Endogenous Growth and Endogenous Business Cycles in an Overlapping Generations Economy with Credit Market Imperfections∗

Takuma Kunieda

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Affiliation: Department of Economics, Brown University.

Address: 64 Waterman Street, Providence, Rhode Island 02912, USA.

Phone: (401) 863-3836.

Fax: (401) 863-1970.

E-mail: Takuma_Kunieda@brown.edu

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Abstract

We study the dynamic properties of growth rates in an overlapping generations economy with credit market imperfections, constructing a Schumpeterian growth model. The analysis demonstrates that: (i) two steady-state equilibria arise as is usual in overlapping generations models with outside money and the growth rate of each increases as credit market imperfections are resolved; (ii) if credit market imperfections are severe or soft and if sunspots do not appear, the economy converges monotonically to a stable steady state; and (iii) if credit market imperfections are moderate, deterministic cycles or chaos would arise in equilibrium.

Keywords: Endogenous growth; Endogenous business cycles; Credit market imperfections; Heterogeneous agents; Chaotic dynamics.

JEL Classification Numbers: E22, E23, O41.
1 Introduction

Endogenous business fluctuations have been studied with overlapping generations models for over twenty years. The dynamic properties of exchange economies have been investigated in a traditional strand initiated by Benhabib and Day [8] and Grandmont [23]. They have paid attention to deterministic cycles or chaotic behavior in equilibrium. In relation to them, Azariadis and Guesnerie [6] and Guesneire [21] studied sunspot equilibria of exchange economies with overlapping generations. In another strand initiated by Diamond [13], the dynamic properties have been studied of overlapping generations economies with production. For instance, Galor and Ryder [19] investigated conditions for the stability of steady-state equilibria. Galor [18] developed a model with two production sectors and made clear the conditions for indeterminacy of equilibrium. Farmer [15], Reichlin [35], Benhabib and Laroque [9], and Rochon and Polemarchakis [36] derived competitive equilibrium cycles.

Meanwhile, sophisticated endogenous growth models have been developed by many researchers since Romer [37], although each of them cannot be cited here. In particular, since Aghion and Howitt’s [2] pioneering work, many have focused on the research and development (R&D) activities of private firms and have developed Schumpeterian growth models. As pointed out by Aghion and Howitt [3], however, productivity growth and the business cycles have been studied separately for several decades.

The objective of this paper is to investigate the dynamic property of growth rates in an overlapping generations economy which faces credit market imperfections, by constructing a Schumpeterian growth model. Although the basic structure of the current model follows traditional models such as Farmer [15] or Benhabib and Laroque [9], few papers in the traditional strand have dealt with credit market imperfections or have taken the Schumpeterian approach. To the best of my knowledge, this paper is the first in that the Schumpeterian approach (combined with credit market imperfections) is incorporated in an overlapping generations model with outside money.

In our model, we assume heterogeneous agents within a generation in terms of their productivities in creating input goods for a R&D sector. This is a different assumption than the traditional approach. Usually, it is difficult to construct a tractable model with credit constraints in a closed economy where
interest rates are determined endogenously. By incorporation of heterogeneous agents into the current economy, the model comes to be tractable in handling credit market imperfections. In our model, each agent can make a deposit in or borrow from an infinitely-lived financial intermediary, which we call the “Bank” following Grandmont [22]. If the productivity of an agent is greater than a cut-off point, he invests in durable goods in order to create the input goods, borrowing from the Bank up to the limit of a credit constraint. Meanwhile, if the productivity of an agent is lower than the cut-off point, he only deposits money in the Bank. Since all of the financial trades are executed via the financial market, each agent unconsciously makes financial transactions with other agents intra- or inter-generationally.

Both economic growth and business cycles are important issues in macroeconomics. In understanding these two macroeconomic phenomena, many researchers have recently emphasized the importance of credit market imperfections. In the literature of economic growth, Galor and Zeira [20] studied the effects of wealth distribution on economic growth, whereas in the literature of business cycles, Kiyotaki and Moore [29] developed a general equilibrium model with credit market imperfections. In the current paper, we investigate the effect of credit market imperfections both on growth rates and on business cycles.

The main findings are as follows. First, two steady-state equilibria arise as is usual in overlapping generations models with outside money. The growth rate of each increases as credit market imperfections are resolved. This finding is consistent with empirical evidence of King and Levine [27,28] and Levine, et al. [30]. Second, equilibrium is indeterminate. Therefore, whenever sunspots arise, the economy fluctuates due to extrinsic stochastic beliefs. While this is also a common property of overlapping generations models with outside money, the economy converges monotonically to a stable steady state if credit market imperfections are severe or soft and if sunspots do not appear. Lastly, if credit market imperfections are moderate, deterministic cycles or chaos would arise in equilibrium.

We employ a Cobb-Douglas utility function. Its merit is not only in the tractability but also in the property. With a Cobb-Douglas utility function, the income and substitution effects of the returns to savings are canceled by each other. In our model, chaotic equilibria would appear for some parameter values. In overlapping generations models, the possibility of chaotic equilibria arising
is much higher than in the models of an infinitely-lived representative agent. However, in almost all the articles which deal with overlapping generations models, the chaotic dynamics is attributed to the elasticity of savings with respect to interest rates. Since we employ a Cobb-Douglas utility function, the savings are not affected by interest rates. Our results are new from a theoretical point of view in the sense that the chaotic dynamics is due to the moderate degree of credit constraints, not to the elasticity of savings with respect to interest rates.

The paper is organized as follows. The model is provided in the next section. In section 3, we study equilibrium, deriving a growth rate function with respect to a cut-off point of agents’ productivities, which divides agents into less capable and more capable agents. We investigate the dynamic property of the economy in section 4. In section 5, we give a discussion about why cycles and chaos appear in equilibrium. In section 6, we execute a numerical analysis and observe that it is when credit market imperfections are moderate that the economy exhibits deterministic cycles or chaos in equilibrium. Section 7 gives concluding remarks.

2 Model

2.1 Individuals

The economy consists of overlapping generations: young and old. Time is discrete and expands from $-\infty$ to $\infty$. Each individual lives for two periods and maximizes his lifetime utility, which is given by:

$$c_{1t}(\phi)\gamma c_{2t+1}(\phi)^{1-\gamma}, \quad 0 < \gamma < 1$$

(1)

where $c_{1t}(\phi)$ and $c_{2t+1}(\phi)$ are consumption in the first and the second periods of his lifetime, respectively. The index $\phi$ expresses the heterogeneity of individuals. As will be explained later, $\phi$ also represents productivity of an individual.

Each individual is born endowed with one unit of labor. When an individual is young, he supplies one unit of labor to an intermediate firm to earn the wages $w_t$, which are spent for consumption, investments or deposits. The budget constraint in the first period is as follows:

$$c_{1t}(\phi) + k_t(\phi) + b_t(\phi) \leq w_t,$$

(2)

1See Benhabib and Day [8] or Grandmont [23] for chaotic equilibria in overlapping generations models and see Nishimura, et al. [34] for chaotic equilibria in the models of an infinitely-lived representative agent.
where \( k_t(\phi) \) is the investments in durable goods and \( b_t(\phi) \) is deposits when positive or debts when negative. He has two kinds of transfer methods of his wealth from the first period of his life to the second period. One is to make a deposit in the Bank. The other is to invest his wealth in durable goods in order to create input goods for a R&D sector.

In our economy, each individual can borrow against future income by selling bonds to the Bank. While he can deposit his income in the Bank as much as he wants to, he can borrow from the Bank only up to some proportion of the investments. He faces a credit constraint. In other words, the following condition for \( b_t(\phi) \) is imposed:

\[
b_t(\phi) \geq -\mu k_t(\phi), \quad 0 \leq \mu < 1.
\]

(3)

We can give a microfoundation for (3). (3) is derived from the incentive compatibility constraints of borrowers and the optimal monitoring behavior of the Bank.\(^2\) We ground the explanation for the credit constraints which are expressed by (3) on asymmetric information. Borrowers never default in equilibrium in this setting.

