Forward Guidance and Optimal Credible Monetary Policy*

Bingbing Dong†

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Abstract

Forward guidance has been seen as a revolution of new instruments for central banks to use to strengthen the recovery of recession in the presence of zero lower bound on nominal interest rates. However, central banks will likely fall back on its words as long as the economy recovers, a time they have only the cost of sticking to zero interests as forward guidance suggested. Through a standard New Keynesian model, this paper sheds light on what the optimal rate policy the central bank can credibly instruct for forward guidance by solving the whole set of sustainable sequential equilibria (SSE). This set depends on discount factor, fiscal subsidy to firms, but not the best sustainable sequential equilibrium. At steady state, the best sustainable sequential equilibrium always achieves the same efficient allocation as those from Ramsey equilibrium. While Ramsey solution implies that forward guidance is an extension of longer duration at zero lower bound, the best SSE comes with a much longer period of higher inflation even after the recession ends, and the latter the credible policy.

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†Economics Department, P.O. Box 400182,University of Virginia, Charlottesville, VA 22904-4182; bd3h@virginia.edu; http://people.virginia.edu/~bd3h/

1 Introduction

Since the Great Recession, many more economies including the US has joined the club of liquidity trap as Japan. Many policies have been introduced to boost the economy and among them forward guidance the one that is recently deemed an effective tool and the Fed's chair Yellen even sees it a revolution (Yellen, 2012).[1] Forward guidance is the statement and communication of the central bank's projected future path of short-term interest rates.[2] The prolonged stay at the ZLB,would then stimulate consumption and increase future inflation, whereas stimulate consumption further. [3] The question of this paper is: what is the optimal, and more importantly, credible duration of ZLB the FG should instruct?[4]

The key for FG to work is for the households to believe what the central bank provides is truly what it will follow. The existing literature has clearly made the point that the optimal policy, either under discretion or commitment, should be the one that interest rate stays even the economy strengthens and there is a pressure to increase interest rate to close inflation and output gaps then. However, policy suggestions via the lens of models under these two regimes have wide discrepancy about the optimal duration of ZLB, which leads to insufficient boost if too short or higher future risk if too long.[5] This paper discuss the policy where no commitment exists between the public and the central bank, and they play an infinitely repeated game and believe what the optimal policy will be when ZLB binds and will follow it.

This paper, in contrast to the concepts of Markov Perfect Equilibria (MPE) and Ramsey Equilibria (RE), answers the question by solving the whole SSE set of a standard New Keynesian workhorse model. The whole equilibrium set is characterized by payoffs to the households and central banks, depending on demand shocks. In particular, we are interested in comparing the BSSE with MPE and Ramsey. The real concern with the Ramsey Equilibria is that in practice, no central bank commits. Take the example of the Fed: the minutes of FOMC tried to make people believe the Fed will follow zero interest rate for a prolonged period when the recession was looking doom while started to prefer increased flexibility as long as the economy is recovered.[6] The Fed definitely does not have commitment.

While SSE is a notion relatively widely used to discuss optimal taxation problems (see Chari and Kehoe (1990), and Phelan and Stacchetti (2001)), it is less discussed in discussing monetary policy, in particular to policy at the ZLB. One exception is Nakata (2014), the key question of which is to what degree the Ramsey plan is sustainable and credible. While he proposes *one* revert-to-discretion plan – if the central bank deviates from RE, the households will follow MPE forever – to support the Ramsey plan, there is no reason to think this plan is *the* one agents in the economy will choose. In contrast, my paper shows all the possibilities by having all equilibria. Compared to a linearized model of Nakata (2014), the model I solved is fully nonlinear and thus could catch higher order effects like precautionary saving effects that is missing in linearized model.[7]

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[1] Forward guidance is not completely new, though. cite works here!

[2] As summarized in Issing (2014), different forms of FG have been adopted by different central banks, including pure qualitative FG, Qualitative forward guidance conditional on a narrative about the macroeconomic conditions under which the present policy will prevail, Calendar based FG, and Outcome-based forward guidance explicitly sets numerical conditions for a future change in policy.

[3] A second effect would be a shorter duration of bad time. check details.

[4] ECB has recently been the first main central bank that introduced negative deposit interest rates while keeping lending rate slightly positive (0.15%). See http://www.ecb.europa.eu/home/html/faqinterestrates.en.html http://www.bbc.com/news/business-27717594

[5] See (Adam and Billi, 2006, 2007) and others.

[6] Cite the minutes change here

[7] Another pioneer paper that uses similar strategies to Nakata (2014) is Ireland (1997), which discussed conditions under which the Friedman rule, the optimal policy under commitment, can be supported when the government lacks

The idea to find the equilibrium payoff set closely follows Feng (forthcoming) and Chang (1998). The central bank and the public (who also own firms) play a dynamic game and the behavior of both are sequentially rational. The reputation mechanism ensures that if the central bank deviates from equilibrium policy at ZLB and pursues different ones, that is, it shortens the duration of zero interest rate when the economy strengthens, though it gets instant benefits by having lower inflation and output gap, it will be punished by a lower continuation value since households will not believe its power to boost economy when recession comes again in the future. The equilibrium path is the one that the central bank will always follow to avoid a lower lifetime utility due to either lower continuation value or current benefits associated with deviations.

Results are as follows: (1) In an economy with fiscal subsidy and without shocks, all three concepts of equilibria give the same results: zero inflation. (2) The BSSE always achieve efficient level regardingless of β since there is no time-inconsistency. (3) The BSSE instructs the central banks' keeping interest rate zero even the economy strengthens, however less than suggested by RE. (4) The BSSE features, however, a much longer period of higher than target inflation after the recession.

The rest of the paper is structured as follows: Section 2 presents the model; Section 3 introduces MPE and RE while Section 4 presents SSE and the strategies to solve the model; Solution methods are discussed in Section 5; Finally, sections 6 and 7 are devoted to results, without and with ZLB, respectively.

2 The Model

This section first presents the concepts of CE and then MPE and RE. Finally it will present the SSE model. The economy is populated by a continum of households and firms. The economy at period t is hit by labor productivity and preference shocks,[8], A_t and β_t, both of which follow a Markov chain. For illustration purpose, I now only present the Markov chain with 2 states for preference shocks. This Markov chain is characterized by the transition matrix:

$$\boldsymbol{P} = \begin{bmatrix} p_{LL} & 1-p_{LL} \\ 1-p_{HH} & p_{HH} \end{bmatrix} \tag{1}$$

and two states $\{\beta^L, \beta^H\}$. The four parameters are chosen to have a 2% frequency of ZLB bindings when preference shock is high.

Let $s_t = (A_t, \beta_t)$ be the shocks at period t, which have finite realizations and finite support $\boldsymbol{S}$. The history of shocks is thus $s^t = (s_0, s_1, ..., s_t)$ for given initial shocks s_0. The probability of each of these histories is given by $\pi(s^t)$. In the following, I will use the Markov chains to present agents' problems and define equilibria.

2.1 The Households

The representative household is to maximize its lifetime utility

$$\sum_{t=0}^{\infty}\sum_{s^t}\left(\prod_{i=0}^{t}\beta(s^i)\right)\pi(s^t)\left\{\log c(s^t) - \psi\frac{l(s^t)^{1+\chi}}{1+\chi}\right\} \tag{2}$$

a commitment technology.

