Government Expenditure and Stochastic Growth: Optimal Policy and the Role of the Risk Premium

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Abstract

This paper employs a stochastic endogenous growth model with productive government expenditure to analyze the macroeconomic effects of income taxation. We demonstrate that in the presence of capital and income risk the impact of taxation on consumption choice as well as on economic growth is ambiguous as it affects the mean as well as the variance of disposable income. We observe that the effects of taxation crucially depend on the degree of risk aversion and on the capital income share. It is possible to solve for welfare maximizing policies, but contrary to the deterministic framework, welfare and growth maximizing policies do not necessarily coincide. In fact, growth maximizing policies may turn out to be welfare minimizing and multiple solutions for optimal tax rates can be found.

JEL classification: D8, D9, E6
1 Introduction

This paper is concerned with the macroeconomic effects of government expenditure and taxation within the context of a stochastic endogenous growth model. In extending a widely-used framework, namely the model of Barro (1990), with aggregate productivity shocks, we show that the introduction of technological uncertainty has interesting implications for the design of an optimal fiscal policy.

In his endogenous growth model with productive government expenditure, Barro makes two important contributions: (a) Intertemporal welfare is maximized by maximizing the growth rate of the economy. (b) An optimal policy is characterized by the choice of a tax rate which equates the marginal damage of taxation with the marginal return of productivity enhancing government expenditure, thereby simultaneously maximizing the growth rate, the propensity to consume out of wealth, and welfare.

As Turnovsky (1999a, b), we argue that these widely applied results cannot be carried over to the stochastic framework in a simple manner. But while Turnovsky (1999a, b) only states that welfare and growth maximizing policies do not necessarily coincide, we are able to show that a growth maximizing policy may in fact minimize welfare. In our model, there is no unique optimal policy, as we find several tax rates satisfying the optimality conditions. We provide conditions under which the standard results continue to hold and identify the equilibrium risk premium to be a key factor in explaining the multiplicity of optimal tax rates.

The results can mainly be ascribed to the influence of uncertainty on the intertemporal decision making of risk averse agents in general and the interaction of risk and real growth in particular. The macroeconomic portfolio approach, as discussed by Eaton (1981), Turnovsky (1993) and Obstfeld (1994), displays the advantage of directly relating fiscal policies to individual wealth accumulation and aggregate growth. The economy is assumed to follow a stochastic trend originating from an aggregate productivity shock. Under this assumption, the individual income risk affects the macroeconomic equilibrium in several ways.

First, the agents will respond to uncertainty in their intertemporal decision on consumption and saving. If the agent is sufficiently risk averse, he will decrease the rate of consumption in favor of additional savings. This motive for precautionary savings can be regarded as self-insurance on capital markets over time and was first discussed by Leland (1968) and Sandmo (1970), or more recently by Kimball (1990), Weil (1993) and Smith (1996a). These considerations immediately give rise to the question of how precautionary savings affect growth, since we are dealing with a Ramsey-type economy. One key factor in explaining our results is, of course, the individual attitude towards risk, as measured by
the Arrow/Pratt index of relative risk aversion. But additionally, we also identify the factor income distribution to be an important determinant of the growth process, because the agents respond differently to the riskiness of accumulable and non–accumulable incomes.

Second, the issue of efficacy and optimality of fiscal policy under uncertainty has to be addressed. From Domar and Musgrave (1944) or Stiglitz (1969) it is well–known that taxation of risky assets may actually increase the demand for these assets, because the government participates in individual income risk and thus provides an insurance. Only recently, a number of studies focused on this result and the interaction between net–returns, individual risk–bearing, portfolio choice, and endogenous growth in the context of stochastic general equilibrium models; see e. g. Turnovsky (1993, 2000a, b, and references therein), Smith (1996b), Corsetti (1997), Clemens and Soretz (1997, 1998), Gokan (2002).

The fiscal means in the Barro (1990) model of productive government expenditure impinge on individual investment in physical capital twofold. Both, the income tax and the productivity enhancing force of government expenditures affect the mean as well as the volatility of disposable incomes. Consequently, the distortionary and insurance effects from fiscal policy may be co– or counter–working in the determination of the growth rate of the economy. Turnovsky (1999a, b) extensively discusses the relationship between risk, productive government expenditures, different modes of financing, and growth from the perspective of optimal public revenue–expenditure schemes. From the viewpoint of our analysis, his main finding is that growth and welfare maximizing policies do not necessarily coincide. Corsetti (1997) stresses the role of government bonds and finds an optimal policy characterized by the government being a net creditor to the public. So a justified question is, what more do we have to add?

Turnovsky (1999a, b) identifies the degree of risk aversion to be the key parameter determining the tradeoff between the equilibrium growth rate and its variance. While this is true for the type of AK–model with constant returns to scale on the aggregate level he applies, we show that this finding cannot be extended to an increasing returns to scale model. By additionally incorporating labor incomes, our framework can be viewed as a direct extension to the model of Barro (1990). Our argument is related to Sandmo’s results regarding the impact of different risky income sources on individual savings (Sandmo, 1970). He explicitly distinguishes between capital risk and income risk, the latter stemming from stochastic income flows from non–accumulable factors. We show that, supplementary to the degree of risk aversion, the factor income distribution plays an important role for the tradeoff between growth and risk. Similar to the ‘learning by doing’ model of Clemens and Soretz (1999), it can be observed that a motive for precautionary savings already emerges for a lower degree of risk aversion in the presence of both risks than in case of a pure capital risk. From this we conclude that our results are valid for a broader class of stochastic endogenous growth models.
Together with the increasing returns technology this gives rise to the nontrivial implications for the design of the optimal tax scheme. Our analysis focuses especially on the roles the certainty equivalent to capital return and the risk premium play for the emergence of multiple optimal tax policies. Another goal of the paper is to discuss the conditions in detail under which welfare and growth maxima stand in conflict or coincide.

In order to keep the model as simple as possible, we narrow our analysis to several respects. We will stick to the discussion of second–best optima for the competitive economy, the income tax rate and productive government spending being the only public revenue and expenditure flows. Extending the model with a non–distortionary consumption tax and/or government bonds à la Corsetti (1997) to internalize the gap between private and social returns is straightforward, but complicates the analysis unnecessarily. Moreover, we assume labor to be inelastically supplied, which is sufficient in order to stress the importance of factor incomes. We will also confine our analysis to the discussion of precautionary savings, which is the empirically more plausible case (Campbell, 1996).

The paper is organized as follows. In section 2, we develop the model, determine the macroeconomic equilibrium of the decentralized economy and compare the results to the corresponding conditions for the Pareto–efficient command optimum. Section 3 discusses the macroeconomic effects of a change in the tax rate and addresses the question of optimal fiscal policy. Section 4 concludes. Technical details are deferred to Appendices A and B.

2 Productive Government Expenditure and Stochastic Growth

The Model  We assume an economy populated by a continuum \([0, 1]\) of identical infinitely–lived individuals who produce a homogeneous good according to the stochastic Cobb–Douglas technology

\[
dY(t) = \gamma K(t)^\alpha L(t)^{1-\alpha} G(t)^{1-\alpha} (dt + dy(t)).
\]

(1)

\(K(t)\) denotes the privately owned capital stock. \(\gamma > 0\) is a productivity parameter. There is no population growth. Labor input \(L(t)\) is supplied inelastically and normalized to unity. The instantaneous output \(dY(t)\) is subject to an aggregate multiplicative productivity shock. \(dy(t)\) is a serially uncorrelated increment to a standard Wiener process with zero mean and variance \(\sigma^2 dt\). The production function of a typical producer displays constant returns to scale with respect to physical capital and labor. Aggregate production is characterized by increasing returns to scale. For simplicity, depreciation is neglected. Following Barro (1990) the government provides public services \(G(t)\) to individual firms. These services may be identified as expenditures on infrastructure and display the characteristics of a pure public good, i. e., are assumed to be non–rival and non–excludable. Hence, we do not discuss congestion as Barro and Sala-I-Martin (1992) or Turnovsky (1996, 1999b). Public spending impacts directly on productivity of physical capital thus assuring ongoing
growth. Yet, agents in the decentralized economy misperceive this productivity enhancing effect on the accumulable factor. According to the productivity shocks generated by the above described continuous–time geometric Brownian motion, the flow of public expenditure is stochastic. The expenditure rule is given by \[ \mathrm{d}G(t) = G(t) \left( \mathrm{d}t + \mathrm{d}y(t) \right). \]

The government imposes a constant tax of \( \tau \in (0, 1) \) on capital and labor incomes in each time increment. Different to Turnovsky (1999a,b) we assume that the government is not able to distinguish between the deterministic and the stochastic movements of output. The stochastic process of government revenues is represented by

\[ \mathrm{d}T(t) = \tau \mathrm{d}Y(t). \] (2)

The agent has two ways of saving income: buying riskless bonds or investing in risky physical capital. The safe asset \( B(t) \) offers a sure instantaneous and constant yield \( i \). Bonds are considered to be perpetuities. With payouts reinvested and continuously compounded \( B(t) \) follows the differential equation

\[ \mathrm{d}B(t) = i B(t) \mathrm{d}t. \] (3)

Wealth \( W(t) \) is the sum of the holdings of the two assets

\[ W(t) = K(t) + B(t). \] (4)

The identical households are characterized by a time–separable utility function in consumption only. \( E_0 \) denotes the mathematical expectation conditional on time 0 information and \( \beta \) is the rate of time preference, positive by assumption.

The objective of a typical agent is to select her rate of consumption as well as her portfolio of assets in order to maximize the expected value of lifetime utility according to the following program, taking prices and the tax rate as given

\[
\max_{C(t), K(t), W(t)} \mathcal{V}(0) = E_0 \int_0^\infty U[C(t)] e^{-\beta t} \mathrm{d}t
\]

s.t. \[ \mathrm{d}W(t) = [i (W(t) - K(t)) + (1 - \tau) (r K(t) + \omega) - C(t)] \mathrm{d}t + \mathrm{d}w(t) \]

with \( K(0) = K_0 > 0 \) given, \( r \) the pre–tax rate of return on physical capital, \( \omega \) the wage rate, and the stochastic process of wealth given by \( \mathrm{d}w(t) = (1 - \tau) (r K(t) + \omega) \mathrm{d}y(t) \).