The budget constraint an individual faces in the second period is given by:

\[
c_{t+1}(\phi) \leq p_{t+1} z_{t+1}(\phi) + r_{t+1} b_t(\phi),
\]

(4)

where \( z_{t+1}(\phi) \) is the input goods for the R&D sector. \( z_{t+1}(\phi) \) is created from \( k_t(\phi) \) which is the investments in the first period. It takes one gestation period to create the input goods. \( p_{t+1} \) is the (real) price of the input goods and \( r_{t+1} \) is the gross interest rate for the deposits or borrowing. Income in the second period is spent only for consumption. The production function of \( z_{t+1}(\phi) \) is given by:

\[
z_{t+1}(\phi) = \phi k_t(\phi),
\]

(5)

where \( \phi \) is the productivity parameter of an individual, as mentioned before. We note that \( \phi \) is different between individuals. \( \phi \) has a distribution, \( G(\phi) \), whose support is \([0, a]\) where \( a \in R_{++} \) and whose density is given by \( G'(\phi) := g(\phi) \). We impose some assumptions on the distribution.

**Assumption 1**

- \( G(\phi) \) is continuously differentiable.

\(^2\)We derive (3), modifying the model given by Aghion, Banergee, and Piketty [1]. The derivation for (3) is available upon request.
\[ \int_0^\infty \phi dG(\phi) < \infty. \]

- For all \( \phi \in [0, a] \), \( g(\phi) > 0 \).

The productivity of each agent is private information, i.e., the third party does not know about his productivity. Accordingly, no one can directly ask another individual whose productivity is greater than his to make input goods for him.

Each individual maximizes his lifetime utility function subject to (2)-(5) and

\[ k_t(\phi) \geq 0. \] (6)

From the first-order conditions, we obtain a lemma as follows:

**Lemma 1**

- If \( r_{t+1} > \phi p_{t+1} \), then \( k_t(\phi) = 0 \) and \( b_t(\phi) = (1 - \gamma)w_t \).
- If \( r_{t+1} < \phi p_{t+1} \), then \( k_t(\phi) = \frac{(1-\gamma)w_t}{1-\mu} \) and \( b_t(\phi) = -\frac{\mu(1-\gamma)}{1-\mu}w_t \).

**Proof:** See appendix.

Let us define \( \phi_t := \frac{r_t + 1}{p_t + 1} \). Then, \( \phi_t \) is a cut-off point at time \( t \). From lemma 1 we note that if the productivity of an agent \( \phi \) is greater than \( \phi_t \), he invests some proportion of his income in durable goods. If the productivity is less than \( \phi_t \), he does not invest in any project but deposits some proportion of his income in the Bank.

### 2.2 Final Production Sector

The general goods are produced from a continuum of intermediate goods, which is distributed uniformly in \([0, 1]\). The CES production function is assumed for the production of the general goods, which is given by:

\[ Y_t = \left[ \int_{i \in [0,1]} A^{1-\alpha}_{it} x^\alpha_{it} di \right]^{1/\epsilon}, \] (7)

where \( A_{it} \) is the quality of the \( i \)th intermediate good and \( x_{it} \) is its quantity. \( \epsilon := \frac{1}{1-\alpha} \) is the elasticity of substitution between input goods, which is greater than one since \( \alpha \in (0, 1) \).
The general good sector is competitive and the representative firm of the final good sector maximizes its (nominal) profits:

\[ p^d_t Y_t = \int_{i \in [0,1]} \hat{p}_{it} x_{it} di, \]  

(8)

where \( \hat{p}_{it} \) is the price of the intermediate good \( i \) and \( p^d_t \) is the price of the general goods. It follows that \( p^d_t = [\int_{i \in [0,1]} A_{it} \hat{p}_{it}^{1-\alpha} di]^{\frac{1}{1-\alpha}} \) in equilibrium. We can think of \( p^d_t \) as the price index at time \( t \). Let \( \tilde{p}_{it} := \hat{p}_{it} / p^d_t \). Then, all the prices are measured by the values of the general goods at each time, i.e., we take the general good as a numeraire at each point in time. From the first-order condition, we have an inverse demand function of the intermediate good \( i \):

\[ \tilde{p}_{it} = A_{it}^{1-\alpha} x_{it}^{\alpha} - w_t x_{it}, \]  

(9)

2.3 Intermediate Sector

The intermediate sector consists of a continuum of firms, which is distributed uniformly in \([0,1]\). This distribution is time-invariant because for the intermediate sector, new innovators come out into the market at each period. Due to the newly invented technologies, new innovators make monopolistic profits. The newly invented technologies may be protected by patents or it may take time for the technologies to be imitated. Therefore, the intermediate firms obtain monopolistic profits. The monopolistic profits, however, will disappear in one period since the next newly invented technologies are introduced into the market by other innovators after one period goes by.

The intermediate goods are produced from labor with a one-for-one technology, i.e., an intermediate firm needs one unit of labor to produce one unit of \( x_{it} \). An intermediate firm chooses \( x_{it} \) in order to maximize its profits, \( \pi_{it} \), by solving:

\[ \pi_{it} = \max_{x_{it}} [\tilde{p}_{it} x_{it} - w_t x_{it}] = \max_{x_{it}} [A_{it}^{1-\alpha} x_{it}^{\alpha} Y_{t}^{1-\alpha} - w_t x_{it}], \]  

(10)

where \( w_t \) is the wage rate. From the first-order condition, we have:

\[ w_t = \alpha A_{it}^{1-\alpha} x_{it}^{\alpha} Y_{t}^{1-\alpha}, \]  

(11)

and

\[ \pi_{it} = (1-\alpha) A_{it}^{1-\alpha} x_{it}^{\alpha} Y_{t}^{1-\alpha}. \]  

(12)
From (9) and (11), we note that $\tilde{p}_{it} = \frac{w_t}{\alpha}$. The prices of all the intermediate goods are the same at each point in time and the mark-up rate is equal to $\frac{1-\alpha}{\alpha}$.

An intermediate firm supplies the intermediate goods with up-to-date quality, which is developed in a R&D department within the firm.\(^3\) The quality of the intermediate goods is improved by the R&D activities. Following empirical evidence by Ha and Howitt [24], we assume a functional form of the quality improvement as follows:

$$\frac{A_{it+1} - A_{it}}{A_{it}} = \eta \left( \frac{z_{it+1}}{A_{it}^{\frac{1-\alpha}{\alpha}}} L_t \right)^{\sigma},$$

(13) where we assume that $0 < \sigma < \min\{1, \frac{\alpha}{1-\alpha}\}$.\(^4\) This assumption guarantees that the demand function of input goods slopes downward. $\eta$ is a productivity parameter of the R&D department of an intermediate firm, $z_{it+1}$ is the input goods for the R&D activities, and $L_t$ is the population of young agents at time $t$. The left hand side is the growth rate of the quality, which is determined by the right-hand side, i.e., the quality-adjusted input goods, $z_{it+1}/A_{it}^{\frac{1-\alpha}{\alpha}}$, and the population of young agents, $L_t$. Following Jones’ [26] critique, the right-hand side is adjusted for scale effects in three respects. First, as $A_{it}$ becomes big, it is difficult for the quality of the intermediate goods to be improved. We divide $z_{it+1}$ by $A_{it}^{\frac{1-\alpha}{\alpha}}$ in order to remove this kind of scale effect and obtain the quality-adjusted input goods. Second, we divide $z_{it+1}/A_{it}^{\frac{1-\alpha}{\alpha}}$ by $L_t$ to deal with a scale effect which comes from the increasing number of individuals. Finally, the production exhibits decreasing return to scale with respect to the input goods, i.e., $0 < \sigma < 1$.

Since each R&D department is in the patent race, the input goods devoted in the R&D activities are determined by the research-arbitrage condition:

$$\pi_{it+1} = p_{it+1} z_{it+1},$$

\(\footnote{As discussed in Aghion and Howitt [2], we may also assume that separate firms are engaged in the R&D activities. In this case, the firms in a R&D sector sell their newly invented technologies to the firms in an intermediate sector, and conclusions will be the same.}

\(\footnote{Using the data for the USA, the UK, France, Germany, and Japan, Ha and Howitt [24] study which model is suitable in reality: the first generation endogenous models, the semi-endogenous models, or the fully-endogenous models with product proliferation. They conclude that a functional form which is used in the fully-endogenous models with product proliferation is most plausible.}

\(\footnote{Although the current economy is closed and thus all the markets clear within the economy, we may allow technology transfers from abroad, by replacing (13) with an equation which expresses the rate of technology adoption as in Aghion, Howitt, and Mayer-Foulkes [4] and Howitt and Mayer-Foulkes [25].}
where \( p_{t+1} \) is the price of \( z_{it+1} \). Using (12), we rewrite this equation as:

\[
(1 - \alpha)A_{it+1}^{1-\alpha}x_{it+1}^\alpha Y_{t+1}^{1-\alpha} = p_{t+1}z_{it+1}.
\]

### 2.4 Governmental Agency: The Bank

The government establishes an intergenerational banking system, by which each agent makes financial transactions in the anonymous financial market. We call the intergenerational banking system simply the “Bank” following Grandmont [22]. This kind of assumption is made by many researchers.\(^6\)

Since young agents are allowed to borrow from the Bank, outside money may become negative. When outside money in an economy is negative, the government has the ownership of some portion of the economy’s stock of capital.