[8]The latter follows Eggertsson and Woodford (2003) and Christiano, Eichenbaum, and Rebelo (2011) and can be viewed as standing in for a wide variety of factors that alter households' propensity to save, for example, financial and uncertainty shocks.

subject to

$$c(s^t) + \frac{b(s^t)}{p(s^t)} = w(s^t)l(s^t) + R(s^{t-1})\frac{b(s^{t-1})}{p(s^t)} + T(s^t) + F(s^t) \tag{3}$$

where w_t is the real wage, R_{t-1} nominal interest rate, T_t a lump-sum transfer or tax, and F_t are profits of the firms in the economy. The household chooses labor to supply (l_t), bonds to buy (b_t) and goods to consume (c_t) to maximize its lifetime utility. The optimality conditions for the households are:

$$\frac{1}{c(s^t)} = \beta_t R(s^t) E_t \left\{ \frac{1}{c(s^{t+1})} \frac{1}{\Pi(s^{t+1})} \right\} \tag{4}$$

$$w(s^t) = \psi l(s^t)^{\chi} c(s^t) \tag{5}$$

2.2 The Firms

The final good producer is to maximize its profits by solving the following problem:

$$\max p(s^t)y(s^t) - \int_0^1 p_i(s^t)y_i(s^t)di \tag{6}$$

$$\text{s.t. } y(s^t) = \int_0^1 y_i(s^t)di \tag{7}$$

Given the prices of final and intermediate goods, the demand for intermediate good i is:

$$y_i(s^t) = \left(\frac{p_i(s^t)}{p(s^t)}\right)^{-\varepsilon} y(s^t) \tag{8}$$

The intermediate goods producer has a linear technology to produce intermediate goods, i.e., $y_i(s^t) = A_t l_i(s^t)$. It maximizes discounted profits by paying a cost to adjust its price, namely

$$\underset{p_{it}}{max}\ E_t \prod_{k=0}^{\infty} \beta_{t+k} \frac{\lambda_{t+k}}{\lambda_t} D_i(s^{t+k}) \tag{9}$$

subject to (8), where $D_i(s^t) = p_i(s^t)y_i(s^t)/p(s^t) - (1-\xi)w(s^t)l_i(s^t) - \frac{\phi}{2}\{\frac{p_i(s^t)}{p_i(s^{t-1})} - 1\}^2 y(s^t)$ is the dividend in period t and the quadratic term captures the cost to adjust prices and λ_{t+k} is the Lagrangian multiplier for the household at date t. Note that ξ is a subsidy to eliminate steady state distortion due to monopolistic pricing. After solving the problem and using symmetry conditions, the behavior of intermediate goods firms can be summarized by the following equation

$$\left[(\epsilon - 1) - (1-\xi)\epsilon\frac{w(s^t)}{A_t} + \phi\left(\Pi(s^t) - 1\right)\Pi(s^t)\right]\frac{y(s^t)}{c(s^t)} = \beta_t E_t \phi\left(\Pi(s^{t+1}) - 1\right)\Pi(s^{t+1})\frac{y(s^{t+1})}{c(s^{t+1})} \tag{10}$$

Equation (10) states that the marginal cost of adjusting prices (LHS) must equate the marginal benefit (RHS).[9]

[9] The higher the marginal benefit of adjusting price, the higher cost of inflation, and higher inflation: quadratic adjustment cost of inflation.

2.3 The Central Bank

It is assumed that the central bank is benevolent and chooses nominal interest rate R to maximize households' lifetime utility (2). However, the central bank cannot reduce the rate below zero.[10] In other words, the nominal interest rate is bound at zero. At each period after the central bank sets the rate, households then make their decisions about consumption and leisure. In the next section, I will detail the credibility of this policy rate.

2.4 Market Clearing

Finally, the markets are clear:

$$y(s^t) = c(s^t) + \frac{\phi}{2}(\Pi(s^t) - 1)^2 y(s^t) \tag{11}$$
$$y(s^t) = A_t l(s^t) = A_t \int_0^1 l_i(s^t) di \tag{12}$$

where (11) is the aggregate demand equation and (12) is the aggregate supply equation. Combining these two and rearranging gives the equilibrium condition in goods market:

$$c(s^t) = (1 - \frac{\phi}{2}(\Pi(s^t) - 1)^2) A_t l(s^t) \tag{13}$$

2.5 Competitive Equilibrium/Outcome

Suppose the economy starts with $\Upsilon\{s_0, R_0\}$, a **CE** or **competitive outcome** for $\Upsilon\{s_0, R_0\}$ is characterized by a sequence (c_t, l_t, w_t, Π_t) such that, for all $t \geq 1$, $s^t \in S^t$, and $R_{max} \geq R_t \geq 0$ and

$$\frac{1}{c_t} = R_t \beta_t E_t \left\{ \frac{1}{c_{t+1}} \frac{1}{\Pi_{t+1}} \right\} \tag{14}$$
$$w_t = \psi l_t^{\chi} c_t \tag{15}$$
$$c_t = (1 - \frac{\phi}{2}(\Pi_t - 1)^2) A_t l_t \tag{16}$$
$$\left[(\epsilon - 1) - (1 - \xi)\epsilon \frac{w_t}{A_t} + \phi \left(\Pi_t - 1\right) \Pi_t\right] \frac{A_t l_t}{c_t} = \beta_t E_t \left[\phi \left(\Pi_{t+1} - 1\right) \Pi_{t+1}\right] \frac{A_{t+1} l_{t+1}}{c_{t+1}} \tag{17}$$

For any outcome, there is an associated state-contingent sequence of values, $h(s^t)$, which will be referred to as a value sequence.

3 Discretion and Full Commitment

From this section on, I present different notions of equilibrium.

3.1 Markov Perfect Equilibria (MPE)

A MPE is the case where the CB at time t maximizes agent's lifetime utility (2) by choosing consumption, labor, wage, inflation and interest rate subject to conditions (14)-(17) and non-negativity of nominal interest rate, taking as given the future behavior of itself. The Bellman

[10] The gross interest rate is bounded below by 1. More notes here!

equation is:

$$V_t(s_t) = \max_{c_t, l_t, w_t, \Pi_t, R_t} u(c_t, l_t) + \beta_t E_t V_{t+1}(s_{t+1}|s_t) \tag{18}$$

subject to

$$\frac{1}{c_t} = R_t \beta_t E_t \left\{ \frac{1}{c_{t+1}} \frac{1}{\Pi_{t+1}} \right\} \tag{19}$$

$$w_t = \psi l_t^{\chi} c_t \tag{20}$$

$$c_t = (1 - \frac{\phi}{2}(\Pi_t - 1)^2) A_t l_t \tag{21}$$

$$\left[(\epsilon - 1) - (1 - \xi)\epsilon \frac{w_t}{A_t} + \phi (\Pi_t - 1) \Pi_t \right] \frac{A_t l_t}{c_t} = \beta_t E_t \left[\phi (\Pi_{t+1} - 1) \Pi_{t+1} \right] \frac{A_{t+1} l_{t+1}}{c_{t+1}} \tag{22}$$

$$R_t \geq 1 \tag{23}$$

The solution of the MPE is characterized by a sequence of time-invariant value function and policy functions of consumption, labor, wage, inflation and interest rate, i.e., $\{c(s_t)\ ,\ l(s_t),\ w(s_t),\ \Pi(s_t),\ R(s_t),\ V(s_t)\}$. Since the discretionary CB reoptimizes every period, MPE is time-consistent by definition.

3.2 Ramsey Equilibrium (RE)

A RE is where the CB at time 0 instructs all the policies of future depending on the possible shocks to maximize the lifetime utility of the household (2). Namely,

$$\max_{\{c_t, l_t, w_t, \Pi_t, R_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^{t} \beta(s^i) \right) \pi(s^t) \left\{ \log c(s^t) - \psi \frac{l(s^t)^{1+\chi}}{1+\chi} \right\} \tag{24}$$

$$s.t. \tag{25}$$

$$(14) - (17) \tag{26}$$

$$R_t \geq 1 \tag{27}$$

$$for\ \forall\ t \tag{28}$$

Note that the RE delivers the highest lifetime utility at time 0, however, does not guarantee that for any given period $t > 1$, the discounted utility $V(s^t)$ coincides with the one if the CB is given the chance to reoptimize.[11]

4 Sequential Sustainable Equilibria

The game is played among households and the central bank.[12] Denote $\Gamma(s_0)$ the game where the economy is initially shocked by s_0. The public history of the game is $\zeta^t = (\zeta_0, \zeta_1, ..., \zeta_t)$, where $\zeta_t = (c_t, l_t, w_t, \Pi_t, R_t, s_t)$. Let σ_H be the strategy of households and σ_B that of the central bank. The value of a strategy $\sigma = (\sigma_H, \sigma_B)$ for the central bank is:

$$\Phi(s_0, \sigma) = \sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^{t} \beta(s^i) \right) \pi(s^t) \left\{ \log c(s^t) - \psi \frac{l(s^t)^{1+\chi}}{1+\chi} \right\} \tag{29}$$

[11] solution mathod follows **?**.