Consumption \( C(t) \) is assumed to be instantaneously deterministic. The current period utility function \( U[C(t)] \) is strictly concave and of the isoelastic form

\[ U[C(t)] = \frac{C(t)^{1-\rho}}{1-\rho} \quad \text{for} \quad \rho > 0, \rho \neq 1, \] (6)

and \( U[C(t)] = \ln C(t) \) for \( \rho = 1 \). The parameter \( \rho \) denotes the Arrow/Pratt–index of relative risk aversion.

\[ ^1 \text{A straightforward extension of the model is to introduce differentiated tax and expenditure rates; see Eaton (1981), Clemens and Soretz (1997) or Turnovsky (1999b). For the purposes followed here it is sufficient to assume flat rates.} \]
Macroeconomic Equilibrium  The optimality conditions for the intertemporal problem (5) are derived in Appendix A.1. The optimal time paths for consumption and the portfolio choice are functions of the derivatives of the time–separable value function $V[W(t),t] = e^{-eta t}J(W) = \max V'(t)$ and form a stochastic differential equation. The solution strategy is trial and error, finding a function $J(W)$ that satisfies the optimality conditions. As household behavior is characterized by constant relative risk aversion, by following Merton (1971), we first conjecture that in macroeconomic equilibrium consumption is a constant, time–invariant fraction of wealth.

$$C(W,t) = \mu W(t),$$

where $\mu$ denotes the marginal propensity to consume out of wealth. Second, we guess that in steady state the portfolio shares are also constant, such that

$$\frac{dW(t)}{W(t)} = \frac{dK(t)}{K(t)} = \frac{dB(t)}{B(t)}.$$  

The results from individual optimization can now be employed to determine the market equilibrium. To derive macroeconomic equilibrium conditions, it is necessary to establish the government budget constraint. The government runs a balanced budget in every period of time. If we do not allow for public deficits or surpluses as in Clemens and Soretz (1997) or Corsetti (1997), government spending equals revenues out of taxation, such that

$$dG = \tau(rK + \omega) (dt + dy).$$

The aggregate resource constraint of the economy includes instantaneously deterministic consumption and government expenditure. Market clearing requires

$$dK = dY - dG - Cdt.$$  

We assume perfect competition in the factor markets. The factor returns can then be obtained by using the first–order conditions of the firm problem. In the decentralized economy agents ignore the productivity enhancing effect of public expenditures on private inputs. The pre–tax values of the rental rate of physical capital and the wage rate are determined by the usual marginal productivity conditions

$$r = \alpha\gamma K^{\alpha-1}G^{1-\alpha} \quad \text{and} \quad \omega = (1-\alpha)\gamma K^{\alpha}G^{1-\alpha}.$$  

Given market returns, the equilibrium value of public spending can be derived from (9)

$$G^* = (\tau \gamma)^{\frac{1}{\alpha}} K.$$  

\footnote{For notational convenience the (\( t \)) part of the variables is omitted.}
From (12) it is obvious that the results from the deterministic model extend to the stochastic setting.

With identical individuals, net trade in assets will be zero. All agents are affected by the technological disturbance to the same extent. Under the additional assumption of a closed economy, the aggregate stock of physical capital equals aggregate wealth, \( K = W \). We now employ the first–order condition (A.2b) together with the conjecture for optimal consumption (7), capital return and the wage rate from (11), and the equilibrium amount of government expenditures (12) in order to solve for the certainty equivalent of real after–tax capital return. For notational simplicity, we define

\[
R(\tau) \equiv \gamma \frac{1}{\alpha} (1 - \tau) \gamma \frac{1}{\alpha},
\]

which will be referred to as tax function. Using (13), the risk–free rate can be obtained as follows

\[
i = \alpha R(\tau) \left[ 1 - \rho \sigma^2 R(\tau) \right].
\]

Depending only on the exogenously given parameters of the model, the risk–free rate is constant as assumed above. From (14) follows that the risk–free rate and the real rate of return on capital differ to the amount of the risk premium, which is given by \( \alpha \rho \sigma^2 R(\tau)^2 \).

Given the functional forms of technology (1) and of instantaneous felicity (6) the first–order conditions (A.2a) and (A.2c) together with the guess (7) can be used to obtain a closed–form solution for the propensity to consume out of capital

\[
\mu = \frac{\beta}{\rho} + R(\tau) \left[ \frac{\rho - \alpha}{\rho} + \sigma^2 R(\tau) \left( \alpha - \frac{\rho + 1}{2} \right) \right].
\]

Individual consumption is a time–invariant function of the exogenously given parameters of the model. The consumption–capital ratio is constant in steady state, thus confirming the conjecture stated above.

The relation for optimum consumption (15) together with the market clearing condition (10) can now be used to determine the stationary growth path. Note, that due to the properties of the underlying stochastic process \( \frac{E[dK]}{dr} = 0 \). The expected growth rate of the economy \( \psi = \left[ \frac{1}{K} E[dK] \right] \) can be derived as follows

\[
\psi = \frac{1}{\rho} \left[ \alpha R(\tau) - \beta \right] + \sigma^2 R(\tau)^2 \left( \frac{\rho + 1}{2} - \alpha \right).
\]

The growth rate of the economy is determined endogenously and depends on the factors that affect aggregate savings. Since we assumed an inelastic labor supply, there are no transitional dynamics and the economy immediately grows along its steady state path.

The expected growth rate (16) is the sum of two components. The first equals the growth rate from the deterministic model. The second, i. e. the diffusion component, describes the individual response to aggregate technological risk. From this can be seen that
current shocks may have long–lasting effects on macroeconomic trend, which are reflected in the second–order effects from the variance of the productivity shock. The expected growth rate will exceed the deterministic one, if the agent has a motive for precautionary saving. The sign of the diffusion component in (16) is determined by the agent’s attitude towards risk and the capital income share \( \alpha \). The importance of the factor income distribution constitutes an important difference to AK–type settings (see Turnovsky, 1999a,b) as well as to approaches, where labor income in terms of human wealth is treated as an (quasi-) accumulable factor; see Corsetti (1997).

In our model, future labor and capital income flows are random due to the productivity shock. Following Sandmo (1970), decreasing absolute risk aversion is a necessary and sufficient condition to save out of precautionary motives in the case of a pure income risk and is satisfied here for any \( \rho > 0 \). Changes in capital risk affect the mean as well as the volatility of future returns. A change in the mean induces a positive income effect, while the change in riskiness gives rise to a negative substitution effect. Precautionary savings can be observed, if the first dominates the latter, which is given here in case of \( \frac{1}{2}(\rho + 1) > \alpha \). Certainty equivalence describes a situation, where both intertemporal effects exactly offset, \( \frac{1}{2}(\rho + 1) = \alpha \), and lies below \( \rho = 1, \forall \alpha \in (0,1) \), which is the case of logarithmic preferences.

The third case, \( \frac{1}{2}(\rho + 1) < \alpha \) is of empirical relevance only, if we consider a comparably low degree of risk aversion combined with \( \alpha \) interpreted as a broad measure of capital. Following Campbell (1996), empirical evidence suggests a comparably high coefficient of relative risk aversion. This argument is supported by Caballero (1990), Hubbard et al. (1993) or Carroll and Samwick (1997) whose empirical findings indicate a strong motive for precautionary savings. For this reason, our analysis will focus on this case. We will illustrate our results with parametric values for the degree of risk aversion of \( \rho > 1 \), where the riskiness of both income sources leads to precautionary savings independent of the factor income distribution.

Equations (15) and (16) can finally be employed to obtain a closed–form solution for intertemporal welfare in macroeconomic equilibrium. Lifetime utility of a representative agent as specified by (5), evaluated along the competitively chosen path, can be derived

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3 Note that the factor income distribution is relevant although labor is assumed to be inelastically supplied. Because the agent responds differently to the riskiness of accumulable and non–accumulable factors, labor income does not trivially cancel out in the determination of the expected growth rate. The diffusion component of the expected growth rate in a pure AK–model equals the one of the Pareto–efficient economy (see eq. (20)) and is given by \( \frac{1}{2} (\rho - 1) \sigma^2 R(\tau)^2 \), with the sign depending entirely on the coefficient of risk aversion. For a detailed discussion, see Clemens (1999).

4 It is an important feature of our model that certainty equivalence does not correspond to logarithmic preferences, which is the case for a pure capital risk for instance in the AK–model. This result holds for expected utility theory with CRRA preferences as well as for non–expected utility with CRRA–CIES preferences; Clemens and Soretz (1999).
as follows (see (A.4) to (A.6) in Appendix A.1):

\[ V[K(0), 0] = \frac{K(0)^{1-\rho}}{1-\rho} \cdot \mu^{1-\rho} \cdot \frac{1}{\beta - (1-\rho) \left( \psi - \frac{1}{2} \rho \sigma^2 R(\tau) \right)^2}. \]  (17)

For notational convenience, we define \( \Delta \equiv \beta - (1-\rho) \left( \psi - \frac{1}{2} \rho \sigma^2 R(\tau) \right)^2 \), which has to be of positive sign in order to avoid unbounded utility.

From (17), we learn that in competitive equilibrium both, the expected propensity to consume out of capital as well as the expected growth rate of the economy, determine lifetime utility. This result is important for the determination of an optimal tax policy. As will be shown below, especially the interaction of consumption and growth effects is responsible for the multiplicity of optimal tax policies.

\textit{Command Optimum} The agents neglect the productivity enhancing effect of government expenditure in the decentralized economy. Only the private marginal product of capital is relevant for the determination of optimal individual investment and there is a wedge between private and social returns. Consider now a benevolent social planner who maximizes the representative agent’s welfare according to (5) while taking account of this distortion.

The equilibrium values of the macroeconomic relationships are denoted with an asterisk and given as follows: \(^5\) The certainty equivalent of risky capital return becomes

\[ i^* = R(\tau) \left[ 1 - \rho \sigma^2 R(\tau) \right]. \]  (18)

The propensity to consume out of capital is given by

\[ \mu^* = \frac{\beta}{\rho} + \frac{\rho - 1}{\rho} R(\tau) \left[ 1 - \frac{1}{2} \rho \sigma^2 R(\tau) \right]. \]  (19)

The expected growth rate of the economy can be determined as

\[ \psi^* = \frac{1}{\rho} [R(\tau) - \beta] + \frac{1}{2} (\rho - 1) \sigma^2 R(\tau)^2, \]  (20)

and finally, welfare of the command economy is given by

\[ V[K(0), 0]^* = \frac{K(0)^{1-\rho} \left( \mu^* \right)^{-\rho}}{1-\rho}. \]  (21)

Note that \( \Delta^* = \mu^* \) in equation (21).