The Bank accommodates the supply of real liabilities to young agents, while following its budget constraint. Let \( M \) be the nominal quantity of outside money, which is assumed to be constant and can be negative or positive.\(^7\) Let \( B_t = \frac{M}{p^d_t} \). Then we have:

\[
p_{t+1}^d B_{t+1} = p_{t+1}^d B_t.
\]

If we put \( r_{t+1} = \frac{p_{t+1}^d}{p_{t+1}^d} \), we obtain the budget constraint of the Bank as follows:

\[
B_{t+1} = r_{t+1} B_t.
\]

The Bank limits its credit provision or liabilities to the private sector following (15). \( B_t \) is the total liabilities to young agents at time \( t \) and \( r_{t+1} \) is the interest rate. (15) also becomes a credit market clearing condition and is consistent with a good market clearing condition in equilibrium. To see this, we aggregate all variables at time \( t \) as follows:

\[
C_t + K_t + B_t = Y_t + r_t B_{t-1},
\]

where \( C_t \) is the aggregate consumption, \( K_t \) is the aggregate durable goods, \( B_t \) is the aggregate government debt and \( Y_t \) is the aggregate output. From the good market clearing condition, we have \( C_t + K_t = Y_t \). Therefore \( B_t = r_t B_{t-1} \) holds.

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\(^6\)For example, in Grandmont [22], Farmer [15], Benhabib and Laroque [9], Azariadis and Smith [7], and Rochon and Polemarchakis [36], young agents can consume more than their income by going into debt from the Bank.

\(^7\)The assumption that \( M \) is constant is unimportant. We may assume the constant growth of outside money.
If $B_{t-1} < 0$, old agents are net borrowers in total and they need to repay to the Bank. The repayment becomes the resources for lending at time $t$. Since $B_{t-1} < 0$, young agents at time $t$ are also net borrowers in total unless $r_t = 0$ in equilibrium.

### 3 Equilibrium

#### 3.1 Growth Rate

We assume away population growth of the economy: we assume that for all $t$, $L_t = 1$. The heterogeneity of the intermediate firms is assumed away as well: all the intermediate firms are symmetrical. Accordingly, from the labor market clearing condition, we have $\int x_{it}di = L_t = 1$. Hereafter, all the variables are independent of $i$. From these assumptions and (7), (11), and (14), we have the following equations:

\[
\begin{align*}
Y_t &= A_t^{\frac{1-\alpha}{\alpha}} \\
w_t &= \alpha A_t^{\frac{1-\alpha}{\alpha}} \\
p_t z_t &= (1-\alpha)A_t^{\frac{1-\alpha}{\alpha}}.
\end{align*}
\]

We note that $Y_t = w_t + p_t z_t$, namely the final output is distributed to the wages and the returns to the investments.

From lemma 1 and $w_t = \alpha A_t^{\frac{1-\alpha}{\alpha}}$, the total supply of the input goods for the R&D sector is given by:

\[
Z_{t+1} := \int_{\phi_t}^{\alpha} \frac{(1-\gamma)w_t}{1-\mu} L_t \phi dG(\phi)
\]

\[
\Leftrightarrow z_{t+1} = \int_{\phi_t}^{\alpha} \frac{(1-\gamma)\alpha A_t^{\frac{1-\alpha}{\alpha}}}{1-\mu} \phi dG(\phi)
\]

\[
\Leftrightarrow z_{t+1} = \frac{(1-\gamma)\alpha A_t^{\frac{1-\alpha}{\alpha}}}{1-\mu} F(\phi_t),
\]

(16)

where $F(\phi_t) = \int_{\phi_t}^{\alpha} \phi dG(\phi)$ and $z_{t+1} = Z_{t+1}/L_{t+1} = Z_{t+1}$ since $L_{t+1} = 1$. Given $\phi_t$, (16) is a supply function of input goods for a R&D sector.

Substituting (16) into (13), we can derive a growth rate as follows:

\[
\Gamma(\phi_t) := \left(\frac{A_{t+1}}{A_t}\right)^{\frac{1-\alpha}{\alpha}} - 1
\]

\[
= \left[\eta \left(\frac{(1-\gamma)\alpha}{1-\mu} F(\phi_t)\right)^{\sigma} + 1\right]^{\frac{1-\alpha}{\alpha}} - 1,
\]

(17)
The demand function of input goods is obtained by the research-arbitrage condition (14):

\[(1 - \alpha)A_{t+1}^{\frac{1-\alpha}{\sigma}} = p_{t+1}z_{t+1}.\] \tag{18}

Substituting (13) into (18), we have a demand function of input goods as follows:

\[p_{t+1} = (1 - \alpha)[\eta A_t^{1-\alpha}z_{t+1}^{-\frac{\sigma-\alpha}{\sigma}} + A_t z_{t+1}^{-\frac{\sigma-\alpha}{\sigma}}]^{\frac{1-\alpha}{\sigma}}.\] \tag{19}

We examine a temporary equilibrium in Figure 1 indicating the relationship between \(\phi_t\), \(z_{t+1}\), \(p_{t+1}\), and \(\Gamma(\phi_t)\). We note that \(\Gamma(\phi_t)\) is an decreasing function with \(\phi_t\). Since the savings of each agent are independent of the type of agent, the aggregate investment in producing input goods falls as the number of agents who produce the input goods decreases (i.e., as \(\phi_t\) increases). Therefore, the growth rate will fall as seen in figure 1. We also note that the price of the input goods is countercyclical to the growth rate. This is because \(p_{t+1}\) is increasing with \(\phi_t\).

[Figure 1 around here]

A recession of this economy is studied in figure 1. Suppose that \(\phi_t\) increases suddenly from a steady state \(\phi_{t-1}^0\) to \(\phi_i^j\) for some reason.\(^8\) Then, as we can see in figure 1, \(z\) decreases, the growth rate falls, and \(p\) increases. More importantly, since \(\phi_t\) goes up, borrowers with low productivities have to leave the market of the input goods and turn into lenders. Since both \(p\) and \(\phi_t\) go up, \(r\) increases as well. This means that other things being equal, the smaller is the growth rate, the greater is the trickle-down effect lenders experience in equilibrium.

Since the support of \(G(\phi)\) is \([0, a]\), the minimal growth rate is \(\Gamma_{\text{min}} = \left[\eta \left(\frac{1-\gamma}{1-\mu} F(a)\right)^\sigma + 1\right]^{\frac{1-\alpha}{\sigma}} - 1 = 0\), whereas the maximal growth rate is given by \(\Gamma_{\text{max}} = \left[\eta \left(\frac{1-\gamma}{1-\mu} F(0)\right)^\sigma + 1\right]^{\frac{1-\alpha}{\sigma}} - 1\). The larger is the mean of \(\phi\), the greater is the maximal growth rate. \(\Gamma : [0, a] \rightarrow [\Gamma_{\text{min}}, \Gamma_{\text{max}}]\) is a homeomorphism. Therefore, the dynamic property of the growth rate is deduced directly from that of the cut-off point.