[12] The firms are eventually owned by households.

Definition of SSE: A strategy profile σ of the game $\Gamma(s_0)$ is an **SSE** if for any $t \geq 0$ and history ζ^{t-1}:

1. $\Phi_B(s_t, \sigma|_{\zeta^{t-1}}) \geq \Phi_B(s_t, (\sigma_H|_{\zeta^{t-1}}, \gamma))$ for any strategy γ in $\sum_B(s_t)$ for the central bank.

2. $\{c(s^j), l(s^j), w(s^j), \Pi(s^j)\}_{j=t}^{\infty}$ is a CE for $\Gamma\{s_t, R_{s^t}\}$, where $R_{s^t} := \{R(s^j)\}_{j=t}^{\infty}$, $R(s^t) \in \sigma_B(\zeta^{t-1}, s_t)$, and $(c(s^t), l(s^t), w(s^t), \Pi(s^t)) \in \sigma_H(\zeta^{t-1}, s_t, R(s^t))$.

In line with Phelan and Stacchetti (2001), $\sigma|_{\zeta^{t-1}}$ denotes the strategy profile in SSE with history ζ^{t-1}, and $(\sigma_H|_{\zeta^{t-1}}, \gamma)$ the strategy profile in which the household plays a SSE strategy under history ζ^{t-1} while the central bank plays an alternative one. The first condition above says that the continuation payoff for the central bank's strategy σ_B is better than that from any deviation to a different strategy. The second condition requires that the household always responds to a central bank strategy with decisions that imply a CE since this is the situation that is compatible with feasibility and optimality.

4.1 Recursive Formulation of SSE

Following Feng (forthcoming) and others[13], define $m_1(s^t) = \frac{1}{c(s^t)\Pi(s^t)}$, $m_2(s^t) = \phi(\Pi(s^t)-1)\Pi(s^t)\frac{y(s^t)}{c(s^t)}$. These two quantities represent , in period t, the expected derivatives of the household's lifetime discounted utility from period $t+1$ on with respect to b_{t+1} and p_{it}, respectively. For any $s^t, m_1(s^{t+1})$ and an arbitrary specified interest rate R, households solve the following problem:

$$max\ log(c) - \psi\frac{l^{1+\chi}}{1+\chi} + \beta_t E_t m_1^+ b(s^{t+1}) \tag{30}$$

subject to the budget constraint (3). It can be shown easily that the recursive problem is equivalent to the sequential problem.[14] The firms also solve a recursive problem appropriately given $m_2(s^{t+1})$.[15]

Definition: Let $\Upsilon\{s, R, \{m_1^+, m_2^+\}\}$ be the static economy in which the current shocks are s, the current interest rate set by the central bank is R, and agents have expectations about the future summarized in $\{m_1^+, m_2^+\}$. (c, l, w, Π) is a CE for $\Upsilon\{s, R, \{m_1^+, m_2^+\}\}$ if and only if the following conditions are satisfied:

$$\frac{1}{c(s)} = R\beta E\{m_1^+\} \tag{31}$$

$$w(s) = \psi l(s)^{\chi} c(s) \tag{32}$$

$$c(s) = (1 - \frac{\phi}{2}(\Pi(s)-1)^2)Al(s) \tag{33}$$

$$\left[(\epsilon-1) - (1-\xi)\epsilon\frac{w(s)}{A} + \phi\left(\Pi(s)-1\right)\Pi(s)\right]\frac{y(s)}{c(s)} = \beta E m_2^+ \tag{34}$$

I denote this equilibrium as $(c, l, w, \Pi) \in \mathbf{CE}^S\{s, R, \{m_1^+, m_2^+\}\}$

This then gives us the following lemma:

Lemma Given a feasible interest rate policy $R = \{R_t\}_{t=0}^{\infty}$, suppose that the sequence $\{c(s^t)$, $l(s^t)$, $w(s^t)$, $\Pi(s^t)\}_{t=0}^{\infty}$ is such that for each t,

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[13] Chang (1998) and Phelan and Stacchetti (2001) show that, though in different model setups, equilibria can be characterized in terms of their value to the government and their marginal value of private variables. see Kydland and Prescott etc for the reason and justification of doing so, how to explain it, and Chang1998

[14] The transversality condition holds because the equilibrium bond position is always zero. In general, the transversality is to prevent the agents from accumulating infinitely large assets like capital and bonds.

[15] See Appendix

$$\{c(s^t), l(s^t), w(s^t), \Pi(s^t)\} \in \mathbf{CE}^S\{s, R, \{m_1(s^{t+1}), m_2(s^{t+1})\}\}$$

where

$$m_1(s^{t+1}) = \frac{1}{c(s^{t+1})\Pi(s^{t+1})} \tag{35}$$

$$m_2(s^{t+1}) = \phi(\Pi(s^{t+1}) - 1)\Pi(s^{t+1})\frac{y(s^{t+1})}{c(s^{t+1})} \tag{36}$$

then $\{c(s^t), l(s^t), w(s^t), \Pi(s^t)\}_{t=0}^{\infty}$ constitutes a competitive equilibrium for $\Upsilon\{s_0, R_0\}$.

The lemma says that the promised marginal value of investment in bonds, cost and benefit adjusting prices, will summarize the expectation of households. By definition, we do not change the problem but now promised marginal value of investment in bonds and promised marginal benefit and cost of changing prices will fully summarize the expectation of households. The equilibria of the economy can be characterized by:

$$V(s) := \{(m_1, m_2, h) | \ \sigma \text{ is a SSE for } \Gamma(s)\}$$

which is a mapping from the values of the states s into set of possible payoffs associated with a strategy profile σ that constitutes a SSE. Note that h equates to the equilibrium continuation payoff of the Fed Φ defined by (29).

4.2 Credible Plans/Self Generation

To recursively characterize $V(s)$, I first introduce two definitions that lead to credible plans.
Consistency A vector $\psi = (R, c, l, w, \Pi, \{m_1^+, m_2^+\})$ is consistent wrt W at s if

$$(c, l, w, \Pi) \in CE^S(s, R, \{m_1^+, m_2^+\})$$

for $(m_1(s, \psi),\ m_2(s, \psi),\ h(\nu, s, \psi)) \in W(s)$, and $(m_1^+, m_2^+, h^+) \in W(s^+)$, where the values of m_1, m_2 and h are given by

$$\begin{aligned} m_1(s, \psi) &= \frac{1}{c\Pi} \\ m_2(s, \psi) &= \phi(\Pi - 1)\Pi y/c \\ h(s, \psi) &= log(c) - \psi\frac{l^{1+\chi}}{1+\chi} + \beta E h^+ \end{aligned}$$

Admissibility

The vector ψ is admissible wrt W if it is consistent wrt W at s and

$$h(s, \psi) \geq h(s, \psi')$$

for any other ψ'.

Consistency guarantees that the vector ψ delivers an allocation that is optimal for households and feasible. In addition, it requires that the promised continuation values (m_1^+, m_2^+, h^+) belong to the same equilibrium set as the implied (m_1, m_2, h). Admissibility says that the interest rate set by the central bank is optimal and it has no incentive to deviate. That is, the central bank cannot increase its payoff by setting a different interest rate R'.

A credible plan thus is the strategy of households and central banks instructed by an admissible ψ.

With these two definitions, let $\mathbb{B}$ be a mapping operator and $W(s) : R \times S \to R^3$ any set of equilibrium payoffs and we can define the operator $\mathbb{B}$ as follows:

For a given set of equilibrium values W,

$$\mathbb{B}(W)(s) = \{(m_1, m_2, h) | \ \psi \text{ admissible wrt } W \text{ at } s\}$$

Following Phelan and Stacchetti (2001) and APS, the mapping operator $\mathbb{B}$ has the following properties:

- 1. If $W \subseteq \mathbb{B}(W)$, then $\mathbb{B}B(W) \subseteq V$;
- 2. V is compact and the largest set of Equilibrium values W such that $W = \mathbb{B}(W)$
- 3. $\mathbb{B}(.)$ is monotone and preserves compactness.
- 4. If we define $W_{n+1} = \mathbb{B}(W_n)$ for all $n \geq 0$, and the equilibrium value correspondence $V \subset W_0$, then $lim_{n\to\infty} W = V$.

