Modeling Expectations with Noncausal Autoregressions

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Consider the AR($p$) process

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \epsilon_t \quad \text{or} \quad \phi(B) y_t = \epsilon_t$$

where $B$ is the usual backshift operator and the error term $\epsilon_t$ is iid($0, \sigma^2$).

If the autoregressive polynomial $\phi(z)$ has all roots outside the unit circle, $y_t$ can be written as

$$y_t = \sum_{j=0}^{\infty} \alpha_j \epsilon_{t-j},$$

i.e., the autoregression is causal (or backward-looking).
Noncausal Autoregression

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Noncausal Autoregression

- Conversely, if \( \phi(z) \) has all roots inside the unit circle, we get

\[
y_t = \sum_{j=0}^{\infty} \beta_j \epsilon_{t+j},
\]

i.e., the autoregression is *purely noncausal* (or forward-looking).

- Finally, if \( \phi(z) \) has roots both outside and inside the unit circle, we have

\[
y_t = \sum_{j=-\infty}^{\infty} \psi_j \epsilon_{t-j},
\]

i.e., the autoregression is *mixed*. 
Conversely, if $\phi(z)$ has all roots inside the unit circle, we get

$$y_t = \sum_{j=0}^{\infty} \beta_j \epsilon_{t+j},$$

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$$y_t = \sum_{j=-\infty}^{\infty} \psi_j \epsilon_{t-j},$$

i.e., the autoregression is mixed.
Noncausal Autoregression

- The presence of causality and noncausality can be interpreted in terms of backward-looking and forward-looking dynamics, which may be of interest in studying expectations.
- Even if \( y_t \) is noncausal, its autocovariance function is indistinguishable from that of a causal AR process.
- Nonidentifiability of causal and noncausal models also appears in Gaussian maximum likelihood estimation.
  - As normality has typically been assumed, it has become a common practice to restrict the parameter space to the causal region.
- In economic applications, the residuals often exhibit deviations from normality, suggesting that alternative distributional assumptions might be preferable. Under nonnormality, it becomes possible to distinguish between causal and noncausal AR models.
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- In economic applications, the residuals often exhibit deviations from normality, suggesting that alternative distributional assumptions might be preferable. Under nonnormality, it becomes possible to distinguish between causal and noncausal AR models.
The previous literature on noncausal AR and related models is relatively small.

- Breidt, Davis, Lii and Rosenblatt (1991)
- Lii and Rosenblatt (1996)
- Huang and Pawitan (2000)
- Rosenblatt (2000)
- Breidt, Davis and Trindade (2001)
- Andrews, Davis and Breidt (2006)

Most of the applications so far are confined to natural sciences and engineering.
Outline

- Noncausal AR Model
  - Economic Interpretation
  - Estimation and Inference
  - Model Selection
  - Simulation Study
  - Empirical Application
  - Conclusion
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Noncausal AR Model

• We write the noncausal AR\((r, s)\) model as

\[
\varphi (B^{-1}) \varphi (B) y_t = \epsilon_t, \quad \epsilon_t \sim iid \ (0, \sigma^2),
\]

where

\[
\varphi (z) = 1 - \phi_1 z - \cdots - \phi_r z^r \neq 0, \quad |z| \leq 1
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and

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\varphi (z) = 1 - \varphi_1 z - \cdots - \varphi_s z^s \neq 0, \quad |z| \leq 1.
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• This formulation differs somewhat from that presented in the previous literature, but it lends itself to more straightforward economic interpretation and has certain statistical advantages.
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Consider, for simplicity, the purely noncausal AR\((0, s)\) model

\[ y_t = \phi_1 y_{t+1} + \cdots + \phi_s y_{t+s} + \epsilon_t. \]

Taking conditional expectation given \(\{y_t, y_{t-1}, \ldots\}\) yields

\[ y_t = \phi_1 E_t (y_{t+1}) + \cdots + \phi_s E_t (y_{t+s}) + E_t (\epsilon_t). \]

- \(y_t\) is affected by expectations of its future values and of the error term \(\epsilon_t\).

Also,

\[ E_t (y_{t+1}) = \phi_1 E_t (y_{t+2}) + \cdots + \phi_s E_t (y_{t+s+1}) + E_t (\epsilon_{t+1}), \]

so that the next period’s expectation is affected by the expected future error.
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Economic Interpretation

- The moving average representation of the AR(0, s) process

\[ y_t = \sum_{j=0}^{\infty} \beta_j \epsilon_{t+j} \]

shows that \( y_t \) is determined by future errors.