A comparison between (16) and (20) suggests that, with appropriate parameter settings, the expected growth rate of the competitive economy can be higher than the growth

\(^5\)Appendix A.2 provides the details of optimization.
rate of the command optimum. The reason for this lies in the diffusion component. Proposition 1 shows that dynamic inefficiency in the decentralized economy can be related to the risk–free rate: 6

**Proposition 1 (Feasibility and the Risk–free Rate)**  
A non–negative certainty equivalent to capital return is a sufficient but not necessary condition that the equilibrium growth path satisfies the transversality condition \( \lim_{t\to\infty} E_t [V(K(t), t)] = 0 \), and a necessary and sufficient condition for \( \psi^* \geq \psi \).

\[
i \geq 0 \iff 1 - \rho \sigma^2 R(\tau) \geq 0.
\] (22)

(Proof: see (A.7) in Appendix A.1)

If we want to replicate one of the main results from the deterministic model in the stochastic setting, which is that growth of the competitive economy is suboptimally low, we have to assure that neither the degree of risk aversion nor the variance per unit of physical capital \( \sigma^2_r / K = \sigma^2 R(\tau)^2 \) is too large. Otherwise, growth of the decentralized economy is suboptimally high.

### 3 Macroeconomic Effects from Tax Policy

The macroeconomic equilibrium of the decentralized economy, which we determined in the previous section is now subject to the analysis of tax incidence. The equilibrium values of the expected growth rate, the propensity to consume, and the risk–free rate form a system that completely describes the equilibrium allocation for a given set of fiscal intervention. The main objective now is to derive conditions, which allow us to characterize an optimal tax policy. We are especially interested in the question, as to whether or not the stochastic model of productive government expenditure mimics the implications for (optimal) tax policy known from the deterministic setting. One major result of the non–stochastic model was to prove that there exists a unique tax rate which simultaneously maximizes growth and welfare. Now, we will demonstrate that this result does not necessarily extend to the stochastic context. We will proceed first with a general discussion of the implications of the tax function \( R(\tau) \), followed by the benchmark results for the efficient economy. We will then discuss the growth, consumption, risk–free rate, and welfare effects of taxation in the decentralized economy and conclude with implications for the design of an optimal tax policy.

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6Contrary to Merton (1971), a positive propensity to consume is not sufficient for the transversality condition to hold. This result is due to the increasing returns to scale technology on the aggregate level.
The Tax Function $R(\tau)$  Taxation affects these macroeconomic variables twofold. If we consider the function $R(\tau)$ as defined in (13) the term $1 - \tau$ reflects the distortionary effect of taxation on accumulation. Here, a rise in the tax rate usually goes along with a decrease in after-tax capital return, thus discouraging growth. The term $\tau \frac{1}{1+\alpha}$ represents the positive effect of an increase in the government spending share $G/Y$ on marginal productivity of physical capital that usually is accompanied by an increase in accumulation. Figure 1(b) displays these counter–working effects of taxation on the function $R(\tau)$.

However, the equilibrium values of the macroeconomic variables of the stochastic economy differ from the deterministic ones to one important respect. While the latter are linear in the tax function, the propensity to consume (15) and the expected growth rate (16) of the stochastic economy are quadratic functions of $R(\tau)$. So, whenever we discuss the question of optimal choice of the tax rate, we have to take into account the second–order effect from the diffusion component of the macroeconomic variables. This is due to the fact, that a change in the tax rate affects the mean as well as the volatility of future income flows.

If we continue to limit our analysis to the case of precautionary savings, the distortionary effect, measured by $1 - \tau$, tends to reduce the expected growth rate in the drift as well as in the diffusion component, but for different reasons. The change in in the drift component is the well–known negative response to taxation as capital investment becomes less attractive. But, in explaining the response of the diffusion term, the argument follows a different line: Taxation reduces the volatility of future returns. The need for individual self–insurance on capital mark ets via precautionary savings diminishes, as the government participates in individual income risk.

The opposite result can be observed, if we consider the productivity enhancing effect of government spending, measured by $\tau \frac{1}{1+\alpha}$. Here, a rise in the tax rate unambiguously increases expected growth. The increase in the drift component can be explained with the positive productivity shift that makes accumulation more attractive. Contrary, the change in the diffusion term reflects the fact that future capital incomes have become riskier. The agent responds to this increase in riskiness with a rise in (precautionary) savings.
Nevertheless it is useful not to treat these distortionary and productivity enhancing effects isolated but in compound, as measured by tax function $R(\tau)$. The tax function responds to changes in the tax rate according to

$$\frac{\partial R(\tau)}{\partial \tau} = \gamma^\alpha \tau^{\frac{1-\alpha}{\alpha}} \left( \frac{1-\tau}{\tau} \cdot \frac{1-\alpha}{\alpha} - 1 \right) = 0 \implies \tau^* = 1 - \alpha$$

$$(23)$$

$$\frac{\partial^2 R(\tau)}{\partial \tau^2} \bigg|_{\tau^* = 1 - \alpha} = -\frac{1-\alpha}{\alpha} \gamma^\alpha \tau^{\frac{1-\alpha}{\alpha}-2} < 0.$$  

The function $R(\tau)$ has a unique maximum in $\tau^* = 1 - \alpha$ for feasible tax rates $\tau \in (0, 1)$. The optimal tax rate $\tau^*$ equals the partial elasticity of production of government expenditure. Figure 1(a) displays the typical curvature of the tax function which is well–known from the deterministic model.

**Command Economy** We first discuss the optimal tax policy a benevolent social planner chooses, in order to get benchmark results. The results of the deterministic setting directly extend to the stochastic framework, if we take account of the results from Proposition 1:

**Proposition 2 (Optimal Policy in the Command Economy)** The expected inter-temporal welfare attains optimal values either at

$$\frac{\partial R(\tau)}{\partial \tau} = 0 \implies \tau^* = 1 - \alpha \quad (24a)$$

or

$$\frac{\partial \mu^*}{\partial R(\tau)} = 0 \implies i^* = 0. \quad (24b)$$

(A) A tax policy implying $i^* = 0$ maximizes welfare for any $\tau \geq \tau^*$, $\tau \in (0, 1)$ and minimizes it for $\tau = \tau^*$.

(B) The tax rate $\tau^* = 1 - \alpha$ implies a welfare

(i) maximum, if $i^*|_{\tau^* = 1 - \alpha} > 0 \quad (24c)$

(ii) minimum, if $i^*|_{\tau^* = 1 - \alpha} < 0. \quad (24d)$

The tax rate $\tau^* = 1 - \alpha$ simultaneously maximizes the expected growth rate, the propensity to consume and welfare in the case of precautionary saving if the certainty equivalent to capital return evaluated at $\tau^*$ is positive.

(*Proof: see (B.9a) to (B.14c) in Appendix B.2*)

The role of the risk–free rate turns out to be a crucial one. If we allow the risk–free rate to be negative, we create a dynamically inefficient situation. A comparably high risk
premium leads to excessive growth out of precautionary motives and a correspondingly low propensity to consume. While the optimal tax rate \( \tau^* = 1 - \alpha \) still maximizes expected growth, it minimizes consumption. According to (21), this last effect dominates in the determination of lifetime utility which also attains a minimum.

Of special interest will now be the second finding from Proposition 2 that \( \tau^* \) simultaneously maximizes growth, consumption and welfare in the command optimum for a positive value of the certainty equivalent. This gives rise to the question whether these results extend to the stochastic competitive economy.

**Expected Growth Rate** The expected growth rate is a quadratic function in \( R(\tau) \). Hence, we additionally have to take the second–order effects from a change in the tax function into account in order to determine the growth effects of a change in the tax rate. This result stems from the underlying stochastic process and is the major extension to the analysis of the deterministic framework, where growth is linear in \( R(\tau) \). The general response of the expected growth rate to a change in \( \tau \) is given by

\[
\frac{\partial \psi}{\partial \tau} = \frac{\partial R(\tau)}{\partial \tau} \times \frac{\partial \psi}{\partial R(\tau)}.
\]

The first term is already known from (23), while the second one reflects the stochastic elements originating from the model extension. Maximization of the growth rate with respect to the tax rate yields:

**Proposition 3 (Growth)** The expected growth rate of the economy attains optimal values either at

\[
\frac{\partial R(\tau)}{\partial \tau} = 0 \quad \implies \quad \tau^* = 1 - \alpha \quad \text{(25a)}
\]

or

\[
\frac{\partial \psi}{\partial R(\tau)} = 0 \quad \implies \quad R(\tau) \bigg|_{\frac{\partial \psi}{\partial R(\tau)} = 0} \equiv R(\tau^*) = \frac{\alpha}{2\rho\sigma^2 \left( \alpha - \frac{\rho + 1}{2} \right)}. \quad \text{(25b)}
\]
The growth rate has a unique interior maximum at \( \tau^* = 1 - \alpha \) in case of \( \frac{1}{2} (\rho + 1) \geq \alpha \), that is the case of certainty equivalence and precautionary savings. A tax rate \( \tau \in (0, 1) \) corresponding to the optimal value \( R(\tau_{\psi}) \) minimizes \( \psi \) in case of precautionary savings.

(Proof: see (B.1a) to (B.2b) in Appendix B.1)

Condition (25a) corresponds to the results of Barro (1990). The expected growth rate attains an optimum when the distortionary effect from income taxation exactly offsets the capital productivity enhancing effect of an increase in government expenditures, that is, when marginal costs and revenues of taxation are equalized. Condition (25b) shows, that the growth rate has another optimum for a tax rate that implies a value of size \( R(\tau_{\psi}) \) for the tax function. From (25b), it immediately becomes obvious that \( R(\tau_{\psi}) \) is negative in case of precautionary savings. This solution is not feasible due the structure of the tax function, since it implies a non–real value for the tax rate \( \tau_{\psi} \).

The results of Proposition 3 are displayed in Figure 2. Figure 2(a) contrasts the expected growth rate of the competitive economy (black) with the expected growth rate of the command optimum (dark grey), and the growth rate of the deterministic decentralized economy (light grey) for the case of logarithmic preferences, \( \rho = 1.7 \). A comparison between the growth rates of the competitive deterministic and stochastic economy shows the effect of precautionary saving. Growth under uncertainty exceeds growth under certainty for each tax rate \( \tau \in (0, 1) \). Furthermore, as expected, Pareto–efficient growth is higher than growth in the competitive equilibrium.

Figure 2(b) displays the expected growth rates of the competitive (black) and the command economy (dark grey) for a comparably high degree of risk aversion, \( \rho = 4 \). This case reflects the results from Proposition 1. The expected growth rate of the decentralized economy exceeds the Pareto–efficient one in the interval \( [\underline{\tau}, \bar{\tau}] \) and the risk–free rate is negative. We get \( \ell = 0 \) in \( \underline{\tau} \) and \( \bar{\tau} \), and according to Proposition 1: \( \psi = \psi^\ell = \frac{1}{2} (\rho + 1) \sigma^2 R(\tau)^2 - \frac{\beta}{\rho} \).

Note that a risk–free rate of \( i \leq 0 \) does not affect the results of Proposition 3 regarding a growth maximizing policy .