Interested readers can refer to their papers for details of the proofs. Numerically, $\mathbb{B}$ is calculated as follows:

$\mathbb{B}(W)(s) = \{(m_1, m_2, h) | \exists R, (c, l, w, \Pi),\ and\ (m_1^+, m_2^+, h^+) \in W(s)$ for all $s^+ \succ s$ such that

$$m_1 = \frac{1}{c\Pi} \tag{37}$$
$$m_2 = \phi(\Pi - 1)\Pi\frac{y}{c} \tag{38}$$
$$h = log(c) - \psi\frac{l^{1+\chi}}{1+\chi} + \beta E h^+ \tag{39}$$
$$(m_1, m_2, h) \in W(s) \tag{40}$$
$$h \geq [u(c', l') + \beta E h^{+'} | (m_1^{+'}, m_2^{+'}, h^{+'})], \forall (m_1^{+'}, m_2^{+'}, h^{+'}) \in W(s^+) \tag{41}$$
$$1/c = R\beta E\{m_1^+\} \tag{42}$$
$$w = \psi l^\chi c \tag{43}$$
$$c = (1 - \frac{\phi}{2}(\Pi - 1)^2)Al \tag{44}$$
$$\left[(\epsilon - 1) - (1 - \xi)\epsilon\frac{w}{A} + \phi\,(\Pi - 1)\,\Pi\right]\frac{y}{c} = \beta E m_2^+ \tag{45}$$
$$R \in [0, \bar{R}] \tag{46}$$

where $s^+ \succ s$ denotes all possible shocks that follow s. Constraints (37) to (40) are called "regeneration constraints", while (41) is an "incentive constraint". Constraints (42) to (45) are necessary to ensure that continuation of a sustainable plan after any deviation is consistent with a CE. Following Feng (forthcoming) and APS, I replace (41) with the following condition,

$$h \geq \tilde{h}(s) \tag{47}$$

where $\tilde{h}(s)$ is the worst possible payoff for the central bank when it announces unexpected interest rate R'. In particular, $\tilde{h}(s)$ is defined as

$$\tilde{h}(s) = \underset{R}{max}\{ \underset{\substack{c,l,w,\Pi, \\ (m_1^+, m_2^+, h^+) \in W(s^+)}}{min} [log(c) - \psi \frac{l^{1+\chi}}{1+\chi} + \beta E h^+]\}$$

such that

$$(c, l, w, \Pi) \in CE^S\{s, R, \Pi, \{m_1^+, m_2^+\}, \forall s^+ \succ s\}$$

The idea of replacing (41) with (47) is that: (1) if the households punish the central bank for deviations from the claimed policy R, they will punish the latter as worse as available. (2) if the central knows the response of households, it will pick the best as long as it decides to deviate. It can be shown that condition (47) is equivalent to (41) in the sense of leading to the same fixed point V by applying the operator $\mathbb{B}$. In my paper, I will use alternative ways to determine $\tilde{h}$. The first is to let it be the one calculated from MPE. In other words, as long as the reputation of the central bank corrupts, both the central bank and the household know that they will revert to the discretionary case.[16]

The MPE payoff is not necessarily the worst payoff and thus punishment to the central bank. Plus, the central bank may not want to complete lose their reputation. In this case, $\tilde{h}$ is endogenous and determined in the equilibrium.[17]

Following Feng (forthcoming), the whole equilibrium can be characterized by the upper and lower boundaries of $W(s)$, which are:

$$\bar{h}(s, m_1, m_2) = \underset{h}{max}\{h | (m_1, m_2, h) \in W(s)\} \tag{48}$$

$$\underline{h}(s, m_1, m_2) = \underset{h}{min}\{h | (m_1, m_2, h) \in W(s)\} \tag{49}$$

I then define the outer approximation of W as follows:

$$\hat{W}(s) = \{(m_1, m_2, h) | h \in [\underline{h}(s, m_1, m_2), \bar{h}(s, m_1, m_2)]\}$$

Proposition For all $(m_1, m_2, h) \in V(s)$,

$$\bar{h}(s, m_1, m_2) = max\ \{u(c, l) + \beta E \bar{h}(s', m_1', m_2')\} \tag{50}$$

$$\underline{h}(s, m_1, m_2) = max\ \{u(c, l) + \beta E \underline{h}(s', m_1', m_2')\} \tag{51}$$

Proof In Appendix.

I then have the lowest payoffs defined as:

$$\tilde{h}(s) = \underset{m_1, m_2}{min}\ \underline{h}(s, m_1, m_2) \tag{52}$$

The working algorithm is to find a new operator F based on $\hat{W}$:

Definition For any convex-valued correspondence $\hat{W}$,

$$F(\hat{W})(s) = \{(m_1, m_2, h) | h \in [\underline{h}^1, \bar{h}^1]\}$$

[16] Nakata (2014)

[17] The higher $\tilde{h}(s)$, the less costly for the central bank to deviate and thus harder to keep promises.

where

$$\bar{h}^1 = max\ \{u(c,l) + \beta E\bar{h}^0\} \tag{53}$$

$$\underline{h}^1 = max\ \{\max_{m_1',m_2'} u(c,l) + \beta E\underline{h}^0, \tilde{h}^0\} \tag{54}$$

$$\tilde{h}^0 = max\{\min_{m_1',m_2'} u(c,l) + \beta E\underline{h}^0\} \tag{55}$$

The details of the algorithm is in Appendix.

4.3 Recovering Strategies

This subsection shows how to find the strategy that supports BSSE. The procedure here can be generalized to find strategies supporting any point belonging to the equilibrium value correspondence.

- Step 1: At $t = 0$, find the highest possible value of $h_0 = sup\{h|(m_{1,0}, m_{2,0}, h_0) \in W^*(s)\}$ and its corresponding $(m_{1,0}, m_{2,0})$. Then search for the central bank's interest rate policy that supports $(m_{1,0}, m_{2,0}, h_0)$. That is, pick R_0 such that

  $$u(c_0, l_0) + \beta h_1 = h_0$$

  where $h_1 = \bar{h}(m_{1,1}, m_{2,1})$, $m_{1,1} = \frac{u_{c,0}}{R\beta}$, $m_{2,1} = \left[(\epsilon - 1) - (1 - \xi)\epsilon\frac{w_0}{A_0} + \phi\,(\Pi_0 - 1)\,\Pi_0\right] A_0 l_0/c_0/\beta$, and $(m_{1,1}, m_{2,1}, h_1) \in W^*(s)$. Values of (c_0, l_0, w_0, Π_0) come from the solution for the following equation system at given $(R_0, m_{1,0}, m_{2,0})$:

  $$m_{1,0} = \frac{1}{c_0\Pi_0} \tag{56}$$

  $$m_{2,0} = (\Pi_0 - 1)\Pi_0\frac{y_0}{c_0} \tag{57}$$

  $$w_0 = \psi l_0^{\chi} c_0 \tag{58}$$

  $$c_0 = (1 - \frac{\phi}{2}(\Pi_0 - 1)^2)A_0 l_0 \tag{59}$$

  Therefore, the above problem is well-defined in terms of $(R_0, m_{1,0}, m_{2,0}, h_0)$.

- Step 2: $t = 1$, $m_{1,1}$, $m_{2,1}$, h_1 are given by the solution in step 1. Now search for the central bank's policy R such that

  $$u(c_1, l_1) + \beta h_2 = h_1$$

  as in step 1.

- Step 3: Repeat step 2 for $t = 2, ...T$, for T be sufficiently large.

5 Solution Method

5.1 Parameterization

The parameterization here follows most of the literature. Frisch elasticity is set to be 1 and the disutility parameter in the current utility function is 1 without loss of generality. Price adjustment cost ϕ is set to 200 to match about 75% of reoptimizing firms in Calvo pricing in linearized forms.

To match an average of 2.5% real annual rate, 2% ZLB bindings, and an average of 3 quarters' persistence of ZLB, I set $p_{HH} = 1/3$, $p_{LL} = 0.9932$, $\beta_H = 1.011$, and $\beta_L = 0.99365$.