\(^8\)This would be due to, say, pessimistic expectations for a real interest rate \(r_{t+1}\). In other words, consider the case in which people hold deflationary expectations.
3.2 Dynamics

The credit market clearing condition is given by:

\[ B_{t+1} = r_{t+1}B_t. \]  

(20)

From lemma 1 and \( w_t = \alpha A_t^{-\alpha} \), we obtain:

\[
B_t = \int_0^{\phi_t} (1 - \gamma)\alpha A_t^{1-\alpha} L_t dG(\phi) + \int_{\phi_t}^a -\mu(1 - \gamma)\alpha A_t^{1-\alpha} L_t dG(\phi) = (1 - \gamma)\alpha A_t^{1-\alpha} L_t dG(\phi) + \mu.
\]

(21)

Substituting (21) and \( r_{t+1} = p_{t+1}\phi_t = \frac{(1-\alpha)(1-\mu)\phi_t Y_{t+1}}{(1-\gamma)\alpha F(\phi_t) Y_t} \) into (20), we have:

\[
\frac{(1 - \gamma)\alpha Y_{t+1}}{1 - \mu} G(\phi t_{t+1}) = \frac{(1 - \alpha)(1 - \mu)\phi_t Y_{t+1}}{(1 - \gamma)\alpha F(\phi_t) Y_t} (1 - \gamma)\alpha Y_t [G(\phi_t) - \mu],
\]

which is reduced to a difference equation of the cut-off point \( \phi_t \) as follows:

\[
G(\phi t_{t+1}) = \frac{(1 - \alpha)(1 - \mu)\phi_t G(\phi_t) - \mu}{\alpha(1 - \gamma) F(\phi_t)} + \mu.
\]

(23)

Let us define a function as follows:

\[
\Psi(\phi) := \frac{(1 - \alpha)(1 - \mu)\phi G(\phi) - \mu}{\alpha(1 - \gamma) F(\phi)} + \mu.
\]

We note that the difference equation (23) is independent of \( \eta, \sigma \), and the level of the quality of input goods \( A_t \). The property of the dynamical system, \( G(\phi t_{t+1}) = \Psi(\phi_t) \), depends upon whether outside money of the economy is positive or negative.

3.3 Steady States

A steady-state equilibrium, \( \phi_t = \bar{\phi} \), solves the following equation:

\[
G(\bar{\phi}) = \frac{(1 - \alpha)(1 - \mu)\bar{\phi} G(\bar{\phi}) - \mu}{\alpha(1 - \gamma) F(\phi)} + \mu.
\]

(24)

We note that there exist two steady-state equilibria, \( \bar{\phi} = \phi^* \) and \( \bar{\phi} = \phi^{**} \), (unless they are repeated values), such that:

\[
G(\phi^*) = \mu
\]

(25)
\[ \frac{\phi^{**}}{F(\phi^{**})} = \frac{\alpha(1 - \gamma)}{(1 - \alpha)(1 - \mu)}, \] 

respectively.

At the steady-state equilibrium with \( \bar{\phi} = \phi^* \), the individuals’ net credit position is always zero in total, whereas at the steady-state equilibrium with \( \bar{\phi} = \phi^{**} \), their net credit position is positive or negative, i.e., the steady-state equilibrium is supported by positive or negative outside money. Gale [17] and Grandmont [22] categorized an economy into two classes. An economy in which the steady-state equilibrium with \( \bar{\phi} = \phi^{**} \) is supported by positive outside money is called the Samuelson case, whereas an economy in which the steady-state equilibrium with \( \bar{\phi} = \phi^{**} \) is supported by negative outside money is called the classical case.

Looking at (25) and (26), we note that if \( \mu = 0 \), then we obtain \( \phi^* = 0 \) and \( \phi^{**} > 0 \). Therefore, by continuity, if \( \mu \) is sufficiently close to zero, the economy is Samuelson. We give an example which demonstrates the boundary between the Samuelson and the classical economies in the following.

**Example 1**: Suppose \( \phi \sim U(0,1) \). In this case, \( G(\phi^*) = \phi^* = \mu \) and \( \frac{\phi^{**}}{F(\phi^{**})} = \frac{2\phi^{**}}{1 - \phi^{**}} = \frac{\alpha(1 - \gamma)}{(1 - \alpha)(1 - \mu)}. \) The economy is Samuelson (classical) if and only if \( \frac{\phi^*}{F(\phi^*)} < (>) \frac{\phi^{**}}{F(\phi^{**})} \), which can be rewritten as:

\[ \frac{2\mu}{1 - \mu^2} < (>) \frac{\alpha(1 - \gamma)}{(1 - \alpha)(1 - \mu)} \]

\[ \iff \ (\mu + 1)(\gamma - \frac{3\alpha - 2}{\alpha}) < (>) \frac{2(1 - \alpha)}{\alpha}. \] 

(27)

Figure 2 gives areas of \((\mu, \gamma)\) in the case of \( \alpha < \frac{1}{4} \), where a Samuelson case or a classical case arises. If the credit constraints are severe or if each individual puts the weight of his utility on the second-period consumption, then it is likely that the economy is a Samuelson case. This is intuitive, because if either of these is the case, every individual does not borrow so much.

King and Levine [27,28] and Levine, et al. [30] give empirical evidence of the positive effect of financial development on economic growth. Our results for the steady states are consistent with their discoveries. The growth rates at
both of the steady states go up if the credit constraints relax, i.e., $\mu$ increases. In proposition 1, we present this claim.

**Proposition 1**
If the credit constraints relax, i.e., $\mu$ increases, then both of the steady-state growth rates go up, i.e., $\frac{\partial \Gamma(\phi^*)}{\partial \mu} > 0$ and $\frac{\partial \Gamma(\phi^{**})}{\partial \mu} > 0$.

**Proof:** See appendix.

4 **Dynamic Properties**

In this section, we investigate the dynamic properties of the economy. Before doing so, we define a dynamical system. We define a compact interval in $\mathbb{R}$ as $X = [0, \max\{\phi^*, \phi^{**}\}]$. We restrict the domain of the dynamical system of (23) to $X$ so as to obtain economically meaningful equilibria. If $\{\phi_t\}_{t=0}^\infty$ starts with $\phi_0 \in (\max\{\phi^*, \phi^{**}\}, a]$, then $G(\phi_t)$ becomes greater than one in finite time. We also assume that the minimum of $\Psi(\phi)$ is no less than zero. If the domain of the dynamical system is restricted to $X$, then the map, $\Psi : X \rightarrow X$, is continuous and maps $X$ into itself. Henceforth, we use the pair $(X, \Psi)$ to denote our dynamical system.

In what follows, we investigate the local and global dynamic properties of the economy. Before doing so, we linearize the difference equation (23) around the steady state:

$$\phi_{t+1} - \bar{\phi} = \Phi(\bar{\phi})(\phi_t - \bar{\phi}),$$

where $\Phi(\phi) = \frac{(1-\alpha)(1-\mu)\phi}{\alpha (1-\gamma) F(\phi)} \left[ \left( \frac{1}{\alpha g(\phi)} + \frac{\phi}{\gamma F(\phi)} \right) (G(\phi) - \mu) + 1 \right]$. The local stability depends upon whether $\phi^{**}$ is greater than $\phi^*$ or not (i.e., whether the economy is Samuelson or classical).

**Proposition 2**
- If $\phi^{**} > \phi^*$, then the steady-state equilibrium with $\bar{\phi} = \phi^*$ is locally stable, whereas the steady-state equilibrium with $\bar{\phi} = \phi^{**}$ is locally unstable.

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9We relax this assumption in the numerical analysis of section 6. If the minimum of $\Psi(\phi)$ is less than zero, then for almost all the initial values $\phi_0 \in X$, the sequence $\{\phi_t\}_{t=0}^\infty$ escaping from $X$ is not an equilibrium because the credit market clearing condition does not hold. The remaining subset of $X$ is a Cantor set, whose Lebesgue measure is equal to zero. Even if the minimum of $\Psi(\phi)$ is less than zero, the equilibria exist. On the remaining Cantor set, we easily observe two steady-state equilibria. There also exists a period-two equilibrium cycle. See section 6.
• If $\phi^{**} < \phi^{*}$, then the steady-state equilibrium with $\bar{\phi} = \phi^{*}$ is locally unstable, whereas the stability of the steady-state equilibrium with $\bar{\phi} = \phi^{**}$ is ambiguous.

**Proof:** If $\phi^{*} < \phi^{**}$, then $|\Phi(\phi^{*})| = \left| \frac{(1-\alpha)(1-\mu)\phi^{*}}{\alpha(1-\gamma)F(\phi^{*})} \right| < \left| \frac{(1-\alpha)(1-\mu)\phi^{**}}{\alpha(1-\gamma)F(\phi^{**})} \right| = 1$ and $|\Phi(\phi^{**})| = \left| \frac{(1-\alpha)(1-\mu)\phi^{**}}{\alpha(1-\gamma)F(\phi^{**})} \right| > 1$. If $\phi^{*} > \phi^{**}$, then $|\Phi(\phi^{*})| = \left| \frac{(1-\alpha)(1-\mu)\phi^{*}}{\alpha(1-\gamma)F(\phi^{*})} \right| > \left| \frac{(1-\alpha)(1-\mu)\phi^{**}}{\alpha(1-\gamma)F(\phi^{**})} \right| = 1$. However, we cannot know about whether $|\Phi(\phi^{**})| = \left| \frac{(1-\alpha)(1-\mu)\phi^{**}}{\alpha(1-\gamma)F(\phi^{**})} \right| + 1$ is greater than one or not. □

[Figure 3 around here]

The phase diagrams of each case are given by figure 3. As seen in panel III, if the equilibrium with $\bar{\phi} = \phi^{**}(\phi^{*})$ is locally unstable, it is possible that the dynamical system will exhibit either cycles or chaos. At minimum, when the equilibrium with $\bar{\phi} = \phi^{**}(\phi^{*})$ is locally unstable, we must have a period-two cycle.