**Propensity to Consume**  The propensity to consume out of capital also is a quadratic function in \( R(\tau) \) but, unlike the expected growth rate, it has an unique maximum in \( R(\tau) \) for feasible \( \tau \) in case of precautionary saving. This is displayed in Figure 3(c) and will be important for explaining the consumption effect of taxation. As before, the response of the macroeconomic variable to a change in the tax rate can be decomposed into the direct effect from a change in \( R(\tau) \) and the indirect effect from a change of \( R(\tau) \) with respect to \( \tau \). Consequently, maximization of the propensity to consume with respect to the tax rate leads to the following results:

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7The parameter values were set according to the following values in order to highlight the analytical findings: \( K(0) = 1, \alpha = 0.35, \beta = 0.05, \gamma = 1.5, \sigma = 0.76 \) (see Smith, 1996b).
Proposition 4 (Consumption) The propensity to consume out of capital attains optimal values either at
\[
\frac{\partial R(\tau)}{\partial \tau} = 0 \quad \implies \quad \tau^* = 1 - \alpha \quad (26a)
\]
or
\[
\frac{\partial \mu}{\partial R(\tau)} = 0 \quad \implies \quad R(\tau)\bigg|_{\frac{\partial \mu}{\partial R(\tau)} = 0} = R(\tau_\mu) = \frac{\rho - \alpha}{2\rho \sigma^2 \left(\frac{\rho + 1}{2} - \alpha\right)}. \quad (26b)
\]

(A) A tax policy according to \( \tau^* = 1 - \alpha \)

(i) maximizes \( \mu \), if \( R(\tau^*) \leq R(\tau_\mu) \) \( (26c) \)

(ii) minimizes \( \mu \), if \( R(\tau^*) > R(\tau_\mu) \). \( (26d) \)

(B) \( \forall \rho > \alpha \), a tax policy \( \tau_\mu \) according to \( R(\tau_\mu) \)

(i) maximizes \( \mu \), if \( \tau_\mu \neq \tau^* \) \( (26e) \)

(ii) minimizes \( \mu \), if \( \tau_\mu = \tau^* \). \( (26f) \)

(Proof: see (B.3a) to (B.4d) in Appendix B.1)

\( R(\tau^*) \) denotes the value of the tax function for \( \tau^* = 1 - \alpha \). Note that the optimal values of the tax function for the expected growth rate \( R(\tau_\psi) \) in (25b) and the propensity to consume \( R(\tau_\mu) \) in (26b) are not identical. \( \rho > \alpha \) is a sufficient condition that a tax rate \( \tau_\mu \neq \tau^* \) according to (26b) maximizes \( \mu \) and includes the case of precautionary saving.\(^8\)

Figure 3(a) depicts the case of logarithmic preferences which stands for a comparably low degree of risk aversion. In this case the optimal value of \( R(\tau_\mu) \) exceeds the value of the tax function for the optimal tax rate \( \tau^* \), that is \( R(\tau_\mu) > R(\tau^*) \). From (26b) and (26c) follows that the two optima of Proposition 4 work into the same direction and Figure 3(a) displays the familiar picture of a unique consumption maximizing tax rate \( \tau^* = 1 - \alpha \). Figure 3(a) also contrasts the propensity to consume of the stochastic competitive economy (black) with the corresponding relationships of the stochastic command (dark grey) and the competitive deterministic economy (light grey). Again we observe (a) the consumption effect from precautionary saving, i.e., that the expected propensity to consume falls short of its deterministic counterpart, and (b) that consumption in the competitive economy is inefficiently large.

Now, what is the new information from Proposition 4? We find that the tax rate \( \tau^* = 1 - \alpha \) indeed might minimize the propensity to consume. This result is more likely the more risk averse the agent is. As can be seen from (26b), a rise in the coefficient of relative risk aversion \( \rho \) affects the optimal value \( R(\tau_\mu) \) of the tax function for a given set of

\(^8\)\( \rho > \alpha \) will also be relevant in the discussion of the welfare effect, see item A of Proposition 6.
the other parameters. The higher the index of risk aversion or the lower the capital income share respectively, the earlier the maximizing effect of the optimal value $R(\tau_{I})$ dominates the effect of $\tau^*$. Figures 3(b) to 3(d) illustrate this result. In Figure 3(c) the consumption maximizing value of the tax function is $R(\tau_{I}) \approx 0.37$. This value is displayed by the dashed line in Figure 3(d). The black graph is the propensity to consume for a varying tax rate, while the grey line is the tax function $R(\tau)$. The propensity to consume has two local maxima at $R(\tau_{I})$ for a comparably high degree of risk aversion of $\rho = 4$, with $\tau_{I} \in \{ \tau, \bar{\tau} \}$. Those maxima are in the intersection of the dashed with the grey line in Figure 3(d), while $\tau^* = 1 - \alpha$ with a corresponding value of $R(\tau^*) \approx 0.5 > R(\tau_{I})$ is a local minimum.

Figure 3(b) contrasts the propensity to consume of the Pareto–efficient (dark grey) with the inefficient stochastic economy (black) for the comparably high degree of risk aversion of $\rho = 4$. In general, the statements of Proposition 4 are also valid for the command optimum (see (B.10a) to (B.12) in Appendix B.2). Yet, they differ to one important respect: The tax rate $\tau^* = 1 - \alpha$ only minimizes the propensity to consume $\mu^*$ if the risk–free rate is negative. As we already know from the discussion of growth effects, $\psi = \psi^*$ for $i = 0$ and consequently $\mu = \mu^*$. This case is reflected in Figure 3(b) by the intersection of the two graphs. The corresponding tax rates are $\underline{\tau}$ and $\bar{\tau}$ which only maximize $\mu^*$, if
As before, in the analysis of the growth effects of taxation, we find that the risk–free rate is negative in the interval \( \tau \in (\bar{\tau}, \overline{\tau}) \) which includes \( \tau^* \). See also Figure 5(b) for an illustration of this result.

It is important to stress that, according to (26b) – (26d), this argument cannot be carried over to the competitive economy in a simple manner. As can be seen from Figures 3(b) and 5(b), the tax rates \( \bar{\tau}, \overline{\tau} \) which maximize \( \mu \) lie outside the interval \( (\bar{\tau}, \overline{\tau}) \) and imply a positive risk–free rate. This difference in results between the competitive and the Pareto–efficient economy is highlighted in Figure 4 for an intermediate degree of risk aversion of \( \rho = 3 \). The tax rate \( \tau^* = 1 - \alpha \) maximizes the propensity to consume \( \mu^* \) but minimizes \( \mu \), while the risk–free rate is positive in the entire interval \( \tau \in (0, 1) \).

**Risk–free Rate**

The analysis of the preceding paragraphs has shown that the certainty equivalent to capital return and the risk premium play a key role in the determination of the macroeconomic effects of taxation in a stochastic environment. We stated in Proposition 1 that, in order to preserve the results from the deterministic model, the risk premium should not be too high to avoid a negative risk–free rate. Otherwise, the Pareto–efficient expected growth rate falls short of the competitive one. We demonstrated that the statements of Proposition 1 are also crucial for the propensity to consume. The lower the risk–free rate is, the more likely does the growth maximizing tax rate \( \tau^* \) minimize consumption, and a negative certainty equivalent to capital return is more likely the higher the risk premium \( \alpha \rho \sigma^2 R(\tau)^2 \) is for a given mean return on physical capital. The tax function and the capital income share are bounded by definition, so the index of risk aversion or the variance of the technological shocks are the most likely candidates to induce a negative risk–free rate.

We will now turn to the question of how changes in the tax rate affect the risk–free rate itself. Similar to the expected growth rate and the propensity to consume, the certainty
equivalent to capital return is a quadratic function in \( R(\tau) \). Maximization with respect to the tax rate leads to the following results:

**Proposition 5 (Risk–free Rate)** The risk–free rate attains optimal values at

\[
\frac{\partial R(\tau)}{\partial \tau} = 0 \quad \Rightarrow \quad \tau^* = 1 - \alpha \quad (27a)
\]

or

\[
\frac{\partial i}{\partial R(\tau)} = 0 \quad \Rightarrow \quad R(\tau)\left|_{\frac{\partial i}{\partial R(\tau)}=0} = R(\tau) = \frac{1}{2\rho\sigma^2} > 0. \quad (27b)
\]

(A) A tax policy according to \( \tau^* = 1 - \alpha \)

(i) maximizes \( i \), if \( R(\tau^*) \leq R(\tau_i) \) \( (27c) \)

(ii) minimizes \( i \), if \( R(\tau^*) > R(\tau_i) \). \( (27d) \)

(B) A tax policy \( \tau_i \) according to \( R(\tau_i) \)

(i) maximizes \( i \), if \( \tau_i \neq \tau^* \) \( (27e) \)

(ii) minimizes \( i \), if \( \tau_i = \tau^* \). \( (27f) \)

(Proof: see (B.5b) to (B.6d) in Appendix B.1)

Equation (27b) shows that the optimal value \( R(\tau_i) \) is a function solely of two parameters, the index of risk aversion and the variance of the technological shocks. The smaller these parameters the higher the optimal value \( R(\tau_i) \) will be and, according to (27c), the more likely \( \tau^* = 1 - \alpha \) will be a maximum of the risk–free rate. This result is displayed in Figure 5(a) for the case of \( \rho = 1 \) which is a comparably low value for the coefficient of risk aversion. Here, the risk–free rate attains an interior maximum at \( \tau^* = 1 - \alpha \). Again, we contrast the stochastic inefficient and efficient economy with the deterministic one. Figure 5(a) shows, that the equilibrium interest rate of the deterministic economy (light grey) exceeds its stochastic counterpart (black) in case of a positive risk premium and equals mean capital return by the arbitrage argument given above. The risk–free rate of the decentralized economy is smaller than the one of the stochastic command optimum (dark grey), which shows that the suboptimally low private return on capital is reflected in the equilibrium value of its certainty equivalent.

Figure 5(b) displays the expected Pareto–efficient and inefficient risk–free rate for a degree of risk aversion of \( \rho = 4 \). Both risk–free rates \( i \) and \( i^* \) have a local minimum in \( \tau^* \) for this comparably large value of the coefficient of risk aversion. Still, \( \tau^* \) maximizes the expected net rate of return on physical capital \( E[(1 - \tau) r] = \alpha R(\tau) \) and hence investment, as can be seen from the analysis of the growth effects. So, what we observe in \( \tau^* \) is a
situation where the agents demand a maximum risk premium on their capital holdings. The capital risk associated with investment is smaller for any tax rate $\tau \geq \tau^*$. According to (27d), we find two local maxima for $i, i^*$ in $R(\tau_i)$ with $\tau_i \in \{\tau, \overline{\tau}\}$. The graphs for the competitive and the command economy intersect in $\tau$ and $\overline{\tau}$ where $i = i^* = 0$, $\mu = \mu^*$ and $\psi = \psi^*$. The risk–free rate is negative in the interval $\tau \in (\tau, \overline{\tau})$ which includes $\tau^*$. Note that $R(\tau_i)$ deviates from $R(\tau_\mu)$, which can also be seen from Figure 5(b). A comparison between the risk–free rate– and the consumption–maximizing tax rates shows that $\tau < \tau^*$ and $\overline{\tau} > \overline{\tau}$.