Table 1: Parameterization

Parameters	Meanings	Values
χ	Frisch elasticity of labor	1
ψ	disutility of labor	1
ϕ	price adjustment cost	200
β	average discount factor	0.994
β^H	high shock of preference	1.011 (2%)
β^L	low shock of Preference	0.99365 (98%)
p_{HH}	persistence of zlb episode	1/3
$1 - p_{LL}$	frequency of zlb episode	0.0068

5.2 Discussions on the Solution Algorithms

6 Optimal Monetary Policies Without ZLB

Since this is the first time to characterize the whole set of SSE of such a model in the literature, I will follow in the order of no shocks, shocks without ZLB and shocks with ZLB in the following for every detail to reach the conclusions of forward guidance with ZLB.

6.1 When there is No Shocks

Figure 1 and Figure 2 show the whole equilibrium set of SSE and a cross-section of the set given m_1 when there is no shocks for the benchmark model. The x- and y-axes are the two auxiliary quantities I introduced above, m_1 and m_2, and the z-axis is the payoff to the central bank, h. The BSSE is the one corresponds to highest payoff h. The first observation is that the BSSE features $(m_1, m_2) = (1, 0)$, of which $m_2 = 0$ means zero inflation; the second observation is that the BSSE is a steady state that can be supported, whereas all equilibria are not. These two then implies a steady of prices and allocation that coincides with MPE and RE, that is, $\Pi = 1$, $c = l = 1$, $w = 1$, and $R = 1/\beta$.

Figure 3 shows the equilibria set when ZLB is absent. It is straightforward to see that the supportive state space shrinks due to the ZLB. Given m_1, the lower bound of m_2 that can be supported shifts up. Figure 4 is cross-section of the equilibria set as a correspondence to m_2 fixing $m_1 = 1$. Therefore, with ZLB, the public won't believe a lower inflation rate policy that can achieved in the absence of ZLB. However, the BSSE and hence the efficient allocation can always be rational and believed by the two agents.

One justification of the CE with Taylor rule and a positive inflation target (2% in the US) is that the fiscal subsidy to eliminate monopolistic pricing is not always working.[18] Without subsidy, the implied optimal inflation from MPE is positive, while that from RE is still zero (See Dong,2013

[18] Another justification argued by Yellen is that it avoids the liquidity trap. See ...

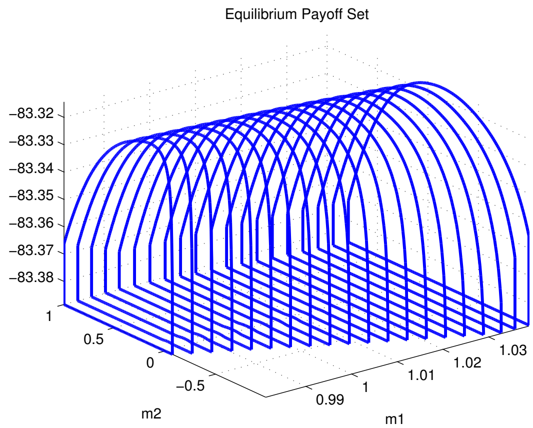

Figure 1: Equilibrium Payoff Set: No Shocks

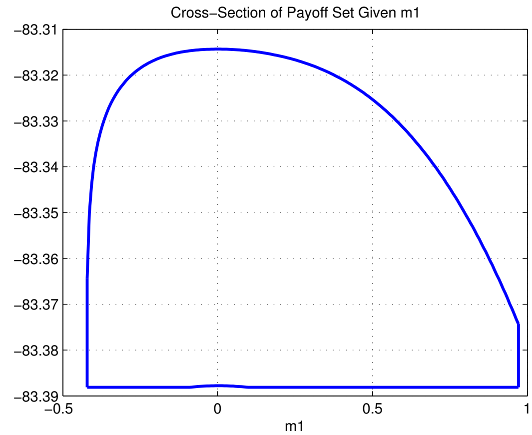

Figure 2: Equilibrium Payoff Set: A Cross Section when $m_1 = 1$

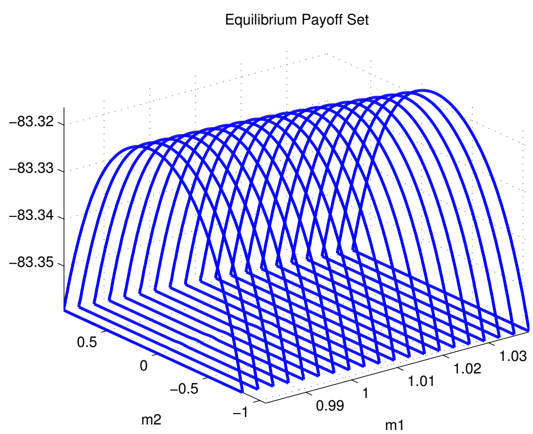

Figure 3: Equilibrium Payoff Set Without ZLB: No Shocks

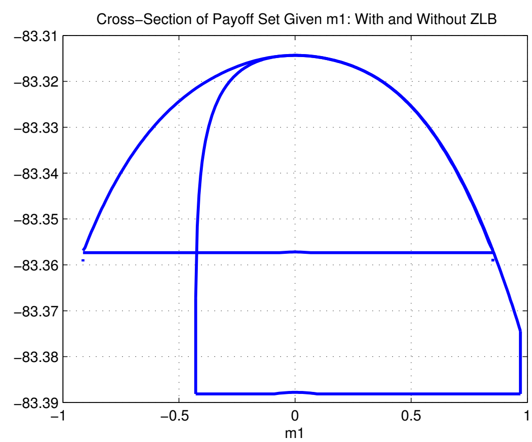

Figure 4: Equilibrium Payoff Set: A Cross Section when $m_1 = 1$

and Nakata, 2013). Figure 5 shows the equilibria set when ζ is set to 0.147 and Figure 6 a cross-section of the BSSE. Not surprisingly, the absence of fiscal policy leads the central bank to pursue in general higher inflation to eliminate the distortion and the households know this, and thus equilibria associated with lower inflation in the presence of subsidy are no long supportive. The BSSE now features $m_1 = 1.004$ and $m_2 = 0.0009$. The positive m_2 corresponds to an inflation of 0.36% annual inflation at steady state.

An interesting question to ask is whether a CE with Taylor rule, where inflation target is set, is supported by SSE. Suppose the Taylor rule is given by:

$$R_t = \bar{R}\left(\frac{y_t}{\bar{y}}\right)^{\rho_y}\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\rho_\Pi} \tag{60}$$

Upon stability by properly choosing the coefficients ρ_y and ρ_Π, it is just to verify if the price and allocation in this case is consistent with BSSE. Suppose the central bank set a target of $\Pi = 0.36\%$, the only thing left to do is to verify the allocation in this case is consistent with that from the

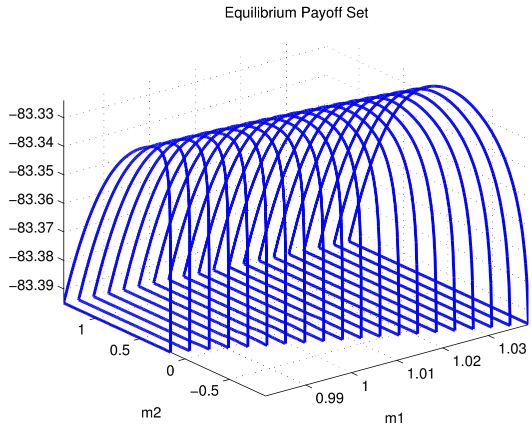

Figure 5: Equilibrium Payoff Set With ZLB and $\zeta = 0.147$: No Shocks

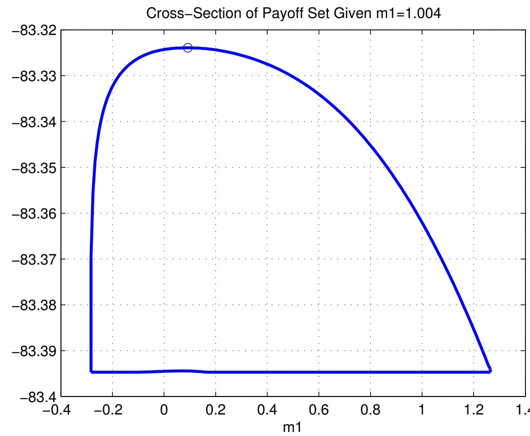

Figure 6: Equilibrium Payoff Set: A Cross Section when $m_1 = 1.004$

BSSE. And it is. With the same subsidy, the inflation target implied by MPE is larger.