- Leading the MA(\( \infty \)) representation and taking conditional expectation yields

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showing the dependence of the expectation of \( y_t \) on expected future errors.

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We propose an approximate maximum likelihood (ML) estimator for the AR\((r, s)\) model and derive the asymptotic properties of this estimator and related tests.

- Under regularity conditions, the ML estimator is asymptotically normal, so the likelihood ratio (LR) and Wald tests have the conventional \(\chi^2\) limiting null distributions.

- The estimator is a slight generalization of Breidt et al. (1991) because we allow the error distribution to depend on an additional parameter \(\lambda\) which facilitates considering a wide variety of alternative distributions.
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Estimation and Inference

- We propose an approximate maximum likelihood (ML) estimator for the AR($r$, $s$) model and derive the asymptotic properties of this estimator and related tests.
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Based on our simulation study and findings in Breidt et al. (1991), we suggest the following model selection procedure.

1. Fit an adequate causal AR($p$) model by Gaussian ML and check for nonnormality of the residuals. If they are not normal, choose a non-Gaussian error distribution. If they look normal, stop here.

2. Estimate all AR($r, s$) models with $r + s = p$, and select the one that maximizes the likelihood function.

3. Check the specification by standard diagnostic tools.
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To examine the finite-sample properties of the ML estimator, related tests and the model selection procedure, we consider AR(1,1) models

\[(1 - \phi_1 B^{-1}) (1 - \phi_1 B) y_t = \epsilon_t,\]
\[\epsilon_t \sim iid \ t(3), \quad E (\epsilon_t^2) = 0.01.\]

- Parameter values: \((\phi_1, \phi_1) = (0.9, 0.9), (0.9, 0.1), (0.1, 0.9)\)
- Sample sizes: \(T = 100, 200, 500\)
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Simulation Study

Findings

- The estimators perform reasonably in terms of bias and standard deviation even for $T = 100$.
- The Wald test tends to overreject even with $T = 500$, while the LR test has good size properties already with $T = 200$, with slight overrejection when $T = 100$. The LR test also has reasonable power properties.
- The model selection procedure performs well for $(\varphi_1, \phi_1) = (0.9, 0.9)$ and $T \geq 200$ (the correct model is selected at least 95% of the time). In the other two cases, the corresponding percentages are 67% ($T = 200$) and 85% ($T = 500$).
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According to typical New Keynesian models, inflation is purely forward-looking, which suggests a noncausal univariate inflation process.

In previous univariate analyses of causal AR models evidence of persistence has been found that has been interpreted as there being a large backward-looking component in inflation.

However, strong persistence, measured by high autocorrelation, need not indicate backward-looking dynamics, but it may actually follow from agents’ forward-looking expectations.

Allowing for noncausality facilitates testing whether inflation dynamics has been backward-looking or forward-looking.
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U.S. Inflation Data

- Annualized quarterly inflation rate computed from the seasonally adjusted U.S. consumer price index (for all urban consumers) published by the Bureau of Labor Statistics.
- The sample period is 1970:1 to 2006:4 (148 observations).
- By visual inspection and unit root tests, the series seems stationary.
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Results

- Of causal Gaussian models, the AR(3) model (AR(3)-N) was selected by conventional methods.
- The Q-Q plot of the residuals indicates nonnormality. In particular, a more leptokurtic distribution, such as the $t$-distribution with a small degree-of-freedom parameter might be suitable.
- By the proposed model selection procedure, a purely noncausal AR(0,3)-$t$ model was found adequate.
- Hence, the results indicate purely forward-looking inflation dynamics.
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<thead>
<tr>
<th></th>
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<th>AR(3,0)-t</th>
<th>AR(2,1)-t</th>
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<td>(0.953)</td>
<td>(0.726)</td>
<td>(0.723)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ljung-Box (4)</td>
<td>0.570</td>
<td>0.494</td>
<td>0.001</td>
<td>0.001</td>
<td>0.278</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.004</td>
<td>0.004</td>
<td>0.057</td>
<td>0.281</td>
<td>0.873</td>
<td></td>
</tr>
</tbody>
</table>
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According to simulation results, the proposed methodology seems to work reasonably well.

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- Applications of noncausal AR models to financial returns; presumably requires extensions that can capture the conditional heteroskedasticity present in these series.
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