**Proposition 3**

Suppose that $\phi^{**} < \phi^{*}$. If the steady-state equilibrium with $\bar{\phi} = \phi^{**}$ is locally unstable, there exists a period-two cycle of $\{\phi_{t}\}_{t=0}^{\infty}$ in equilibrium.

**Proof:** Let $\bar{\Psi}(\phi) = G^{-1}(\Psi(\phi))$. Then (23) is written as $\phi_{t+1} = \bar{\Psi}(\phi_{t})$. We can take $\phi_{0}$ close to $\phi^{*}$ so that $\bar{\Psi}^{2}(\phi_{0}) < \phi_{0} < \phi^{*}$ because $\Phi(\phi^{*}) > 1$. If the steady-state equilibrium with $\bar{\phi} = \phi^{**}$ is locally unstable, then $\Phi(\phi^{**}) < -1$. So we can take $\phi^{*}_{0}$ close to $\phi^{**}$ so that $\phi^{**} < \phi^{*}_{0} < \bar{\Psi}^{2}(\phi^{*}_{0}) < \bar{\Psi}^{2}(\phi_{0})$. Therefore, by continuity, there exists $\hat{\phi}_{0}$ such that $\phi^{**} < \hat{\phi}_{0} = \bar{\Psi}^{2}(\hat{\phi}_{0}) < \phi^{*}$, which means that there exists a period-two cycle. □

A graphical analysis gives a proof of proposition 3 as well. Since $\phi^{**} < \phi^{*}$, if the steady-state equilibrium with $\bar{\phi} = \phi^{**}$ is locally unstable, $\Phi(\phi^{**}) < -1$ holds. The configuration of $\phi_{t+1} = G^{-1}(\Psi(\phi_{t}))$ in this case is drawn in figure 4. Its mirror image relative to the 45 degree line is drawn as well. As seen in the figure, there exists at least one pair $\hat{\phi}_{0}$ and $\hat{\phi}_{1}$, where $\hat{\phi}_{1} \neq \hat{\phi}_{0}$, such that $\bar{\phi}_{1} = G^{-1}(\Psi(\hat{\phi}_{0}))$ and $\bar{\phi}_{0} = G^{-1}(\Psi(\hat{\phi}_{1}))$.

[Figure 4 around here]
Since the initial value $\phi_0$ is determined by the individuals’ expectations, it can jump. Therefore, the equilibrium is indeterminate globally, i.e., for any initial value $\phi_0 \in X$, the sequence $\{\phi_t\}_{t=0}^{\infty}$ is an equilibrium of this economy. If $\phi_0 = \phi^*$, then the sequence $\{\phi_t\}_{t=0}^{\infty}$ stays at $\phi^*$. If $\phi_0 = \phi^{**}$, then the sequence $\{\phi_t\}_{t=0}^{\infty}$ stays at $\phi^{**}$ as well. If $\phi_0 \in (\phi^*, \phi^{**})$, then the sequence $\{\phi_t\}_{t=0}^{\infty}$ is monotonically decreasing and converges to $\phi^*$. In this case, we note that the growth rate, $\Gamma(\phi_t)$, is monotonically increasing and converges to $\Gamma(\phi^*)$. If the equilibrium is indeterminate depending upon the individuals’ expectations, this also implies the possibility of sunspot equilibria. \(^{10}\)

$\Gamma : X \rightarrow [\Gamma(\text{max}\{\phi^*, \phi^{**}\}), \Gamma(0)]$ is a homeomorphism. This means that if the equilibrium sequence, $\{\phi_t\}_{t=0}^{\infty}$, exhibits cyclical behavior, then so does the equilibrium sequence of the growth rates, $\{\Gamma(\phi_t)\}_{t=0}^{\infty}$, and that if the equilibrium sequence, $\{\phi_t\}_{t=0}^{\infty}$, exhibits chaos, then the equilibrium sequence of the growth rates, $\{\Gamma(\phi_t)\}_{t=0}^{\infty}$, exhibits chaos as well.

5 Discussion

5.1 Cycles and Chaos

In the previous section, we have examined the dynamical system analytically and we have seen the appearance of cycles in equilibrium. Before proceeding to the numerical analysis, we consider intuitively why cycles come about in the economy.

In what follows, we have to consider a configuration of the dynamical system satisfying (22) or (23). We note that the number of agents who invest in durable goods (or equivalently the number of agents who hold positive deposits) is determined by $\phi_t$. Since $Y_t$ and $Y_{t+1}$ are canceled out in (22), the number of agents who invest in durable goods at time $t + 1$ is determined by the number of such agents at time $t$, which in turn determines the growth rate at time $t+1$. $\phi_t$ creates two different forces: one is the force which is effective via the interest rate $r_{t+1}$ and the other is the force which works through the aggregate liability.

As we see in the right-hand side of (22), both $r_{t+1}$ and $B_t$ are increasing with $\phi_t$. Nevertheless, an increase in $\phi_t$ at the beginning of period $t$ has an ambiguous effect on $\phi_{t+1}$. This is because $B_t$ could be negative. If $B_t$ is negative and if the

---

\(^{10}\)Farmer and Woodford [16] study a similar dynamical system to that of figure 3-I or 3-II and derive sunspot equilibria.
effect of an increase in $\phi_t$ on $r_{t+1}$ is greater than its effect on $B_t$, then $r_{t+1}B_t$ is decreasing with $\phi_t$. As a result, $\phi_{t+1}$ is decreasing with $\phi_t$. On the other hand, needless to say, when $B_t$ is positive, $r_{t+1}B_t$ is increasing with $\phi_t$; accordingly, $\phi_{t+1}$ is increasing with $\phi_t$.

We go a step further and interpret the effect of $\phi_t$ on $\phi_{t+1}$ in terms of (i) the supply and demand of liabilities, (ii) the supply of input goods for the R&D sector, and (iii) the economy-wide productivity in producing the input goods. We focus on the case in which $B_t < 0$.

(i) The effects of the supply and demand of liabilities. When old agents are net borrowers in total ($B_t < 0$), young agents in the next generation are net borrowers in total as well ($B_{t+1} < 0$) unless $r_{t+1} = 0$. If $\phi_t$ increases, the number of agents who make a deposit in the Bank increases but the number of agents who borrow from the Bank falls. As a result, $B_t$ goes up.

The effect of an increase in $\phi_t$ on $r_{t+1}$ is divided into two kinds. One is the effect of the supply of input goods for a R&D sector and the other is the effect of the economy-wide productivity of input goods:

$$\frac{\partial r_{t+1}}{\partial \phi_t} \propto \left( \frac{\partial F(\phi_t)^{-1}}{\partial \phi_t} \right) \phi_t + \frac{\partial F(\phi_t)^{-1} \partial \phi_t}{\partial \phi_t}$$

(ii) The effect of the supply of input goods. This effect is an indirect effect of an increase in $\phi_t$ on $r_{t+1}$. If $\phi_t$ increases, then the number of agents who invest in durable goods decreases. If this is the case, the supply of input goods for the R&D sector falls and thus $p_{t+1}$ rises (see figure 1).

(iii) The effect of the economy-wide productivity in producing the input goods. If $\phi_t$ increases, relatively less capable producers are excluded from producing input goods for the R&D sector. Accordingly, the economy-wide productivity of the input goods rises and less capable agents enjoy the trickle down effect. The $\phi_t$ in the $r_{t+1}$ term is a direct effect of an increase in the economy-wide productivity.

Whether loanable resources at time $t + 1$ fall or rise depends upon the effects of (i), (ii) and (iii). If the effect of (ii) multiplied by the effect of (iii) is so strong
that it dominates the effect of (i), then loanable resources go up. Accordingly, the number of agents who invest in durable goods in the next period must increase and then \( \phi_{t+1} \) decreases. In particular, if \( \phi_t \) is close to zero, then the marginal effect of (iii) is so strong that it dominates the effect of (i). We can show from (23) that \( \frac{d\Psi(\phi_t)}{d\phi_t} \bigg|_{\phi_t=0} < 0 \) if \( \mu > 0 \). On the other hand, if \( \phi_t \) is close to \( a \), \( \phi_{t+1} \) is increasing with \( \phi_t \) and \( \lim_{\phi_t \to a} \frac{d\Psi(\phi_t)}{d\phi_t} = +\infty \) holds. Therefore, by continuity, \( \Psi(\phi_t) \) has a minimum in \([0, a)\).