Finally, I discuss the effects of discount factor on the equilibria set. As in the literature of fiscal policy, higher discount factor means agents are more patient and thus continuation value plays a bigger role, which enforces the central bank not to deviate. Thus, a higher β extends the supportive space. However, since higher β also represents a weaker demand and the central bank uses higher inflation to boost economy, the space should shrink. It turns out that the latter effect dominates and is shown in Figures 7 and 8, where β is set to be 0.9941. Note also that the BSSE with efficient allocation is supported. The reason is that when shocks are shut down, there is no time-inconsistency.

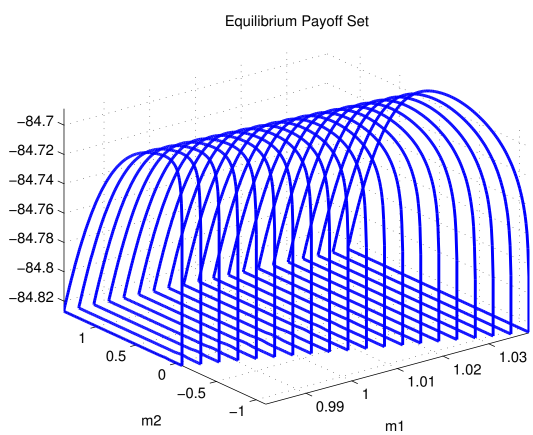

Figure 7: Equilibrium Payoff Set With ZLB: No Shocks

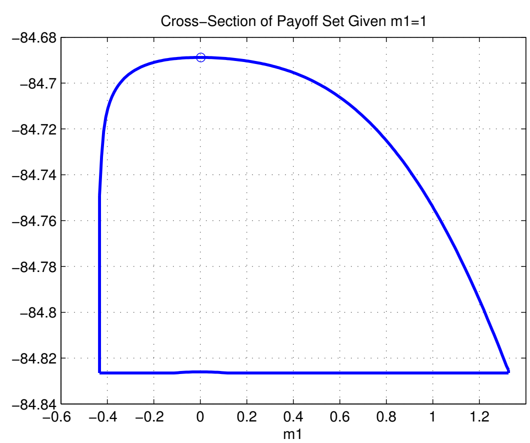

Figure 8: Equilibrium Payoff Set: A Cross Section when $m_1 = 1$

6.2 When Shocks Exist

Figures 9 and 11 show equilibria set for good and bad states (low and high β, respectively). The two sets are isomorphic. The only difference is that with high shock, the supportive range of m_2

shifts up, consistent with what are found above when β changes in the absence of shocks.

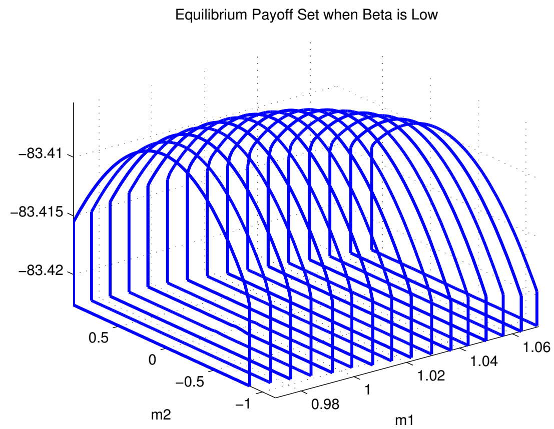

Figure 9: Equilibrium Payoff Set Without ZLB: Low Shock

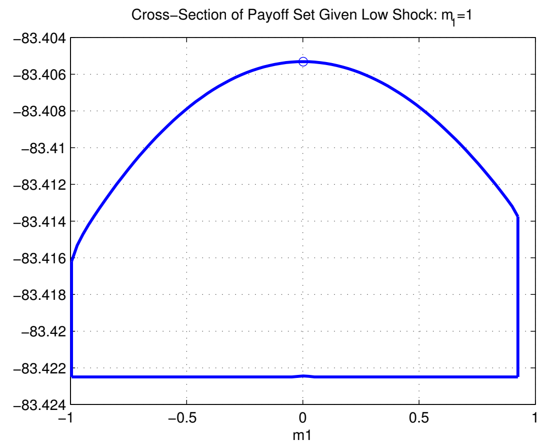

Figure 10: Equilibrium Payoff Set: A Cross Section when Shock is Low $m_1 = 1$

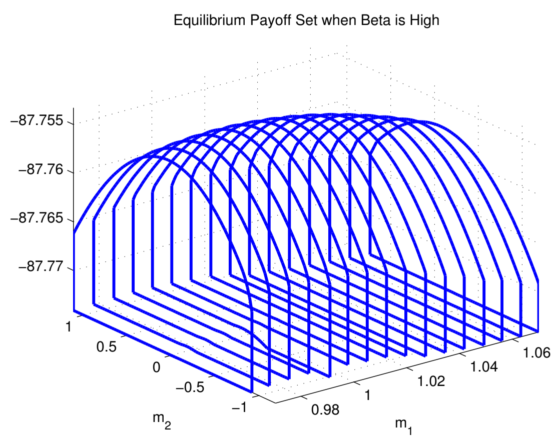

Figure 11: Equilibrium Payoff Set Without ZLB: High Shock

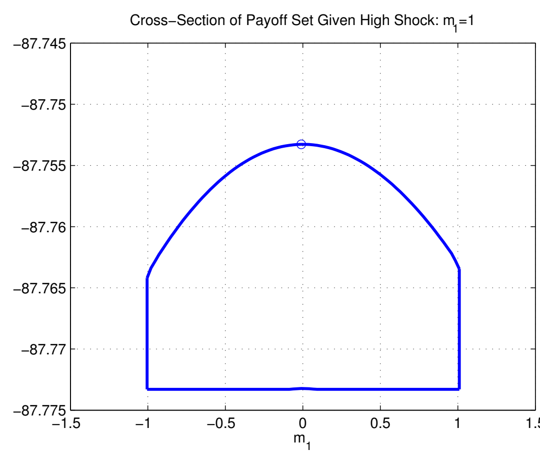

Figure 12: Equilibrium Payoff Set: A Cross Section when Shock is High $m_1 = 1$

Figures 10 and 12 show the BSSEs which feature the efficient allocation. The feature of zero inflation in high state is complimentary to the findings above for no shock case: the BSSE always achieves the efficient allocation as long as there is no constraints on the interest rate tool.

To show how BSSEs are supported, and more importantly, how the tool of interest rate is used, I experiment with a one-period shock to the discount factor after being at the steady state for a long time. The dynamics of payoffs to households, interest rates, and inflation are drawn in Figure 13.[19]

There are several things to pay attention. First, it takes about 20 years for the economy to return its normal state after one-period shock. In the case of MPE, the economy goes back immediately when the shock disappears. In the case of RE, this return-normal takes, though longer than that of MPE, much less time than SSE. Second, there is a spike of inflation at the time of shock, compared

[19]The wiggles are due to solution accuracy.

