Expectations, Learning, and Forward Guidance

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Abstract

Unconventional monetary policies, such as forward guidance, operate through the management of expectations about future states of interest rates. This paper examines the link between expectations formation and the effectiveness of forward guidance. A standard Dynamic Stochastic General Equilibrium (DSGE) model from Preston (2005) is extended to include forward guidance shocks in the monetary policy rule following Del Negro, Giannoni, and Patterson (2012) and Laseen and Svensson (2011). Agents either form expectations about future macroeconomic variables from a rational expectations model or from an imperfect information model. The novel result is that agents with an imperfect information forecasting model misvalue the effects forward guidance has on their expectations. The impulse responses of the output gap and inflation rate display more persistence when agents form expectations from an incomplete model than a rational expectations model. The reason is that agents with rational expectations precisely understand the effects forward guidance has on their expectations, while agents with an imperfect information forecasting model revise their beliefs every period.

Keywords: Forward Guidance, Adaptive Learning, Expectations, Infinite Horizon.

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1 Introduction

Since U.S. short-term interest rates effectively have reached the zero lower bound (ZLB) due to the 2007-2009 global financial crisis, the conventional policy tool of changing interest rates by monetary policymakers has been exhausted. In response, central banks have pursued "unconventional" policies, such as forward guidance. A central bank uses forward guidance to communicate the future course of the policy rate to the public. An example of forward guidance was given when the Bank of Canada in April 2009 announced "the target overnight rate can be expected to remain at its current level until the end of the second quarter of 2010." Woodford (2012) argues that committing to an interest rate path that is lower than what one would commit to in a normal policy rate period can have positive effects on an economy that is in a persistent liquidity trap. Woodford (2012) describes a channel where anticipating the economy will expand in the future, agents' consumption will increase today. However, they expect that a conventional monetary policy rule will restrain the economy from "overheating," and thus, limit current consumption. If a forward guidance statement, instead, keeps a low policy rate through part of the expansion, Woodford (2012) explains consumption today will not be as limited.

Because the effectiveness of forward guidance hinges on how expectations respond to forward guidance, it is of interest to investigate whether the efficacy depends on the rational expectations assumption, which is the standard benchmark in macroeconomic models. According to Evans and Honkapohja (2001), agents with rational expectations are assumed to know the values of the model's deep parameters, structure of the model, and distribution of the error terms. However, it may not be realistic to assume that agents in an economy are endowed with superior information. A more realistic scenario might assume agents form expectations from an incomplete information forecasting model. Thus, from a policy maker's viewpoint, the effectiveness of forward guidance hinges on the expectations formation process that agents are assumed to hold.

The main contribution of this paper is to demonstrate that the effects of forward guidance depend on the manner in which agents form expectations. In particular, a micro-founded model derived under (potentially) non-rational expectations is formed following Preston (2005). Households, firms, and a central bank occupy the economy. The log-linearized equations demonstrate that the current output gap and inflation rate depend on future expectations of the endogenous variables. The central bank operates a monetary policy rule augmented with anticipated shocks as in Del Negro, Giannoni, and Patterson (2012) and Laseen and Svensson (2011). The anticipated shocks represent a distinctive way to model forward guidance by the central bank. The shocks also represent what the Bank of England (2013) describes as "time-contingent guidance" (p. 40), where the central bank communicates a definitive forward guidance end date.

Agents are assumed to form expectations about future macroeconomic variables from a rational expectations model or from an imperfect information model. Under a rational expectations, Evans and Honkapohja (2001) describe agents as being endowed with knowledge of the model's parameters, distribution of the error terms, and form of the model. As described in Preston (2005), agents with rational expectations also know the beliefs of other agents, and thus, are able to infer the proper aggregate probabilities and processes of the state variables. A popular alternative to rational expectations is to assume agents have an imperfect knowledge of the economy when forming expectations. Agents with incomplete knowledge of the economy form expectations through a process labeled adaptive learning. Adaptive learning agents know the form of the laws of motion for the endogenous variables, but lack knowledge of the parameters of the economy and the knowledge of the beliefs of other agents, as described in Evans and Honkapohja (2001) and Preston (2005). Adaptive learning agents behave as econometricians by estimating the parameters of the economy every period using econometric techniques in order to forecast future macroeconomic variables.