**Welfare Effects** Let us now turn towards the welfare effects of a change in the tax rate. From (17), it can be seen that the effect of any tax policy on welfare of the competitive economy can be assessed in terms of its impact on the propensity to consume, the expected growth rate, and indirectly, by its effects on the certainty equivalent to capital return. The central point of interest will be the design of the optimal, that is, the welfare maximizing policy. In general, an optimal policy will be characterized by a tax scheme that equates marginal (social) costs of government expenditure to its marginal benefits in terms of an increase of productivity.

If we summarize the results from the analysis of the macroeconomic variables, we can state the following: (a) the growth rate is maximized for $\tau^* = 1 - \alpha$, (b) $\tau^*$ minimizes the propensity to consume and the risk–free rate for a comparably high risk premium, which is not necessarily accompanied by a negative risk–free rate, and (c) welfare is a function of $\psi$ and $\mu$. Other things equal, welfare increases in $\mu$ and $\psi$, but additionally the indirect (negative) effect from consumption on growth has to be taken into account. Maximization of (17) with respect to the tax rate leads to the following result:
**Proposition 6 (Optimal Policy in the Competitive Economy)**  
The expected lifetime utility attains optimal values either at

\[ D(\tau) = \mu - (1 - \alpha) R(\tau) \frac{\partial \mu}{\partial R(\tau)} = 0 \]  
(28a)

or

\[ \frac{\partial R(\tau)}{\partial \tau} = 0 \quad \implies \quad \tau^* = 1 - \alpha \]  
(28b)

or

\[ i = 0 \quad \implies \quad \tau|i=0 \equiv \tau_v \in \{\bar{\tau}, \underline{\tau}\} \]  
(28c)

(A) A tax policy according to \( D(\tau) = 0 \) is not consistent with feasible solutions of the model.

\[ D(\tau) > 0, \quad \forall \rho > \alpha. \]  
(28d)

\[ D(\tau) = \Delta = \mu^* > 0 \text{ in the command economy.} \]

(B) A tax policy according to \( \tau^* = 1 - \alpha \).

(i) maximizes \( V(0) \), if \( i|\tau^*=1-\alpha \geq 0 \)  
(28e)

(ii) minimizes \( V(0) \), if \( i|\tau^*=1-\alpha < 0 \).  
(28f)

(C) A tax policy according to \( \tau_v \in \{\bar{\tau}, \underline{\tau}\} \)

(i) maximizes \( V(0) \), if \( \tau_v \in \{\bar{\tau}, \underline{\tau}\} \neq \tau^* \)  
(28g)

(ii) minimizes \( V(0) \), if \( \tau_v \in \{\bar{\tau}, \underline{\tau}\} = \tau^* \).  
(28h)

(Proof: see (B.7a) to (B.8d) in Appendix B.1)

Contrary to the non–stochastic model where \( \tau^* \) is a unique optimum, we find three candidates for an optimal policy. Referring to part (A) of Proposition 6, we can neglect a tax policy according to (28a). The term \( D(\tau) \) is an artefact which stems from the result that, unlike in the command optimum, in the competitive economy \( \Delta \neq \mu \). Under condition (28d), and depending on the parameterization of the model, the term \( D(\tau) \) is bounded from below, either by the propensity to consume or by the transversality condition.

One major result from Proposition 6 is that a growth maximizing tax rate \( \tau^* \) does not necessarily represent a welfare maximizing policy. Condition (28g) establishes the existence of other welfare maximizing policies, which do not maximize the expected growth rate. This outcome contradicts the result from the deterministic model of productive government expenditure and can be ascribed to the second–order effects of the technological disturbance on individual decision–making.
As before in the discussion of the consumption effects, the results are related to the size of the risk premium. From (28e) and (28h) follows that non-negativity of the risk free rate, evaluated at the optimal tax rate $\tau^* = 1 - \alpha$, is a necessary and sufficient condition to ensure that this tax rate constitutes a welfare maximizing policy. Here, the impact from the expected growth rate on welfare always more than compensates any impact from the propensity to consume. This is the only case, where the findings from the non–stochastic environment extend to the stochastic model. A tax policy according to the tax rate $\tau^*$ maximizes all the macroeconomic relationships and subsequently maximizes welfare.

Figure 6(a) depicts this result for the case of logarithmic preferences. We contrast the stochastic competitive (black) and the command economy (dark grey) with their respective deterministic counterparts (medium grey, light grey). In all of the four settings $\tau^* = 1 - \alpha$ maximizes welfare. Of course, the highest welfare can be realized in the non–stochastic command optimum, because a risk averse agent always prefers a safe outcome to any risky prospect with the same mathematical expectation of payoffs.9

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9Logarithmic preferences imply certainty equivalence in the command optimum with respect to $\mu^*$ and $\psi^*$, but of course not with respect to $V[K(0)]^*$. 
If we compare the two stochastic economies, we find the results from the deterministic model of productive government expenditure extended to the stochastic setting. The intertemporal lifetime utility in the competitive economy falls short of the Pareto–efficient utility level. But, if we compare the two competitive economies, we see another interesting result, namely that welfare in the stochastic environment exceeds welfare in the deterministic one. This results stems from the fact that, even for logarithmic preferences, the agent of the competitive economy has a motive for precautionary saving. The risk–induced accumulation moves the expected growth rate of the decentralized economy closer towards its Pareto–efficient level and thus improves welfare.

Even if we consider a tax policy \( \tau^* = 1 - \alpha \) where, according to condition \((26d)\) of Proposition 4, the propensity to consume is minimized, this effect is more than offset by a maximized growth rate, as long as the risk–free rate, evaluated at \( \tau^* \), is positive. Figure 6(b) illustrates this result for a value of the index of risk aversion of \( \rho = 3 \), which we also used in Figure 4. A comparison between Figure 4 and 6(b) shows that a tax policy according to \( \tau^* \) maximizes growth and welfare in the competitive economy but minimizes the propensity to consume and the risk–free rate. From item (B) of Proposition 2 follows that this outcome cannot appear in the command optimum.

In contrast to this, Figure 6(c) displays the intertemporal welfare of the decentralized economy for the comparably high coefficient of risk aversion of \( \rho = 4 \). According to \((28f)\) of Proposition 6, a government choosing \( \tau^* = 1 - \alpha \) minimizes welfare. Although this tax rate maximizes growth, it would actually minimize the propensity to consume and the risk–free rate. This latter effects dominate in the determination of welfare, because \( \tau^* \) entails more risk, than the agents are willing to bear. The economy would be better off instead by the choice of a tax rate \( \tau_v \in \{\tau, \overline{\tau}\} \). The government can choose between taxation of incomes at a low rate \( \tau < \tau^* \) or a comparably high rate \( \overline{\tau} > \tau^* \). While both tax rates imply identical optimal welfare levels and a zero risk–free rate, the lower of the two values is closer to tax rates observed in real tax schemes.

This kind of optimal policy displays a peculiar characteristic: Differences in returns between the competitive and the command economy, which arise due to the misperception of the productivity enhancing effect of government expenditure, are no longer relevant for the determination of the equilibrium values of the macroeconomic relationships. A tax policy according to \( \tau_v \in \{\tau, \overline{\tau}\} \) maximizes welfare in the competitive as well as in the command economy\(^{10} \) because the two economies are identical. Figure 6(d) highlights this result. It displays the grey shaded area of Figure 6(c) and compares the change in welfare due to a rise in the tax rate for the competitive (black) with the Pareto–efficient stochastic economy (dark grey). For \( \tau < \overline{\tau} \) and \( \tau > \overline{\tau}, \) the expected growth rate of the competitive economy is suboptimally low. It is suboptimally high for \( \overline{\tau} < \tau < \overline{\tau}, \) which includes \( \tau^* \); see Figure 2(b). Accordingly, welfare of the competitive economy falls below its Pareto–

\(^{10}\)See Proposition 2.
efficient value in those areas. The expected growth rate, the propensity to consume, and welfare are identical in both economies, if the tax rate takes on either the value $τ$ or $\bar{τ}$.

Hence, we find an important difference to the previously discussed optimal policy where $i|τ=1-α ≥ 0$. Under this condition, the optimal tax rate $τ^* = 1 - α$ maximizes the macroeconomic relationships in both, the competitive as well as the command economy, but the equilibrium values itself differ from each other. Although the government of the decentralized economy fixes its expenditure at the optimal level, it cannot close the gap between private and social returns. This second–best optimum stems from the distortion of the income tax.\textsuperscript{11}

To sum up the results, we can state the following: (a) The agents respond to the riskiness of their income flows in different ways, depending on whether the income stems from accumulable or non–accumulable factors. For this reason, the results derived for the stochastic competitive economy not only depend on the degree of risk aversion but also on the factor income distribution. This is not the case for the Pareto–efficient economy, where only capital risk is relevant for intertemporal decision–making. There, the size of the coefficient of risk aversion solely decides on precautionary motives. (b) The stochastic endogenous growth model with productive government expenditure can replicate the results from the deterministic setting, if we additionally impose a non–negativity constraint on the certainty equivalent to capital return. Under this condition, an optimal tax policy exists that simultaneously maximizes growth, consumption, and welfare for a tax rate, which equals the partial elasticity of production of government expenditure. (c) Under the same condition, we also find an optimal policy for the stochastic competitive economy which maximizes growth, but minimizes the propensity to consume. This policy maximizes welfare, because the growth effect more than compensates the consumption effect. It cannot appear neither in the Pareto–efficient nor in the non–stochastic economy. (d) Maximized welfare in the risky competitive environment can be higher than in the deterministic one due to the presence of precautionary saving, which drives the expected growth rate closer to its efficient value. (e) If the equilibrium risk premium is sufficiently large, a tax policy, where the tax rate equals the partial elasticity of production of government expenditure, still maximizes growth, this, however, to a dynamically inefficient degree, since it minimizes welfare, the propensity to consume and the risk–free rate. The latter even becomes negative. An optimal policy instead is characterized by a tax rate, which leads to a zero certainty equivalent to capital return. In this case, the competitive and the Pareto–efficient economy are identical. Moreover, the optimal tax scheme is not unique, since we find two tax rates, which satisfy the optimality condition. This welfare maximizing policy does not maximize the expected growth rate.\textsuperscript{11}

\textsuperscript{11}A first–best allocation can only be achieved, if we implement another fiscal instrument, for instance a consumption tax or government bonds, see Turnovský (1996) or Clemens and Soretz (1997, 1998).
4 Summary and Discussion

In this paper we developed a continuous–time stochastic endogenous growth model with a positive production externality due to productive government expenditure. A separate analysis of the drift and the diffusion components of the macroeconomic variables lead to the conclusion that, especially in the diffusion component, taxation may be either growth enhancing or growth depressing. Contrary to a pure AK–type model, the direction of effects is determined by two factors: the coefficient of relative risk aversion and the factor income distribution.