In order to consider why cycles arise in equilibrium, we study two polar cases. The first is the case in which \( \mu \) is very close to one. When \( \mu \) is very close to one, we note that the effect of \( \phi_t \) on \( r_{t+1} \) goes to zero. This happens because the supply of input goods for the R&D sector becomes very big (see (16)) and thus \( p_{t+1} \) is very close to zero. If this is the case, cycles do not appear because \( G(\phi_{t+1}) \) is almost equal to \( \mu \). The second polar case is the one in which \( \mu \) is equal to zero. In this case, the effect of \( \phi_t \) on \( r_{t+1} \) is strongest, whereas \( B_t \) is always positive because no one is allowed to borrow from the Bank. Therefore, \( \Psi'(\phi_t) > 0 \) for \( \phi_t > 0 \) and no cycles arise.

Meanwhile, what happens if \( \mu \) is an intermediate value? If \( \mu \) is an intermediate value, the effect of \( \phi_t \) on \( r_{t+1} \) will be significant, while it is likely that \( B_t \) will become negative. Depending upon other parameters, if \( \phi_t \) is relatively small, the effects of (ii) and (iii) are so strong that the minimum of \( \Psi(\phi) \) will be close to zero. If this is the case, for the dynamical system of the sequence \( \{\phi_t\}_{t=0}^\infty \), there exists \( \phi_0 \in [0, a) \) such that \( \phi_0 > \phi_1 \) where \( \phi_1 \) is close to zero. However, if \( \phi_1 \) is close to zero, \( \phi_2 \) is close to \( G^{-1}(\mu) \) from (23), which is greater than \( \phi_1 \). If \( \phi_2 \) is close to zero, then \( \phi_3 \) is smaller than \( \phi_2 \). If this kind of law of motion of \( \phi_t \) continues, we have a cyclical equilibrium in this economy. If \( \phi_3 \) happens to be equal to \( \phi_0 \), a period-three cycle is obtained, which means that the dynamical system exhibits topological chaos.

5.2 Sunspots and Cycles

As discussed before, the equilibrium is indeterminate and sunspot equilibria can arise in this economy. On the other hand, if the degree of credit market imperfections is moderate, cycles can appear in equilibrium. If sunspot equilibria arise and the economy fluctuates, the fluctuation is caused by extrinsic stochastic beliefs, whereas if cycles appear, the fluctuation is created endogenously by
the nonlinearity of the economy.

There is a possibility that the current economy is subject to self-fulfilling fluctuations due to sunspots under any value of $\mu$. However, if agents do not hold stochastic beliefs, such self-fulfilling fluctuations do not appear in equilibrium. On the other hand, if $\mu$ is an intermediate value, endogenous fluctuations will still occur. Therefore, we can safely say that it is when credit market imperfections are moderate that the growth rates are more highly volatile than otherwise.

6 Credit Constraints and Chaos

The growth rate is a decreasing function of the cut-off point and has a one-to-one relationship with the cut-off point. Therefore, investigating the dynamic property of $\{\phi_t\}_{t=0}^{\infty}$ is equivalent to studying that of the equilibrium growth rates, $\{\Gamma(\phi_t)\}_{t=0}^{\infty}$. To investigate the dynamic property of the sequence $\{\phi_t\}_{t=0}^{\infty}$ concretely, we use example 1 and assume that $\phi \sim U(0,1)$. Accordingly, the difference equation associated with the dynamical system $(X, \Psi)$ is given by:

$$\phi_{t+1} = \Psi(\phi_t) = \frac{2(1-\alpha)(1-\mu)}{\alpha(1-\gamma)} \frac{\phi_t(\phi_t - \mu)}{1-\phi_t^2} + \mu. \quad (28)$$

We can verify that $\Psi : X \to X$ is a (upside-down) unimodal map. Let $m$ be a critical point of this map, i.e., $\Psi'(m) = 0$. $m$ is easily obtained: $m = \frac{\mu}{1 + \sqrt{1-\mu^2}}$. In order for $\Psi$ to map $X$ into itself, $\Psi(m)$ should be no less than zero. If we relax this assumption, i.e., if $\Psi(m) < 0$, then for almost all the initial values $\phi_0 \in X$, the sequence, $\{\phi_t\}_{t=0}^{\infty}$, escapes from $X$.

If a sequence $\{\phi_t\}_{t=0}^{\infty}$ escapes from $X$ in finite time, it cannot be an equilibrium. The reason is as follows. Suppose $A_0 = \{\phi \in X : \Psi(\phi) < 0\}$ (see figure 5). Pick any $\phi_t$ in $A_0$. Then $\Psi(\phi_t) < 0$, which implies that $B_{t+1} > r_{t+1}B_t$ holds even if $\phi_{t+1} = 0$. Therefore, the market clearing condition never holds. In order for the market clearing condition to hold, $r_{t+1}$ must decrease (since $B_t < 0$ now) because $\phi_{t+1}$ cannot be smaller than zero. In order for $r_{t+1}$ to decrease, $\phi_t$ should be smaller than the original value. Therefore, a sequence $\{\phi_t\}_{t=0}^{\infty}$ such that $\phi_t$ enters $A_0$ in finite time is not an equilibrium. As studied by Boldrin, et al. [12] in detail, the remaining subset of $X$ is a Cantor set whose Lebesgue measure is equal to zero. On the Cantor set, there exist steady-state equilibria or there exist equilibrium cycles. At minimum, $\phi^*$ and $\phi^{**}$ are equi-
A period-two cycle in figure 5 is an equilibrium as well. Moreover, on the Cantor set, the equilibrium path would even display complex dynamics and would exhibit sensitive dependence on initial conditions.11 In the following numerical analysis, we relax the assumption that $\Psi(m)$ is no less than zero. Since the measure of the Cantor set is zero, we will see empty windows in bifurcation diagrams if $\Psi(m) < 0$.

With a unimodal map, we can use Mitra’s [33] sufficient condition for the existence of topological chaos. Another sufficient condition is given by the well-known Li-Yorke theorem. Mitra proposes a weaker sufficient condition than the Li-Yorke theorem, which is useful for identifying the existence of topological chaos without knowing the appearance of a period-three cycle. Mitra’s condition is convenient and simple.

**Lemma 2**
Suppose that $\Psi(m) \geq 0$. If $\Psi$ satisfies $\Psi^2(m) > m$ and $\Psi^3(m) > \phi^{**}$, then $(X, \Psi)$ exhibits topological chaos.

**Proof:** See Mitra [33].

We should note that we consider an upside-down case, whereas Mitra gives a theorem for the usual unimodal case. Therefore, the inequalities in lemma 2 are the reverse of those of Mitra [33]. We should also note that the condition $\Psi(m) \geq 0$ is the one which guarantees that the minimum of $\Psi(\phi)$ is no less than zero and that the dynamical system $(X, \Psi)$ can be defined. If an economy satisfies Mitra’s condition, the dynamical system, $(X, \Psi)$, has a cycle with a period which is not a power of two. We can paraphrase lemma 2 by proposition 4.

**Proposition 4**
Suppose that the parameters of the dynamical system, $(X, \Psi)$, satisfy the following three conditions:

- $\alpha(1 - \gamma)(\Psi^3(m))^2 + 2(1 - \alpha)(1 - \mu)\Psi^3(m) - \alpha(1 - \gamma) > 0$.

11To the best of my knowledge, in the literature of economics, only Boldrin, et al. [12] deal with equilibria on Cantor sets. See also Devaney [14].
\( \alpha \mu (1 - \gamma) - (1 - \sqrt{1 - \mu^2})(1 - \mu)(1 - \alpha) \geq 0. \)

\( \Psi^2(m) > m. \)

Then \((X, \Psi)\) exhibits topological chaos.

**Proof:** See appendix.

There are uncountably many parameters which satisfy these three conditions. However, due to their nonlinearity, it is difficult to draw a figure indicating the parameter space. We give an example for the dynamical system \((X, \Psi)\) which exhibits topological chaos.