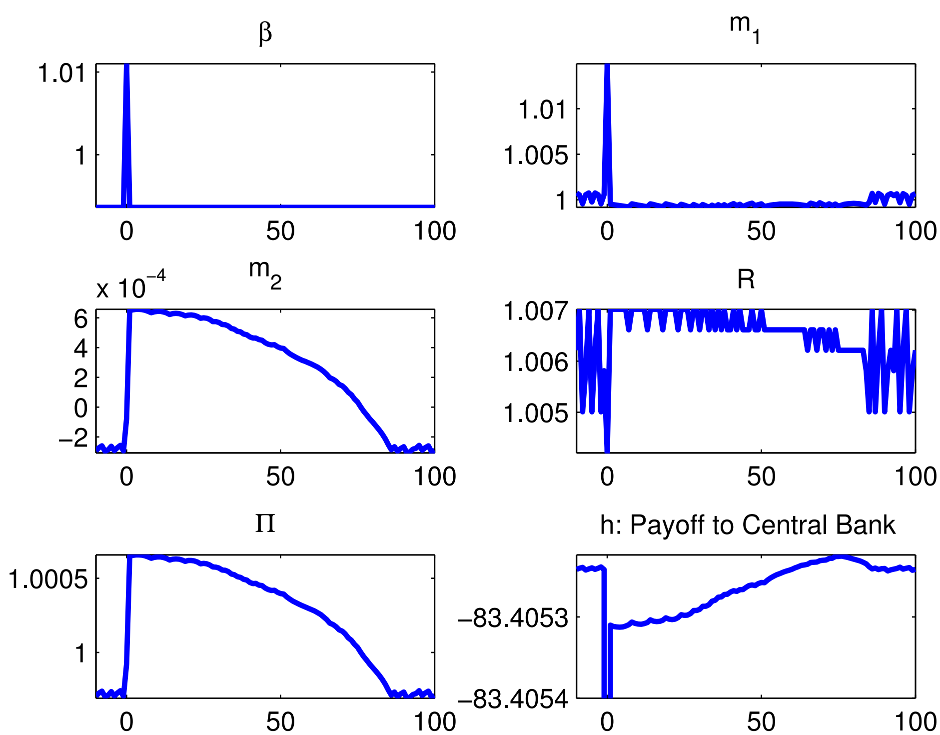

Figure 13: Simulated Path after One Bad Shock

to a deep deflation in the case of MPE. Third, the decrease of nominal interest rates is much milder than that from MPE. The reason for this is that expected inflation path changes a lot as the central bank wants and on-impact rates do not have to adjust too much. This gives us the intuition of how optimal policy will work when ZLB binds, which will be detailed in the following section.

7 Forward Guidance and Optimal Credible Monetary Policy

Figure 14 and Figure 15 are the equilibrium set and cross-section of BSSE with ZLB when the state is normal.

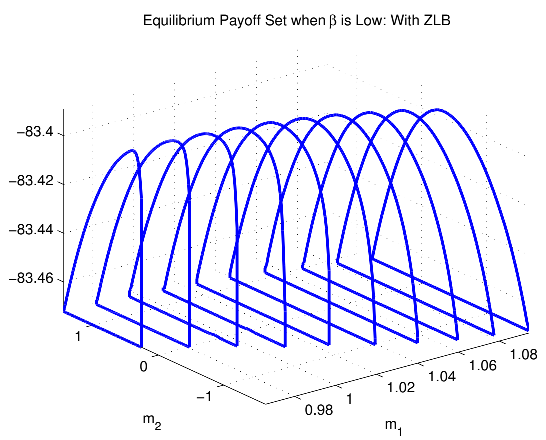

Figure 14: Equilibrium Payoff Set with ZLB: Low Shock

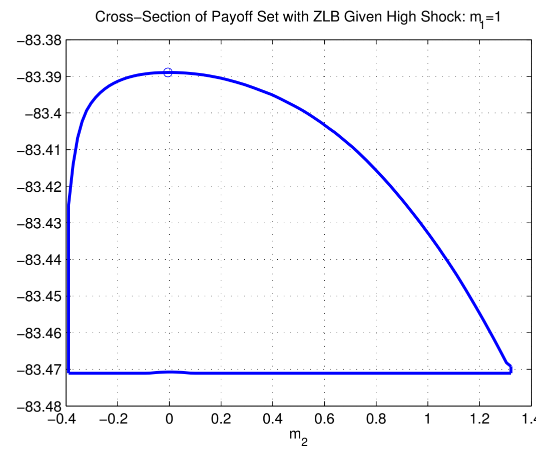

Figure 15: Equilibrium Payoff Set with ZLB: A Cross Section when $m_1 = 1$

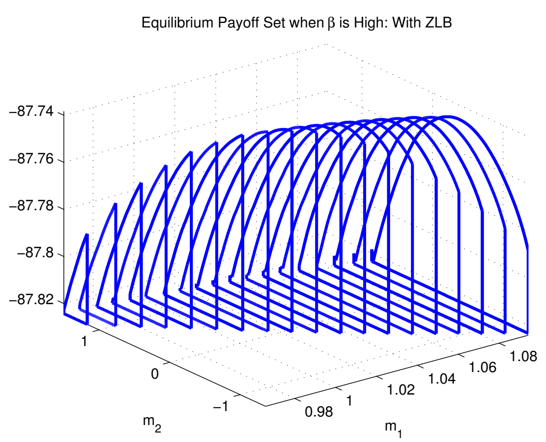

Figure 16: Equilibrium Payoff Set with ZLB: High Shock

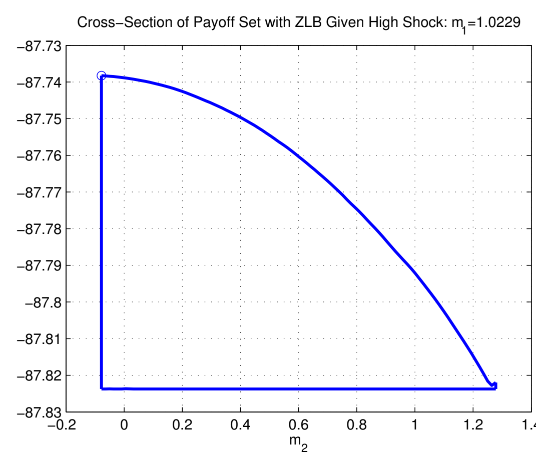

Figure 17: Equilibrium Payoff Set with ZLB: A Cross Section when $m_1 = 1.0229$

Figure 16 and Figure 17 are the equilibrium set and cross section of BSSE with ZLB when the state is in recession. The most prominent difference is that the equilibrium set of high shock cut off and the BSSE features a sudden cliff of central bank's payoff along the dimension of m_2. The dynamics will be detailed and added after I refine the solution by adding more grid points and having finer step size for policy instrument.

8 Discussions

9 Conclusion

This paper explores the optimal and credible duration of ZLB as forward guidance instructs via the workhorse of a standard New Keynesian model by characterizing first the whole set of SSE. The BSSE, in contrast to RE and MPE, features a strong reliance of the expected inflation for a very long time. Following a deep recession, there can be as long as 20 years inflation higher than target. This implies that the recent recession will likely be followed an inflation period in the next several years.

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A Appendix

A.1 Firm's Dynamic Problem

This subsection shows that given the promised values about adjusting prices, the firm solves the same recursive problem as without it. The dynamic problem of an individual firm can be written as follows:

$$V(p_{it-1}) = \lambda_t D_{it} + \beta_t E_t V(p_{it})$$

subject to constraint (8) and the definition of dividend D_{it}. λ_t is the Lagrangian multiplier of households' budget constraint. Solving the problem gives the following Euler equation:

$$\left[(\epsilon-1)(\frac{p_{it}}{p_t})^{-\epsilon} - (1-\xi)\epsilon\frac{w_t}{A_t} + \phi\left(\frac{p_{it}}{p_{it-1}}-1\right)\frac{p_t}{p_{it-1}}\right] y_t\lambda_t = \beta_t E_t\left[\left(\frac{p_{it+1}}{p_{it}}-1\right)\frac{p_{it+1}p_t}{p_{it}^2}\right]\phi y_{t+1}\lambda_{t+1}$$

Imposing symmetry gives the equation (10) in the text:

$$\left[(\epsilon-1)-(1-\xi)\epsilon\frac{w_t}{A_t}+\phi\left(\Pi_t-1\right)\Pi_t\right]y_t\lambda_t=\beta_t E_t\left[\left(\Pi_{t+1}-1\right)\Pi_{t+1}\right]\phi y_{t+1}\lambda_{t+1}$$

Now let $m_2^+=\left[\left(\Pi_{t+1}-1\right)\Pi_{t+1}\right]\phi y_{t+1}\lambda_{t+1}$ be the marginal value (payoff) of adjusting price relative to aggregate price. Then the firm's problems becomes:

$$V(p_{it-1})=\lambda_t D_{it}+\beta_t E_t m_2^+ p_{it}/p_t$$

Solving and imposing symmetry gives exactly the same solution as the original recursive problem.