The novel results of this paper are as follows. Agents forming expectations through adaptive learning misvalue the effects forward guidance has on their expectations of future macroeconomic variables. This benchmark result is shown via impulse responses of the macroeconomic variables to one-unit forward guidance shocks. First, the effects after the initial forward guidance announcements are more persistent under adaptive learning, and is evidenced with impulse response trajectories under adaptive learning displaying a more persistent path than under rational expectations. Second, agents with rational expectations are quicker to adjust their expectations once the forward guidance shock has been realized upon the economy. However, the effect of the realization of the forward guidance shock is slower to dissipate under agents with adaptive learning. These results occur because of the difference in information assumed between the two expectations formation. The adaptive learning agents know the form of the laws of motion for the endogenous variables, but do not know the values of the coefficients. They update their coefficient estimates, and thus, their beliefs every period as new information arrives. Agents with rational expectations know precisely the model's coefficients and the equilibrium probability distribution. They are able to understand how the anticipated changes in monetary policy will affect the values of the endogenous variables at future dates. Thus, rational expectations agents understand the effect forward guidance has on the economy, while adaptive learning agents fail to appropriately understand this effect.

Previous Literature–This paper contributes to the recently growing unconventional monetary policy literature. Eggertsson and Woodford (2003) emphasize that the expectations channel plays a key role on the economy when interest rates are at the ZLB and at any level. Specifically, Eggertsson and Woodford (2003) emphasize that the future path of short-term interest rates affects long-term interest rates and asset prices, and thus, the management of expectations about future interests rates affects agents' optimal decisions. In addition, the recent literature has found large effects from forward guidance. Carlstrom, Fuerst, and Paustian (2012) show that standard New Keynesian Dynamic Stochastic General Equilibrium (DSGE) models with the interest rate fixed for a finite period of time result in unusually extreme responses of output and inflation. Del Negro et al. (2012) construct a DSGE model with forward guidance and show that their model produces unusually large responses of macroeconomic variables to forward guidance. Del Negro et al. (2012) state that the long-term bond yield drives these unusually high responses, and thus, resolve this issue by constraining the long-term interest rate.

The model utilizes time-contingent forward guidance since there has been recent evidence of its effectiveness. Swanson and Williams (2012) show that recent Federal Reserve forward guidance announcements have affected medium and longer-term interest rates. Woodford (2012) explains that forward guidance has had an impact on market participants. Using overnight interest rate swaps (OIS) to measure market expectations about the policy rate in Canada, Woodford (2012) shows that OIS rates immediately changed upon release of the Bank of Canada's forward guidance statement. The work of Chang and Feunou (2013) showed that the Bank of Canada's forward guidance statement in 2009 changed market expectations about future interest rates. In particular, Chang and Feunou (2013) explain that the volatility of future interest rates decreased upon issuance of the Bank of Canada's forward guidance statement.

By analyzing the role of expectations formation on forward guidance, this paper builds on previous literature on the role of expectations on macroeconomic models. Evans, Honkapohja, and Mitra (2012) develop a Real Business Cycle (RBC) model with both the assumption of rational expectations and adaptive learning. Both types of agents credibly know the future path of government spending and taxes. A temporary change in fiscal policy leads to different effects on adaptive learning agents and rational expectations agents. This outcome is due to adaptive learning agents failing to understand the effect that future fiscal policy has on key macroeconomic variables and wealth. Eusepi and Preston (2010) use an infinite-horizon macroeconomic model and show that increased central bank communication can lead to increased macroeconomic stability. Milani and Rajbhandari (2012) examine the effects of four types of expectations–rational expectations, rational expectations with "news" shocks, adaptive learning, and survey expectations–on a New Keynesian model. Eusepi and Preston (2011) utilize an RBC model and show that the assumption of adaptive learning has a more favorable fit to the data than the assumption of rational expectations. Williams (2003) examines the effects of different expectation assumptions on a simple New Keynesian and RBC model. Williams (2003) found that adaptive learning had an effect on the impulse response functions when using a basic New Keynesian model without forward guidance. Moreover, this current paper analyzes the role of expectations with a micro-founded model, but applied to the mainstream issue of forward guidance.

The remaining sections of the paper are organized as follows. Section two presents the DSGE model with forward guidance. Section three discusses expectations formation under both rational expectations and adaptive learning. Section four presents the results of the methodology. Within section four, the benchmark results of forward guidance under both rational expectations and adaptive learning are presented via impulse response functions.

2 Model

The aggregate dynamics of the economy are described by a DSGE model derived under subjective expectations (see Preston (2005)). There is a continuum of households each of whom maximizes expected future discounted utility by choosing sequences of consumption, labor, money holdings, and bonds.[1] Household's optimal decisions satisfy a sequence of Euler equations and the intertemporal budget constraint. Each household's optimal consumption decision depends on wealth and expectations about future state variables (e.g. income, inflation). The resulting log-linearized equation for the output gap is given by

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}[(1-\beta)x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T^n] \tag{1}$$

where

$$r_t^n = \rho_n r_{t-1}^n + \varepsilon_t^n \tag{2}$$

and $\varepsilon_t^n \overset{iid}{\sim} N(0, \sigma_n^2)$. All variables are in terms of log deviations from steady state. Equation (1) relates the current output gap x_t to current and future expected values of the output gap, interest rate i_t, inflation rate π_t, and natural real interest rate shock r_t^n. β describes the household's discount rate and is bounded between zero and one. $\sigma > 0$ defines the intertemporal elasticity of substitution of consumption between periods. $\hat{E}_t$ denotes (potentially) non-rational expectations. Households take into account the future values of the endogenous variables infinitely far into the