**Example 2:** For (28), if \(\alpha = \frac{1}{3}, \gamma = \sqrt{3} - 1,\) and \(\mu = \frac{1}{2}\), then the dynamical system exhibits topological chaos. The proof of this claim is given in the appendix. The phase diagram is given by figure 6.

Our interest is in the relationship between the severity of credit constraints and the appearance of chaos. With what degree of credit constraints does the economy experience chaotic dynamics? In other words, to what extent for values of \(\mu\) does the economy exhibit erratic behavior? As shown in lemma 2, if the dynamical system \((X, \Psi)\) has a cycle with a period which is not a power of two, then we can say that it is topological-chaotic; however, if the dynamical system has a “globally stable” cycle, we will not observe chaos.\(^{12}\) To investigate if the economy has a “globally stable” cycle or not, we analyze the dynamical system \((X, \Psi)\) numerically, observing bifurcation diagrams and the Liapunov exponents (the LEs). We relax the assumption of \(\Psi(m) \geq 0\) in the numerical analysis, and create the bifurcation diagrams for the dynamical system \((X, \Psi)\), iterating 10000 times. We set the initial condition to \(\phi_0 = 0.01.\(^{13}\)

\[^{12}\text{See Boldrin, et al. [12] for details.}\]

\[^{13}\text{It is well known that if for a given } \mu, \text{ a Schwarzian derivative, } S(\Psi) := \frac{\Psi'''(\phi)}{\Psi''(\phi)} - \frac{3}{2} \left( \frac{\Psi''(\phi)}{\Psi'(\phi)} \right)^2 < 0 \text{ for all } \phi \in X, \text{ the orbits starting from almost all the initial values of } \phi_0 \in X \text{ have the same asymptotic behavior. Therefore, the choice of initial conditions is not important. See for example Medio and Gallo [32]. Our Schwarzian derivative is too complicated to investigate its sign analytically. We plotted the values of our Schwarzian derivatives and numerically confirmed that the signs are negative. In addition, to make sure, we examined various initial values: we found that the asymptotic behavior of the dynamical system is invariant to the initial values.}\]
Figures 7a-7e give bifurcation diagrams with respect to the degree of credit constraints, i.e., $\mu$. All the cases are with $\alpha = 0.25$. For all the diagrams, if $\mu$ is small or large, $\{\phi_t\}_{t=0}^\infty$ converges to a stable steady-state equilibrium, which is consistent with our prediction in section 5. The economy does not exhibit erratic dynamics in these cases. As seen in figure 7b, if individuals put equal weight on consumption in the first and second periods, i.e., $\gamma = 0.5$, and if credit market imperfections are moderate, the economy converges to a period-two cycle asymptotically. In figure 7c where $\gamma = 0.55$, as $\mu$ increases from zero, the first period doubling bifurcation occurs around $\mu = 0.324$ and the second period doubling bifurcation occurs around $\mu = 0.45$. After the third period doubling bifurcation around $\mu = 0.51$, the economy converges to a period-eight stable cycle asymptotically until $\mu = 0.57$.

More erratic results are obtained in figures 7d where $\gamma = 0.6$. As $\mu$ increases from zero, we observe the first period doubling bifurcation around $\mu = 0.27$. The second bifurcation follows around $\mu = 0.34$. These period doubling bifurcations are repeated over and over again and, eventually, the economy enters a so-called “chaotic region”.

[Figure 7 around here]

In figure 8a, we plotted the Liapunov exponents when $\gamma = 0.6$. A Liapunov exponent expresses the complexity of a dynamical system, i.e., if a Liapunov exponent is less than zero, then the system converges to a stable cycle (including a steady state). Meanwhile, if it is greater than zero, the system is significantly sensitive to the initial conditions, i.e., in our case, to $\phi_0$s. With a slight change of the initial value, there is a big change of the future values. In this sense, the sequence of the cut-off points is unpredictable, and as a result, so is the sequence of the growth rates. Looking at figure 8a, we note that the LE is equal to zero around $\mu = 0.071$; however, there is no period doubling bifurcation around this value. This means that the economy enters a classical case from a Samuelson case around this value. Around $\mu = 0.27$, the LE is zero again and the economy experiences the period doubling bifurcation as already seen. Every time the graph touches to the $LE = 0$ horizontal line, a period doubling bifurcation occurs; however, around $\mu = 0.36$, the LE becomes positive and hence the economy enters a chaotic region. When $\mu$ decreases from one, the same things happen: the period doubling bifurcation occurs over and over again.
and eventually the economy enters a chaotic region around 0.71. If an economy is in a chaotic region, the future growth rate is unpredictable. For example, suppose two such economies have arbitrarily close but different initial values of $\phi_0$. Even if this is the case, the growth rates of the two economies would diverge in finite periods.

If $\mu$ is in between 0.487 and 0.60, $\Psi(m) < 0$ holds. In this case, we observe that there is an empty window in a bifurcation diagram (see 7d). We will say again that if $\Psi(m) < 0$, then for almost all the initial values $\phi_0 \in X$, the sequence, $\{\phi_t\}_{t=0}^\infty$, escapes from $X$ and the remaining set is a Cantor set. On the Cantor set, there still exist steady-state equilibria and equilibrium cycles. The economy still exhibits complex dynamics with sensitive dependence on initial conditions.

Figure 7e gives a case with $\gamma = 0.7$, in which the shape of the bifurcation diagram is similar to the one with $\gamma = 0.6$. The difference is in that the empty window is wider than that of the case with $\gamma = 0.6$.

[[Figure 8 around here]]

7 Concluding Remarks

Credit market imperfections sometimes have serious effects on macroeconomic phenomena, especially on the dynamic properties of economies as studied in this paper. Our main findings are as follows. First, two steady state equilibria arise and each of their growth rates goes up as credit market imperfections are resolved. This finding is consistent with empirical evidence of King and Levine [27,28] and Levine, et al. [30]. Second, since equilibrium is indeterminate, the economy fluctuates due to extrinsic stochastic beliefs whenever sunspots appear. Third, while indeterminacy is a usual property of overlapping generations models with outside money, the economy converges monotonically to a stable steady state if credit market imperfections are severe or soft and if no sunspots occur. Lastly, if credit market imperfections are moderate, deterministic cycles or chaos would arise in equilibrium.

While endogenous business cycles have been studied with overlapping generation models for over twenty years, to the best of my knowledge, this paper
is the first one in which the relationship between endogenous business fluctuations and credit market imperfections has been made clear by an overlapping generations model with production and with outside money.

**Appendices**

**Proof of lemma 1**

The first-order conditions for an individual \( \phi \)'s maximization problem are given by:

\[
\begin{align*}
-\frac{u_t(\phi)}{c_t(\phi)} &+ \frac{(1-\gamma)p_{t+1}u_t(\phi)}{c_{2t+1}(\phi)} + \kappa_t(\phi) + \lambda_t(\phi)\mu = 0 \quad (29) \\
-\frac{u_t(\phi)}{c_t(\phi)} &+ \frac{(1-\gamma)r_{t+1}u_t(\phi)}{c_{2t+1}(\phi)} + \lambda_t(\phi) = 0 \quad (30) \\
\kappa_t(\phi)k_t(\phi) &= 0, \quad \kappa_t(\phi) \geq 0 \quad (31) \\
\lambda_t(\phi)[b_t(\phi) + \mu k_t(\phi)] &= 0, \quad \lambda_t(\phi) \geq 0, \quad (32)
\end{align*}
\]

where \( \kappa_t(\phi) \) and \( \lambda_t(\phi) \) are the Lagrange multipliers associated with the non-negativity constraint of \( k_t(\phi) \) and the credit constraint (3), respectively. \( u_t(\phi) = c_{1t}(\phi)^\gamma c_{2t+1}(\phi)^{1-\gamma} \).

Suppose that \( \kappa_t(\phi) > 0 \) and \( \lambda_t(\phi) > 0 \) for some \( \phi \). Then we have \( k_t(\phi) = 0 \) and \( b_t(\phi) = 0 \) from (31) and (32), which implies \( c_{2t+1}(\phi) = 0 \). This contradicts the optimality condition.