Our analysis was confined to the case of precautionary saving. For this case, we found that there is no unique growth, consumption and welfare maximizing income tax scheme, as the second–order effects from the technological disturbance have to be taken into account. This ambiguity with respect to the optimal tax scheme is the main result of our paper and constitutes the major difference to the non–stochastic model of productive government expenditure.

We first derived conditions under whom the optimal tax scheme is identical to the one of the deterministic framework. According to this, the share of government purchases is chosen optimally if the marginal costs of government spending equals its marginal benefit. The optimal tax rate equals the partial elasticity of production of government expenditure. While this tax scheme still maximizes growth and welfare, it minimizes the expected propensity to consume and the certainty equivalent to capital return, if the equilibrium risk premium is sufficiently large. Moreover, this policy turns out to be welfare minimizing for an even larger risk premium as expected growth is suboptimally high. This last result only occurs for a negative risk–free rate.

So the question is, what does an negative risk–free rate imply? The equilibrium value of the risk–free rate was derived via an arbitrage argument for an economy with identical individuals. The credit market is closed as none of the agents is willing to sell or buy bonds at this equilibrium value. Wealth is invested entirely in risky physical capital and the agents demand a risk premium in order to be compensated for the riskiness of their investment. The tax rate $\tau^*$ maximizes the risk premium for any given attitude towards risk, and the risk premium itself increases linearly in the degree of risk aversion.

In the case of a negative certainty equivalent to capital return, we have a situation where an agent is risk averse to a degree that he would prefer and accept a devaluation of wealth by an investment in a safe asset which yields a negative rate of return rather than to invest in physical capital, which yields a positive return. So, if a third party offered the agents of our model a safe asset with a negative rate of return slightly above the equilibrium value of the risk–free rate, they would entirely switch away from capital investment.\textsuperscript{12} Yet, a safe asset is not available in the underlying economy. As long as the

\textsuperscript{12}Accordingly there would be no production and no real growth.
relevant macroeconomic relationships such as the propensity to consume and the expected growth rate are positive, and utility is bounded, the risk–free rate might as well be negative.

The second type of optimal policy was characterized by a zero risk–free rate. This specific case gives rise to a situation where differences between private and social returns on investment play no role for the choice of the optimal tax rate. The Pareto–efficient and the competitive economy are identical. Welfare and growth maxima do not coincide. Due to the second–order effects of the productivity shocks on the equilibrium values of the macroeconomic relationships, we find two tax rates, one below the other above the growth maximizing tax rate, which satisfy the optimality conditions and maximize welfare.

Again, these results are associated with a risk premium being sufficiently large. A natural question then would be to ask, whether they are likely to occur for a realistic parameterization of the model. Besides the tax rate, the size of the risk premium essentially depends on the coefficient of risk aversion, the capital income share, and the variance of the productivity shock. If we consider the empirical estimates for the latter, obtained for example by Prescott (1986) or Ramey and Ramey (1995), the degree of risk aversion has to be unrealistically large in order to explain the theoretical findings; see Mehra and Prescott (1985).

Nevertheless, we think our results regarding the importance of the risk–premium are more than a theoretical curiosity, closely depending on the framework discussed here. They can also be observed in the context of another widely applied endogenous growth model, the stochastic version of the ‘learning by doing’–approach by Romer (1986), which displays equivalent properties (see Clemens, 1999): a technology with increasing returns to scale on the aggregate level and a Pareto–inefficient competitive allocation. Although, in the deterministic setting, both approaches are considered to be of the AK–type if labor is inelastically supplied and normalized to unity, this argument does not carry over to the stochastic framework in a simple manner, because the risk–averse agent responds to the riskiness of incomes from accumulable and non–accumulable factors differently.

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A Optimization and Derivation of Lifetime Utility

A.1 Decentralized Economy

Let $V[W(t), t]$ represent the maximum feasible level of expected lifetime utility. Positing the time-separable form $V[W(t), t] = e^{-\beta t} J(W)$, application of Itô’s Lemma leads to the following objective function

$$\max_{C, K, W} \mathcal{L} = U(C) e^{-\beta t} + V_t + V_W [i (W - K) + (1 - \tau) (r K + \omega) - C] + \frac{1}{2} V_{WW} \sigma_W^2. \quad (A.1)$$

The optimality conditions with respect to $C, K$ and $W$ are

$$0 = U'(C) - J'(W), \quad (A.2a)$$

$$0 = J'(W) [i (1 - \tau) - i] + \frac{1}{2} \frac{\partial \sigma_W^2}{\partial K} J''(W), \quad (A.2b)$$

$$0 = J'(W) (i - \beta) + J''(W) [i (W - K) + (1 - \tau) (r K + \omega) - C] + \frac{1}{2} J''(W) \sigma_W^2, \quad (A.2c)$$

with the variance of wealth given by $\sigma_W^2 = \mathbb{E} (dW)^2 / dt$.

Condition (A.2a) displays the well-known result of equalized marginal utility of consumption over time and determines the accumulation process together with (A.2c). Condition (A.2b) sets up the optimal portfolio choice.

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Condition (A.2c) replaces the Bellman equation that is usually used in this kind of setting. Since we are dealing with suboptimal outcomes in the decentralized economy and increasing returns to scale on the aggregate level, Bellman’s principle of optimality cannot be applied here. For details see Clemens (1999, Chapter 2).
Furthermore, the transversality condition

$$\lim_{t \to \infty} E_t [V(W(t), t)] = 0$$  \hspace{1cm} (A.3)

has to be satisfied to assure that lifetime utility is bounded.

According to (5), lifetime utility is a function of instantaneously deterministic consumption. From substitution of (15) follows

$$V'(0) = E_0 \int_0^{\infty} \frac{\mu^{1-\rho} K(t)^{1-\rho}}{1-\rho} e^{-\beta t} \, dt.$$  \hspace{1cm} (A.4)

By assumption the stochastic process of production is a geometric Brownian motion. From this follows that the aggregate stock of capital is log–normally distributed. Given the variance of physical capital $\sigma^2_K = \sigma^2 R(\tau)^2$, the expected capital stock of time 0 can then be derived as

$$E_0 [K(t)] = \exp ((1-\rho) (\ln K(0) + \psi t - \frac{1}{2} \sigma^2 R(\tau)^2) + \frac{1}{2} (1-\rho)^2 \sigma^2 R(\tau)^2 t)$$

$$= K(0)^{1-\rho} e \left\{(1-\rho) \left(\psi - \frac{1}{2} \rho \sigma^2 R(\tau)^2\right) t\right\}. \hspace{1cm} (A.5)$$

With the initial capital stock exogenously given, a constant growth rate and propensity to consume, combination of (A.4) and (A.5), and integration leads to the expression (17) for maximized individual welfare

$$V[K(0), 0] = \frac{K(0)^{1-\rho}}{1-\rho} \cdot \frac{\mu^{1-\rho}}{\beta - (1-\rho) \left(\psi - \frac{1}{2} \rho \sigma^2 R(\tau)^2\right)}, \hspace{1cm} (A.6)$$

and $V[K(0), 0] = \frac{1}{\beta} (\beta \ln K(0) + \beta \ln \mu + \psi - \frac{1}{2} \sigma^2 R(\tau)^2)$ for $\rho = 1$.

**Proof of Proposition 1** The transversality condition (A.3) is satisfied if the denominator of the second term in (A.6) $\Delta = \beta - (1-\rho) \left(\psi - \frac{1}{2} \rho \sigma^2 R(\tau)^2\right)$ is positive. This is equivalent to

$$\Delta \equiv \beta + (\rho - 1) \left[\alpha R(\tau) + \rho \sigma^2 R(\tau)^2 \left(\frac{1}{2} - \alpha\right)\right] > 0. \hspace{1cm} (A.7)$$

Rearranging the term in square brackets yields $\alpha R(\tau) (1 - \rho \sigma^2 R(\tau)) + \frac{1}{2} \rho \sigma^2 R(\tau)^2$. From this becomes obvious that the transversality condition can only be violated in case of precautionary saving if $1 - \rho \sigma^2 R(\tau) < 0$, i.e., if the risk free rate (14) is sufficiently negative. The same condition applies for $\psi - \psi^* > 0$.

**A.2 Command Optimum**

For simplicity, the analysis is restricted to the closed–economy case, where $K = W$. The planner chooses the consumption–capital ratio and savings as to maximize expected lifetime utility (5) of the representative agent subject to the capital accumulation equation

$$\frac{dK}{dt} = [\gamma K^\alpha L^{1-\alpha} G^{1-\alpha} - G - C] \, dt + [\gamma K^\alpha L^{1-\alpha} G^{1-1-\alpha} - G] \, dy. \hspace{1cm} (A.8)$$

He accounts for the equilibrium value of public (12). Substitution into (A.8) leads to

$$\frac{dK}{dt} = \left[R(\tau) - \mu^*\right] K \, dt + R(\tau) K \, dy. \hspace{1cm} (A.9)$$

As before, the value function $V[K(t), t] = e^{-\beta t} J(K)$ represents maximized lifetime utility. Application of Itô’s Lemma and taking expectations implies the following objective function:

$$\mathcal{L} = U(C) e^{-\beta t} + V_t + V_K [R(\tau) - \mu^*] K + \frac{1}{2} V_{KK} \sigma^2_K, \hspace{1cm} (A.10)$$

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Taking account of \( V[K(t), \tau] = e^{-\beta t} J(K) \) and taking partial derivatives with respect to \( C \) and \( K \) leads to the necessary conditions:

\[
0 = U'(C) - J'(K) \quad \text{(A.11a)}
\]

\[
0 = J'(K) [R(\tau) - \beta] + J''(K) R(\tau) K \left[ 1 - \mu^* + \sigma^2 R(\tau) \right] + \frac{1}{2} \sigma^2 J''(K) K^2 R(\tau)^2. \quad \text{(A.11b)}
\]

The conjecture for the optimal program is, as before, that the consumption–capital–ratio \( C/K = \mu^* \) is constant in equilibrium. From (A.11a) then follows immediately

\[
J'(W) = C^{-\rho}, \quad J''(W) = -\rho \mu^* C^{-(\rho+1)}, \quad J'''(W) = \rho (\rho + 1) \mu^* C^{-(\rho+2)}. \quad \text{(A.12)}
\]

Substitution into (A.11b) and solving for \( \mu^* \) leads to expression (19) of the text. The expected growth rate of the economy (20) is then obtained by substituting (19) into the capital accumulation equation (A.9) and taking expectations.