[1] As is standard in many macroeconomic models, money holdings do not alleviate any transaction frictions. Thus, the optimal decision for each household is to hold zero money balances.

future when choosing optimal consumption today. Intuitively, the expected course of a household's consumption pattern matters to its optimal consumption today. A household also knows future consumption patters are affected by future values of income, interest rates, and inflation. Thus, expectations of these variables are important for household's optimal decisions today.

The production side of the economy is populated by firms that take into account expectations of variables infinitely far into the future. A firm operates in a monopolistically competitive environment where each good is produced using labor from households. It is also subject to a Calvo (1983) pricing scheme, and thus, has an α probability of not being able to change their price every period. The representative firm chooses its price each period to maximize expected present discounted value of profits. The resulting log-linearized equation for inflation is

$$\pi_t \quad = \quad \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} [\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + \mu_T] \tag{3}$$

where

$$\mu_t \quad = \quad \rho_\mu \mu_{t-1} + \varepsilon_t^\mu \tag{4}$$

and $\varepsilon_t^\mu \overset{iid}{\sim} N(0, \sigma_\mu^2)$. All variables are in terms of log deviations from steady state. Equation (3) defines the inflation rate which is a function of current and future values of the output gap, inflation rate, and cost-push shock μ_t. Monopolistically competitive firms in this economy have an α probability of not being able to change their price every period. $\kappa \equiv \frac{(1-\alpha)}{\alpha}\frac{(1-\alpha\beta)}{(1+\omega\theta)}(\omega + \sigma^{-1}) > 0$. Preston (2005) defines the parameter ω as "the elasticity of a firm's real marginal cost function ... with respect to its own output" (p. 93). θ measures the elasticity of substitution between differentiated goods. $\hat{E}_t$ denotes (potentially) non-rational expectations. The optimal decisions by firms are shown to depend on the expected path of macroeconomic variables. Since prices are sticky, a firm must be concerned that it will not be able to adjust its price in the future regardless of future economic conditions. Thus, optimal pricing decisions today requires firms to forecast future states and values of economic variables.

Preston (2005) also argues that expectations multiple periods ahead matter to agents with subjective expectations due to a lack of information. If agents have rational expectations, he explains they would know other agents have similar beliefs and would be able to compute aggregate probability laws, and thus, equations (1) and (3) would simplify to only one period ahead expectations. Under subjective expectations, however, agents do not have this additional information. Another approach to modeling learning in macroeconomic models regards the "euler-equation" method presented in Evans and Honkapohja (2001), where only one-period ahead forecasts of the endogenous variables show up in the model's equations.[2] This paper does not suggest one method better than the other. Instead, since forward guidance affects the values of future variables, the infinite-horizon approach provides an appropriate way to include forward guidance.

The model is closed by describing the central bank of the economy. The central bank follows a monetary policy rule that takes the following form

$$i_t = \chi_\pi \pi_t + \chi_x x_t + \varepsilon_t^{MP} + \sum_{l=1}^{L} \varepsilon_{l,t-l}^{R} \tag{5}$$

The central bank adjusts the short-term nominal interest rate to changes in the output gap and inflation rate. ε_t^{MP} defines a monetary policy shock and is *i.i.d.* In order to incorporate forward guidance into the model, the monetary policy rule is augmented with anticipated shocks following Laseen and Svensson (2011) and Del Negro et al. (2012). Each anticipated or forward guidance shock ($\varepsilon_{l,t-l}$) is contained in the last term in equation (5) and is *i.i.d.*. Intuitively, the forward guidance shock can be thought of as an announcement by the central bank in period $t-l$ that the interest rate will change l periods later, or in period t. If the central bank has been communicating guidance on the interest rate for L periods ahead, there would be $1, 2, 3, \ldots, L$ forward guidance shocks that affect the monetary policy rule in period t. Thus, L corresponds to the length of the forward guidance horizon announced by the central bank. The last term in equation (5) can also be thought of as the sum of all forward guidance commitments stated by the central bank $1, 2, ...,$ and L periods ago that affect the nominal interest rate in period t. Following Laseen and Svensson (2011) and Del Negro et al. (2012), the system is also augmented with L state variables $v_{1,t}, v_{2,t}, ..., v_{L,t}$. The law of motion for each of these state variables is given by

$$v_{1,t} = v_{2,t-1} + \varepsilon_{1,t}^{R} \tag{6}$$

$$v_{2,t} = v_{3,t-1} + \varepsilon_{2,t}^{R} \tag{7}$$

$$v_{3,t} = v_{4,t-1} + \varepsilon_{3,t}^{R} \tag{8}$$

$$\vdots$$

$$v_{L,t} = \varepsilon_{L,t}^{R} \tag{9}$$

In other words, $v_{1,t}, v_{2,t}, ..., v_{L,t}$ are the sum of all central bank forward guidance commitments known in period t that affect the interest rate $1, 2, ...,$ and L periods into the future, respectively.[3] It should be noted that equations (6) − (9) can be simplified to find that $v_{1,t-1} = \sum_{l=1}^{L} \varepsilon_{l,t-l}^{R}$. In