Suppose that \( \kappa_t(\phi) = 0 \) and \( \lambda_t(\phi) = 0 \). Then we have \( r_{t+1} = \phi p_{t+1} \) from (29) and (30), which contradicts the conditions in the lemma. From (29) and (30), we have:

\[
\frac{u_t(\phi)(1-\gamma)[p_{t+1} - r_{t+1}]}{c_{2t+1}(\phi)} + \kappa_t(\phi) + \lambda_t(\phi)[\mu - 1] = 0. \quad (33)
\]

Suppose that \( r_{t+1} > \phi p_{t+1} \). Then, from (33), we have \( \kappa_t(\phi) > [1 - \mu] \lambda_t(\phi) \), which implies \( \kappa_t(\phi) > 0 \) and \( \lambda_t(\phi) = 0 \). Then \( k_t(\phi) = 0 \). From (2), (4), and (30), we have \( b_t(\phi) = (1-\gamma)w_t \). On the other hand, suppose that \( r_{t+1} < \phi p_{t+1} \). Then, from (33), we have \( \kappa_t(\phi) < [1 - \mu] \lambda_t(\phi) \), which implies \( \kappa_t(\phi) = 0 \) and \( \lambda_t(\phi) > 0 \). Since \( \lambda_t(\phi) > 0 \), we have \( b_t(\phi) = -\mu k_t(\phi) \). From (29) and (30), we have

\[
c_{2t+1} = \frac{(1-\gamma)(\phi p_{t+1} - \mu r_{t+1})}{\gamma(1-\mu)} c_{1t}. \quad (34)
\]

Substituting \( b_t(\phi) = -\mu k_t(\phi) \) and \( c_{2t+1}(\phi) = \frac{(1-\gamma)(\phi p_{t+1} - \mu r_{t+1})}{\gamma(1-\mu)} c_{1t} \) into (4), we have \( c_{1t}(\phi) = \frac{2(1-\mu)}{1-\gamma} k_t(\phi) \). Substituting this into (2), we have \([1 - \mu]k_t(\phi) = (1-\gamma)w_t \). \( \Box \)
Proof of proposition 1

We take four steps to show proposition 1.

**Step 1:** If \( \bar{\phi} \in [0, a) \), then \( F(\bar{\phi}) > \bar{\phi}(1 - G(\bar{\phi})) \), since \( F(\bar{\phi}) = \int_{\bar{\phi}}^{a} \phi dG(\phi) > \int_{\bar{\phi}}^{a} \bar{\phi} dG(\phi) = \bar{\phi}(1 - G(\bar{\phi})) \).

**Step 2:** \( \frac{\partial \phi^*}{\partial \mu} = \frac{1}{g(\phi^*)} \), which follows from \( G(\phi^*) = \mu \).

**Step 3:** \( F(\phi^{**}) - (1 - \mu)\phi^{**} g(\phi^{**}) \frac{\partial \phi^{**}}{\partial \mu} = (1 - \mu) \frac{F(\phi^{**})}{\phi^{**}} \frac{\partial \phi^{**}}{\partial \mu} \). This is because from \( \frac{\phi^{**}}{F(\phi^{**})} = \frac{\alpha(1 - \gamma)}{(1 - \alpha)(1 - \mu)} \), we have:

\[
\log \phi^{**} - \log F(\phi^{**}) = \log \left[ \frac{\alpha(1 - \gamma)}{(1 - \alpha)} \right] - \log(1 - \mu).
\]

Therefore, we have:

\[
\left[ \frac{1}{\phi^{**}} + \frac{\phi^{**} g(\phi^{**})}{F(\phi^{**})} \right] \frac{\partial \phi^{**}}{\partial \mu} = \frac{1}{1 - \mu} \quad (34)
\]

\[
\Leftrightarrow \quad F(\phi^{**}) - (1 - \mu)\phi^{**} g(\phi^{**}) \frac{\partial \phi^{**}}{\partial \mu} = (1 - \mu) \frac{F(\phi^{**})}{\phi^{**}} \frac{\partial \phi^{**}}{\partial \mu}.
\]

**Step 4:** Case 1: \( \bar{\phi} = \phi^* \). From step 1,

\[
\frac{\phi^*}{F(\phi^*)} < \frac{1}{1 - G(\phi^*)} = \frac{1}{1 - \mu} \quad (35)
\]

holds. From step 2 and (35), we have:

\[
\frac{-\phi^*(1 - \mu) + F(\phi^*)}{(1 - \mu)^2} > 0
\]

\[
\Leftrightarrow \quad \frac{-\phi^* g(\phi^*)(1 - \mu) \frac{1}{g(\phi^*)} + F(\phi^*)}{(1 - \mu)^2} > 0
\]

\[
\Leftrightarrow \quad \frac{-\phi^* g(\phi^*)(1 - \mu) \frac{\partial \phi^*}{\partial \mu} + F(\phi^*)}{(1 - \mu)^2} > 0
\]

\[
\Leftrightarrow \quad \frac{\partial}{\partial \mu} \left[ \frac{F(\phi^*)}{1 - \mu} \right] > 0
\]

\[
\Leftrightarrow \quad \frac{\partial \Gamma(\phi^*)}{\partial \mu} > 0.
\]

Case 2: \( \bar{\phi} = \phi^{**} \). From (34), \( \frac{\partial \phi^{**}}{\partial \mu} > 0 \) holds. Then, from step 3, we have:

\[
\frac{(1 - \mu) \frac{F(\phi^{**})}{\phi^{**}} \frac{\partial \phi^{**}}{\partial \mu}}{(1 - \mu)^2} > 0
\]
\[
\iff -\phi^{**} g(\phi^{**})(1 - \mu) \frac{\partial \phi^{**}}{\partial \mu} + F(\phi^{**}) > 0
\]
\[
\iff \frac{\partial}{\partial \mu} \left[ \frac{F(\phi^{**})}{1 - \mu} \right] > 0
\]
\[
\iff \frac{\partial \Gamma(\phi^{**})}{\partial \mu} > 0.
\]

Proof of proposition 4

From (28), \(\phi^{**}\) satisfies:
\[
\alpha (1 - \gamma)(\phi^{**})^2 + 2(1 - \alpha)(1 - \mu)\phi^{**} - \alpha(1 - \gamma) = 0.
\]

Therefore, \(\Psi^3(m) > \phi^{**}\) is equivalent to:
\[
\alpha (1 - \gamma)(\Psi^3(m))^2 + 2(1 - \alpha)(1 - \mu)\Psi^3(m) - \alpha(1 - \gamma) > 0.
\]

Furthermore with some calculation, we can easily verify that \(\Psi(m) \geq 0\) is equivalent to:
\[
\alpha \mu (1 - \gamma) - (1 - \sqrt{1 - \mu^2})(1 - \mu)(1 - \alpha) \geq 0. \Box
\]

The dynamical system of Example 2

To prove the current dynamical system exhibits topological chaos, we use a theorem given by Mitra [33]. Without loss of generality, the domain of the dynamical system can be restrict to \(X = [0, \mu]\). Then the map, \(\Psi : X \rightarrow X\), is continuous and maps \(X\) into itself unless the minimum of \(\Psi(\phi)\) is less than zero.

Definition 1

The dynamical system \((X, \Psi)\) is called turbulent if there exist points \(\phi_0, \phi_1, \text{ and } \phi_2\) such that
\[
\Psi(\phi_1) = \Psi(\phi_2) = \phi_2, \quad \Psi(\phi_0) = \phi_1 \quad \text{and} \quad \phi_1 < \phi_0 < \phi_2.
\]

Theorem 1 (Mitra)

If the dynamical system, \((X, \Psi)\), is turbulent, then \((X, \Psi)\) has a period-three cycle.

Since if we have a period-three cycle in the dynamical system, \((X, \Psi)\), then by the well-known Li-Yorke theorem, the economy exhibits topological chaos,
we only need to show that the dynamical system, \((X, \Psi)\), is turbulent. For the dynamical system, if the economy starts with \(\phi_0 = 2 - \sqrt{3}\), then we have \(\phi_1 = 0 < \phi_0 < \phi_2 = \frac{1}{2}\). Therefore, the system \((X, \Psi)\) is turbulent and thus the economy exhibits topological chaos. □

References


Figure 1
Figure 2: Categorization of an economy
Figure 3-I: $\phi^{**} > \phi^*$

Figure 3-II: $\phi^{**} < \phi^*$, No Cycles
Figure 3-III: $\phi^{**} < \phi^*$, Cycles or Chaos

Figure 4: Period-Two Cycle
Figure 5: Case of $\Psi(m) < 0$

Figure 6: Topological Chaos
Figure 7a: Bifurcation Diagram Gamma=0.3
Figure 7d: Bifurcation Diagram Gamma=0.6
Figure 7e: Bifurcation Diagram Gamma=0.7
Figure 8a: Liapunov Exponent $\Gamma = 0.6$
Figure 8b: Liapunov Exponent Gamma=0.7