A.2 Numerical implementation of operater $\mathbb{F}$

Let $\boldsymbol{S}\times\boldsymbol{M_1}\times\boldsymbol{M_2}\times\boldsymbol{H}$ denote the space of all equilibrium state vectors and associated payoffs to the central bank (s,m_1,m_2,h). $\boldsymbol{W}:\boldsymbol{S}\rightarrow\boldsymbol{M_1}\times\boldsymbol{M_2}\times\boldsymbol{H}$ is a correspondence from $\boldsymbol{S}$ to $\boldsymbol{M_1}\times\boldsymbol{M_2}\times\boldsymbol{H}$.

With an initial guess $\boldsymbol{W}^0(s)=\{(m_1(s),m_2(s),h(s))\}$ and a pre-determined tolerance level ϵ, the algorithm goes as follows:

- Step 1: For $\forall s\in\boldsymbol{S}$, find $\boldsymbol{\Omega}(s):=\{(m_1,m_2,h)|(m_1,m_2,h)\in\boldsymbol{W}^0(s),\exists R\in[\underline{R},\bar{R}]$ and $(m_1',m_2',h')\in\boldsymbol{W}^0(s')$ such that :

$$\begin{aligned}
h &= u(c,l)+\beta\mathbb{E}h'\geq\tilde{h}^0(s) && (61)\\
h &= u(c,l)+\beta\mathbb{E}\bar{h}^0(s',m_1',m_2')\geq\underline{h}^0(s,m_1,m_2) && (62)\\
Em_1' &= \frac{1}{c}\frac{1}{R\beta} && (63)\\
Em_2' &= \frac{1}{\beta}\left[(\epsilon-1)-(1-\xi)\epsilon\frac{w}{A}+\phi\left(\Pi-1\right)\Pi\right]\frac{Al}{c} && (64)\\
m_1 &= \frac{1}{c\Pi} && (65)\\
m_2 &= \phi(\Pi-1)\Pi\frac{Al}{c} && (66)\\
w &= \psi l^\chi c && (67)\\
c &= (1-\frac{\phi}{2}(\Pi-1)^2)Al && (68)
\end{aligned}$$

where

$$\begin{aligned}
\bar{h}^0(s,m_1(s),m_2(s)) &= \max_h\{h|(m_1,m_2,h)\in\boldsymbol{W}^0(s)\} && (69)\\
\underline{h}^0(s,m_1(s),m_2(s)) &= \min_h\{h|(m_1,m_2,h)\in\boldsymbol{W}^0(s)\} && (70)\\
\tilde{h}^0(s) &= \min_{(m_1,m_2)}\underline{h}^0(s,m_1,m_2) && (71)
\end{aligned}$$

- Step 2: For $\forall s\in\boldsymbol{S}$, and $\boldsymbol{\Omega}(s)$, denote $\boldsymbol{\Omega}^M(s,h):=\{(m_1,m_2)|(m_1,m_2,h)\in\boldsymbol{\Omega}(s),h=$

$h(s, m_1, m_2)\}$, and define

$$\bar{h}^1(s, m_1, m_2) = \max_{R} \max_{\substack{c,l,\Pi,w, \\ (m_1', m_2', h') \in \boldsymbol{W}^0(s')}} u(c,l) + \beta \mathbb{E} \bar{h}^0(s', m_1', m_2') \tag{72}$$

$$\underline{h}^1(s, m_1, m_2) = \max\{\max_{R} \min_{\substack{c,l,\Pi,w, \\ (m_1', m_2', h') \in \boldsymbol{W}^0(s')}} u(c,l) + \beta \mathbb{E} \underline{h}^0(s', m_1', m_2'), \tilde{h}^0(s)\} \tag{73}$$

$$\tag{74}$$

for all $(m_1, m_2) \in \boldsymbol{\Omega}^M(s, h)$. Otherwise, set

$$\bar{h}^1(s, m_1, m_2) = +\infty \tag{75}$$

$$\underline{h}^1(s, m_1, m_2) = -\infty \tag{76}$$

Further, let

$$\tilde{h}^1(s) = \min_{(m_1, m_2) \in \boldsymbol{\Omega}^M(s,h)} \underline{h}^1(s, m_1, m_2) \tag{77}$$

- Step 3: Define $\boldsymbol{W}^1(s) = \{(m_1, m_2, h) | (m_1, m_2) \in \boldsymbol{\Omega}^M(s, h), h \in [min\{\bar{h}^0(s, m1, m2), \underline{h}^1(s, m_1, m_2)\}, max\{\underline{h}^0(s, m_1, m_2), \bar{h}^1(s, m_1, m_2)\}]\}$
- Step 4: Set $\boldsymbol{W}^* = \boldsymbol{W}^1$ if $\|\boldsymbol{W}^1 - \boldsymbol{W}^0\| < \epsilon$; otherwise, set $\boldsymbol{W}^0 = \boldsymbol{W}^1$ and repeat the steps above.

A.3 Proofs

Proof of Proposition 2 The proof follows Feng (forthcoming) and first shows that the sequence of $\hat{W}_n$ is decreasing and $\hat{W}_n\ \hat{W}_{n+1}$. Since W_n is convex-valued, I show that the upper boundary decreasing and the lower one increasing. The UB's decreasing is due to the fact that $\bar{h}^1(s, m_1, m_2)$ is defined as $max_R u(c,l) + \beta \mathbb{E} \bar{h}^0(s', m_1', m_2')$ such that $\psi = (R, c, l, w, \Pi, \{m_1', m_2', h'\})$ is admissible wrt $\hat{W}^0$ at s. The lower bound's increasing can be proved similarly.

A.4 Solution Accuracy and Computaional Errors

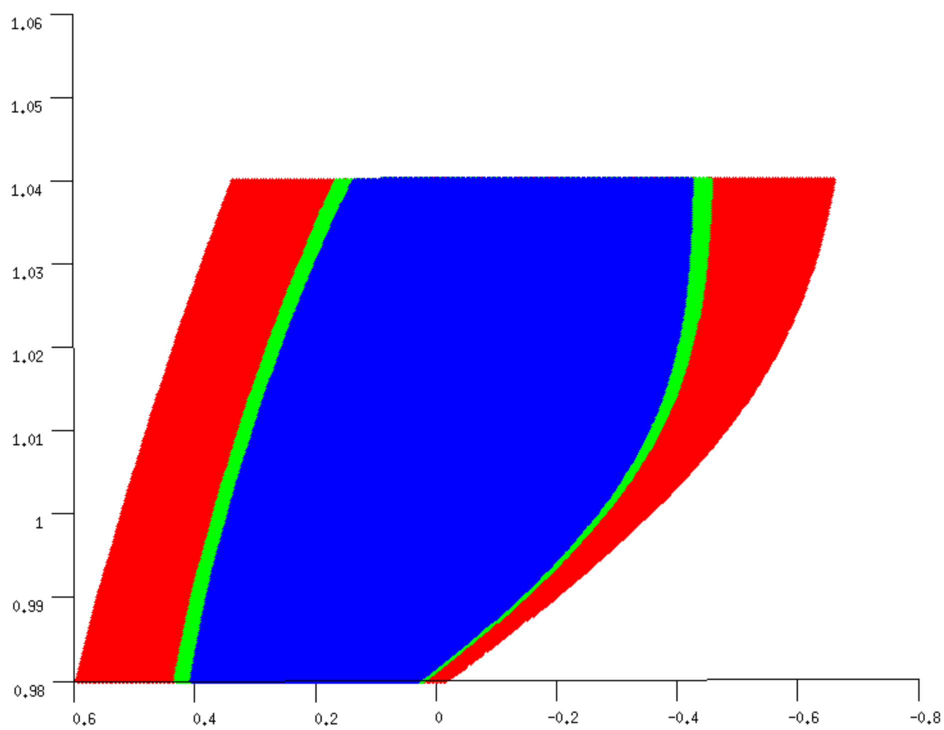

Figure 18: State space difference due to iteration accuracy(Red: 1×10^{-4},Green:5×10^{-5},Blue:2×10^{-5})