The solution procedure for lifetime utility equals the one described in the preceding section. By substitution of the consumption–capital–ratio and the initial capital stock of the command economy, welfare can be derived as

\[
V[K(0), 0^*] = \frac{K(0)^{1-\rho}}{1-\rho} \cdot \left( \frac{\mu^*}{\beta - (1-\rho) (\psi - \frac{1}{4} \rho \sigma_\psi^2)} \right)^{1-\rho} = \frac{K(0)^{1-\rho}(\mu^*)^{-\rho}}{1-\rho}, \quad \text{(A.13)}
\]

and \( V[K(0), 0^*] = \frac{1}{1-\rho} \left( \beta \ln K(0) + \beta \ln \beta + \psi - \frac{1}{4} \rho \sigma_\psi^2 R(\tau)^2 \right) \) for \( \rho = 1. \)

## B Optimal Tax Policy

### B.1 Decentralized Economy

**Growth Effects** The first- and second-order conditions of optimizing the expected growth rate with respect to the tax rate are given by:

\[
\frac{\partial \psi}{\partial \tau} = \frac{\partial R(\tau)}{\partial \tau} \times \frac{\partial \psi}{\partial R(\tau)} = \frac{\partial R(\tau)}{\partial \tau} \times \frac{\partial \psi}{\partial R(\tau)} = \frac{1-\tau}{\tau} \times \frac{1-\alpha}{\alpha} \quad \text{(B.1a)}
\]

\[
\frac{\partial^2 \psi}{\partial \tau^2} = \frac{\partial \psi}{\partial R(\tau)} \times \frac{\partial^2 R(\tau)}{\partial \tau^2} + \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 \times \frac{\partial^2 \psi}{\partial R(\tau)^2} = \frac{1-\tau}{\tau} \times \frac{1-\alpha}{\alpha} \quad \text{(B.1b)}
\]

The necessary condition (25a) of the text is derived out of (B.1a) by setting \( A = 0 \), while condition (25b) corresponds to \( B = 0 \) and solving for \( R(\tau_{\psi}) \).

**Proof of Proposition 3** Evaluation of (B.1b) at \( \tau^* = 1-\alpha \) and \( R(\tau) = R(\tau_{\psi}) \) respectively leads to the following results:

\[
\frac{\partial^2 \psi}{\partial \tau^2} \big|_{\tau^* = 1-\alpha} = -\frac{\partial \psi}{\partial R(\tau)} \times \frac{1-\alpha}{\alpha} \quad \text{(B.2a)}
\]

\[
\frac{\partial^2 \psi}{\partial \tau^2} \big|_{R(\tau) = R(\tau_{\psi})} = \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 \sigma^2 \rho \left[ \rho + 1 - 2\alpha \right]. \quad \text{(B.2b)}
\]

\( \partial \psi/\partial R(\tau) \) of (B.2a) as well as the second expression \( \rho + 1 - 2\alpha \) of (B.2b) is of positive sign in case of precautionary savings. From this follows immediately that \( \tau^* = 1-\alpha \) maximizes the growth rate. \( R(\tau_{\psi}) \) is a minimum in case of precautionary savings for \( \tau_{\psi} \neq \tau^* \) and in case of \( \tau = \tau^* \). \( \square \)
**Consumption Effects** The first- and second-order conditions of optimizing the expected propensity to consume out of capital with respect to the tax rate are given by:

\[
\frac{\partial \mu}{\partial \tau} = \tau^{1-a} \gamma^\frac{1}{\alpha} \left[ \frac{1 - \tau}{\tau} \cdot \frac{1 - \alpha}{\alpha} \right] \times \left[ \frac{\rho - \alpha}{\rho} + 2\sigma^2 R(\tau) \left( \alpha - \frac{\rho + 1}{2} \right) \right]
\]  

(B.3a)

\[
\frac{\partial^2 \mu}{\partial \tau^2} = \frac{\partial \mu}{\partial R(\tau)} \frac{1 - \alpha}{\alpha} \gamma^\frac{1}{\alpha} \tau^{\frac{1-a}{\alpha}-1} \left[ \frac{1 - \tau}{\tau} \cdot \frac{1 - \alpha}{\alpha} - \frac{1}{\tau} \right] + \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 \sigma^2 \left[ 2\alpha - (\rho + 1) \right].
\]  

(B.3b)

The necessary condition (26a) of the text is derived out of (B.3a) by setting \( A = 0 \), while condition (26b) corresponds to \( C = 0 \) and solving for \( R(\tau_i) \).

**Proof of Proposition 4** Evaluation of (B.3b) at \( \tau^* = 1 - \alpha \) and \( R(\tau) = R(\tau_i) \) respectively leads to the following results:

\[
\frac{\partial^2 \mu}{\partial \tau^2} \bigg|_{\tau^* = 1 - \alpha} = -\frac{\partial \mu}{\partial R(\tau)} \frac{1 - \alpha}{\alpha} \gamma^\frac{1}{\alpha} \tau^{\frac{1-a}{\alpha}-2} \quad (B.4a)
\]

\[
\frac{\partial^2 \mu}{\partial \tau^2} \bigg|_{R(\tau)=R(\tau_i)} = \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 \sigma^2 \left[ 2\alpha - (\rho + 1) \right]. \quad (B.4b)
\]

From (B.4b) it becomes obvious that the choice of a tax rate \( \tau \) implying the optimal value \( R(\tau_i) \) maximizes the consumption-capital ratio in case of precautionary saving if \( \tau_i \neq \tau^* \). From (B.4a) follows that \( \tau^* = 1 - \alpha \) is a maximum or a minimum for the propensity to consume, if

\[
\frac{\partial^2 \mu}{\partial \tau^2} \bigg|_{\tau = 1 - \alpha} \leq 0 \quad \text{for} \quad \frac{\partial \mu}{\partial R(\tau)} \leq 0 \quad \implies \quad R(\tau^*) \leq R(\tau_i) \quad (B.4c)
\]

We find

\[
\frac{\partial^3 \mu}{\partial \tau^3} \bigg|_{\tau^* = 1 - \alpha} = \frac{\partial^3 \mu}{\partial \tau^3} \bigg|_{\tau = 1 - \alpha} = 0,
\]

\[
\frac{\partial^3 \mu}{\partial \tau^3} \bigg|_{\tau^* = 1 - \alpha} < 0 \quad \text{and} \quad \frac{\partial^3 \mu}{\partial \tau^3} \bigg|_{\tau = 1 - \alpha} > 0 \quad \text{for} \quad \rho > \alpha. \quad (B.4d)
\]

**Risk–free Rate Effects** The first- and second-order conditions of optimizing the risk–free rate with respect to the tax rate are given by:

\[
\frac{\partial i}{\partial \tau} = \tau^{\frac{1-a}{\alpha}} \gamma^\frac{1}{\alpha} \left[ \frac{1 - \tau}{\tau} \cdot \frac{1 - \alpha}{\alpha} \right] \times \alpha \left[ 1 - 2\rho \sigma^2 R(\tau) \right]
\]  

(B.5a)

\[
\frac{\partial^2 i}{\partial \tau^2} = \frac{\partial i}{\partial R(\tau)} \frac{1 - \alpha}{\alpha} \gamma^\frac{1}{\alpha} \tau^{\frac{1-a}{\alpha}-1} \left[ \frac{1 - \tau}{\tau} \cdot \frac{1 - \alpha}{\alpha} - \frac{1 - \tau}{\tau} \right] - \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 2\alpha \rho \sigma^2
\]  

(B.5b)

**Proof of Proposition 5** Evaluation of (B.5b) at \( \tau^* = 1 - \alpha \) and \( R(\tau) = R(\tau_i) \) respectively leads to the following results:

\[
\frac{\partial^2 i}{\partial \tau^2} \bigg|_{\tau^* = 1 - \alpha} = -\frac{\partial i}{\partial R(\tau)} \frac{1 - \alpha}{\alpha} \gamma^\frac{1}{\alpha} \tau^{\frac{1-a}{\alpha}-2} \quad (B.6a)
\]

\[
\frac{\partial^2 i}{\partial \tau^2} \bigg|_{R(\tau)=R(\tau_i)} = -2\alpha \rho \sigma^2 \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2. \quad (B.6b)
\]
From (B.6b) it follows that the choice of a tax rate $\tau$ implying the optimal value $R(\tau_i)$ maximizes the risk–free rate. From (B.6a) follows that $\tau^* = 1 - \alpha$ is a maximum or a minimum for the risk–free rate, if

$$\frac{\partial^2 i}{\partial \tau^2} \bigg|_{\tau = 1 - \alpha} \leq 0 \quad \text{for} \quad \frac{\partial i}{\partial R(\tau)} \leq 0 \quad \implies \quad R(\tau^*) \leq R(\tau_i) \quad (B.6c)$$

We find

$$\frac{\partial^3 i}{\partial \tau^3} \bigg|_{\tau = 1 - \alpha} \text{for } \frac{\partial^3 i}{\partial \tau^3} = 0, \quad \frac{\partial^4 i}{\partial \tau^4} \bigg|_{\tau = 1 - \alpha} < 0 \quad \text{and} \quad \frac{\partial^4 i}{\partial \tau^4} \bigg|_{\tau = 1 - \alpha} > 0. \quad (B.6d)$$

**Welfare Effects**

Maximization of lifetime utility (17) with respect to the tax rate yields:

$$\frac{\partial V(0)}{\partial \tau} = K(0)^{1 - \rho} \mu^{-\rho} \frac{D(\tau)}{\Delta^2} \times \left( 1 - \rho \sigma^2 R(\tau) \right) \times \left[ \mu - (1 - \alpha) R(\tau) \frac{\partial \mu}{\partial R(\tau)} \right] \quad (B.7a)$$

$$\frac{\partial^2 V(0)}{\partial \tau^2} = K(0)^{1 - \rho} \mu^{-\rho} \frac{D(\tau)}{\Delta^3} \times \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 \times \left[ (1 - \rho \sigma^2 R(\tau)) \left( \frac{\partial \mu}{\partial R(\tau)} \left( \alpha - \frac{\rho D(\tau)}{\mu} \right) - (1 - \alpha) R(\tau) \frac{\partial^2 \mu}{\partial R(\tau)^2} \right) - \rho \sigma^2 D(\tau) \right] + (1 - \rho \sigma^2 R(\tau)) D(\tau) \left[ \frac{\partial^2 R(\tau)}{\partial \tau^2} + 2 (\rho - 1) \left( \frac{\partial R(\tau)}{\partial \tau} \right)^2 \left( 1 - \rho \sigma^2 R(\tau) - \frac{\partial \mu}{\partial R(\tau)} \right) \right] \quad (B.7b)$$

with $\Delta$ as defined on page 8 or in (A.7) on page 27. Additionally, we employed the relationship $\frac{\partial \mu}{\partial R(\tau)} = 1 - \frac{\partial \mu}{\partial \mu(\tau)}$. The second term of (B.7a) is the familiar derivative of the tax function with respect to the tax rate implying the optimal value $\tau^* = 1 - \alpha$. The third term is related to equation (22) of Proposition 1. The last term is the first necessary condition (28a) of Proposition 6.

