discrete distribution of the random variable $X$, is given by

$$\text{Prob}(X = k) = p_k$$

$$H(X) = - \sum p_k \log(p_k)$$

Select the distribution that maximizes entropy subject to model constraints
A = \{a_1, a_2, b_1, b_2\}

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M1, M2 hold for both

\(Q_1\) has maximal entropy

Finite populations: can show existence and uniqueness (concave programming)

Not so intuitive with infinite populations
CONCLUSIONS

- Foundations of random matching (proportionality)
- Existence and nonuniqueness
- Economic implications of nonuniqueness
- Approaches to deal with nonuniqueness
Further work on entropy for infinite populations is needed

Can we assume no aggregate uncertainty?

Relationship between matching and optimality

Applications to monetary theory and experimental economics