**Proof of Proposition 6**

ad (A) Equation (28a) implying $D(\tau) \equiv \mu - (1 - \alpha) R(\tau) \frac{\partial \mu}{\partial R(\tau)} = 0$ can be rearranged to

$$D(\tau) = \frac{\beta}{\rho} + \frac{\rho - \alpha}{\rho} \alpha R(\tau) \left( 1 - \rho \sigma^2 R(\tau) \right) + \sigma^2 R(\tau)^2 \left( 1 - \alpha \right)^2 + \frac{1}{2} (\rho - 1) \quad (B.8a)$$

By (B.8a), the size of $D(\tau)$ is related to the sign of the risk–free rate. We find

(a) for $i = 0$: $D(\tau) > \Delta = \mu > 0$,

(b) for $i > 0$: $D(\tau) > \Delta > 0$. $D(\tau)$ is bounded from below by the transversality condition,
(c) for \( i < 0 : D(\tau) > \mu > 0 \) for \( \rho > \alpha \). \( D(\tau) \) is bounded from below by the propensity to consume.

From (B.8a) follows \( D(\tau) > 0, \forall \rho > \alpha \), since we only deal with feasible solutions of the model, that is, \( \Delta, \mu > 0 \).

ad (B) Evaluation of (B.7b) at \( \tau^* = 1 - \alpha \) leads to the following result

\[
\frac{\partial^2 V(0)}{\partial \tau^2} \bigg|_{\tau^*} = \frac{K(0)1 - \rho \mu - \rho D(\tau)}{\Delta^2} \times \frac{\partial^2 R(\tau)}{\partial \tau^2} \times (1 - \rho \sigma^2 R(\tau)) \leq 0 \quad \text{for } i|_{\tau^*} \leq 0
\]

(B.8b)

ad (C) Evaluation of (B.7b) at \( i = 0 \) yields

\[
\frac{\partial^2 V(0)}{\partial \tau^2} \bigg|_{i=0} = \frac{-K(0)1 - \rho \mu - \rho D(\tau)}{\Delta^2} \times \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 < 0 \quad \text{for } \tau \neq \tau^*
\]

(B.8c)

Regarding (B) and (C), we find

\[
\frac{\partial^3 V(0)}{\partial \tau^3} \bigg|_{\tau^*} = \frac{\partial^3 V(0)}{\partial \tau^3} \bigg|_{i=0 \text{ for } \tau^* = 1 - \alpha} = 0
\]

\[
\frac{\partial^4 V(0)}{\partial \tau^4} \bigg|_{\tau^*} < 0 \quad \text{and} \quad \frac{\partial^4 V(0)}{\partial \tau^4} \bigg|_{i=0 \text{ for } \tau^* = 1 - \alpha} > 0.
\]

(B.8d)

From this we conclude that \( \tau^* = 1 - \alpha \) is a welfare maximizing policy for \( i \geq 0 \) and welfare minimizing for \( i < 0 \). A tax policy implying \( i = 0 \) maximizes welfare for any \( \tau \geq \tau^* \).

\( \square \)

### B.2 Command Optimum

**Growth Effects** The first–order condition of maximizing the expected growth rate \( \psi^* \) with respect to the tax rate is given by \( \frac{\partial \psi^*}{\partial \tau} = \frac{\partial \psi^*}{\partial R(\tau)} \times \frac{\partial R(\tau)}{\partial \tau} = 0 \), which, as before, is satisfied either if the first or the second term of the product equals zero. We solve \( \frac{\partial \psi^*}{\partial R(\tau)} = 0 \) for the optimal value of \( R(\tau) \) and define \( R(\tau) \bigg|_{\tau^*} \equiv R(\tau_{\psi^*}) \). Evaluation at the optimal values \( \tau^* = 1 - \alpha \) and \( R(\tau_{\psi^*}) \) leads to results that qualitatively do not differ from the results obtained for the competitive economy for the case of precautionary saving:

\[
\frac{\partial^2 \psi^*}{\partial \tau^2} \bigg|_{\tau^*} = -\frac{\partial \psi^*}{\partial R(\tau)} \frac{1 - \alpha}{\alpha} \rho^2 \frac{\tau^2}{\rho^2} \leq 0
\]

(B.9a)

\[
\frac{\partial^2 \psi^*}{\partial \tau^2} \bigg|_{R(\tau) = R(\tau_{\psi^*})} = \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 \alpha^2 (\rho - 1) > 0
\]

(B.9b)

As \( \frac{\partial \psi^*}{\partial R(\tau)} > 0, \forall \rho \geq 1 \) and \( \rho > 1 \) in case of precautionary savings, \( \tau^* = 1 - \alpha \) is a unique interior maximum while \( R(\tau_{\psi^*}) \) is a minimum.

**Consumption Effects** The first– and second–order conditions of optimizing the propensity to consume of the command optimum with respect to the tax rate are given by:

\[
\frac{\partial \mu^*}{\partial \tau} = \frac{\partial \mu^*}{\partial R(\tau)} \rho \left[ \frac{1 - \tau}{\tau} \cdot \frac{1 - \alpha}{\alpha} - 1 \right] \times \frac{\rho - 1}{\rho} (1 - \rho \sigma^2 R(\tau))
\]

(B.10a)

\[
\frac{\partial^2 \mu^*}{\partial \tau^2} = \frac{\partial \mu^*}{\partial R(\tau)} \left[ \frac{1 - \alpha}{\alpha} \frac{1}{\tau^2} \frac{\tau^2}{\rho^2} - 1 \right] \left[ \frac{1 - \tau}{\tau} \cdot \frac{1 - \alpha}{\alpha} - 1 \right] + \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 \alpha^2 (1 - \rho)
\]
Evaluation of (B.10b) at $\tau^* = 1 - \alpha$ and $R(\tau)|_{\tau = 0} = R(\tau_{\mu^*})$ leads to:

\[
\frac{\partial^2 \mu^*}{\partial \tau^2} \bigg|_{\tau = 1 - \alpha} = \frac{\partial \mu^*}{\partial R(\tau)} \left(1 - \frac{\alpha}{\rho} \gamma \frac{1}{\tau} \right)^2 \left(1 - \frac{\alpha}{\rho} \gamma \frac{1}{\tau} \right) \alpha \gamma \frac{1}{\tau} \tau - 2
\] (B.11a)

From (B.11a) follows that $\mu^*$ is maximized at $\tau = 1 - \alpha$ for $\rho > 1$, if

\[
\frac{\partial^2 \mu^*}{\partial \tau^2} \bigg|_{\tau = 1 - \alpha} < 0 \quad \text{for} \quad R(\tau^*) < \left(\frac{\rho - 1}{\rho} \cdot \frac{\rho + 1}{\mu^*} \left(1 - \rho \sigma^2 R(\tau)^2 + \rho \sigma^2 \right) \right)
\] (B.11b)

From (B.11b) it is obvious that $R(\tau_{\mu^*})$ maximizes $\mu^*$ in case of precautionary savings. From (B.11a) follows that $\tau^* = 1 - \alpha$ is a maximum or a minimum for $\rho > 1$, if

\[
\frac{\partial^2 \mu^*}{\partial \tau^2} \bigg|_{\tau = 1 - \alpha} \leq 0 \quad \text{for} \quad R(\tau^*) \leq R(\tau_{\mu^*}) \quad \Rightarrow \quad \tau^* \leq 0.
\] (B.12)

Risk–free Rate Effects The qualitative properties of the effects from a change in the tax rate on the risk–free rate are identical to the ones discussed for the competitive economy in section B.1. They differ only with respect to the parameter $\alpha$.

Welfare Effects The first– and second–order conditions of (A.13) with respect to the tax rate are given by

\[
\frac{\partial V(0)^*}{\partial \tau} = K(0)^{1-\rho} \mu^{-(p+1)} \frac{\partial R(\tau)}{\partial \tau} \left(1 - \rho \sigma^2 R(\tau) \right)
\] (B.13a)

\[
\frac{\partial^2 V(0)^*}{\partial \tau^2} = K(0)^{1-\rho} \mu^{-(p+1)} \left\{ \left(1 - \rho \sigma^2 R(\tau) \right) \frac{\partial^2 R(\tau)}{\partial \tau^2} - \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 \left[ \rho - 1 \cdot \frac{\mu}{\rho} \left(1 - \rho \sigma^2 R(\tau)^2 + \rho \sigma^2 \right) \right] \right\}
\] (B.13b)

An optimal policy is characterized either by $\tau^* = 1 - \alpha$ or by $i^* = 0$.

Proof of Proposition 2 Evaluation of (B.13b) at $\tau^* = 1 - \alpha$ and $i^* = 0$ leads to the following results:

\[
\frac{\partial^3 V(0)^*}{\partial \tau^3} \bigg|_{\tau = 1 - \alpha} = K(0)^{1-\rho} \mu^{-(p+1)} \left(1 - \rho \sigma^2 R(\tau) \right) \frac{\partial^2 R(\tau)}{\partial \tau^2} \leq 0 \quad \text{for} \quad \tau^* = 1 - \alpha \leq 0
\] (B.14a)

\[
\frac{\partial^3 V(0)^*}{\partial \tau^3} \bigg|_{\tau = 0} = -K(0)^{1-\rho} \mu^{-(p+1)} \rho \sigma^2 \left[ \frac{\partial R(\tau)}{\partial \tau} \right]^2 < 0 \quad \text{for} \quad \tau \neq \tau^*.
\] (B.14b)

We obtain

\[
\frac{\partial^3 V(0)^*}{\partial \tau^3} \bigg|_{\tau = 1 - \alpha} = \frac{\partial^3 V(0)^*}{\partial \tau^3} \bigg|_{\tau = 0} = 0
\]

\[
\frac{\partial^4 V(0)^*}{\partial \tau^4} \bigg|_{\tau = 1 - \alpha} = \frac{\partial^4 V(0)^*}{\partial \tau^4} \bigg|_{\tau = 0} = 0
\] (B.14c)