[2] For a comparison between the "infinite-horizon" and euler-equation approach to learning, see Evans, Honkapohja, and Mitra (2013)

[3] In the terminology of Laseen and Svensson (2011), $v_{1,t}, v_{2,t}, ..., v_{L,t}$ are described as central bank "projections" (p. 10) of what $\sum_{l=1}^{L} \varepsilon_{l,t-l}^{R}$ will be $1, 2, ...,$ and L periods into the future, respectively.

addition, equations $(5)-(9)$ show a tractable method to model forward guidance. Since the forward guidance shocks equal $v_{1,t-1}$, the forward guidance shocks can be put into a vector of predetermined variables in standard state-space form. As described by Laseen and Svensson (2011), standard solution techniques then can be used to solve the final system of equations. This method also relieves the concern of indeterminant solutions. As described in Honkapohja and Mitra (2005) and Woodford (2005), indeterminacy can arise if forward guidance is modeled as pegging the interest rate to a certain value.[4]

In order to gain further intuition about $v_{1,t}, v_{2,t}, ..., v_{L,t}$, consider the case where the central bank's forward guidance horizon is 2 periods ahead, i.e. $L = 2$. The model's system of equations consists of $v_{1,t}$ and $v_{2,t}$ whose laws of motion are defined as

$$v_{1,t} = v_{2,t-1} + \varepsilon_{1,t}^{R} = \varepsilon_{2,t-1}^{R} + \varepsilon_{1,t}^{R} \tag{10}$$

$$v_{2,t} = \varepsilon_{2,t}^{R} \tag{11}$$

Thus, $v_{1,t}^{R}$ defines the sum of all forward guidance commitments by the central bank known in period t that affect the interest rate one period later. $v_{1,t}^{R}$ consists of current period forward guidance affecting the interest rate one period later, $\varepsilon_{1,t}^{R}$, and previous period's forward guidance affecting the interest rate two periods later, $v_{2,t-1} = \varepsilon_{2,t-1}^{R}$. $v_{2,t}$ is the sum of all forward guidance commitments by the central bank known in period t that affect the interest rate two periods later. Since the forward guidance horizon is two periods, $v_{2,t}$ consists of current period forward guidance affecting the interest rate two periods later, $\varepsilon_{2,t}^{R}$.

To summarize, the aggregate dynamics of the economy with forward guidance are defined by the output gap, inflation rate, AR(1) shock processes, monetary policy rule with forward guidance, and the laws of motion of the sum of central bank commitments, that is, equations $(1)-(4)$ and $(5)-(9)$.

3 Expectation Formation

This paper assumes agents form expectations following either the rational expectations hypothesis or adaptive learning. The difference between the two types of expectation formations regards the amount of information agents hold. Under rational expectations, agents know the structure of the model, parameters of the model (e.g. σ, κ, etc.), distribution of the error terms, and beliefs of other agents. Under adaptive learning, agents lack the amount of information available under rational

[4] Carlstrom, Fuerst, and Paustian (2012) show that determinacy can arise from a interest rate peg if terminal conditions are known and a standard monetary policy rule is followed after the interest rate peg. However, unusually large responses of the output and inflation are found through this process.

expectations, and instead, they operate as econometricians (See, for example, Marcet and Sargent (1989), Evans and Honkapohja (2001), and Evans, Honkapohja, and Mitra (2009).) They know the variables in the rational expectations solution and the distribution of the error terms. However, they do not know the beliefs of other agents and the values of the model's parameters. In order to compute the coefficients, they run a least squares regression each period using updated information. This additional step implies that expectations form slower relative to the assumption of rational expectations.

Rational Expectations–The model defined by equations (1) – (4) and (5) – (9) can be simplified under the assumption of rational expectations. Agents with rational expectations understand the beliefs of other agents and are able to compute the aggregate probabilities of the model. As shown in Preston (2005), this additional information simplifies the infinite horizon model to the "benchmark" one-step-ahead New Keynesian model. Specifically, equations (1) and (3) become

$$x_t = E_t x_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}) + r_t^n \tag{12}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \mu_t \tag{13}$$

The model with rational expectations can be solved using standard techniques, such as one suggested by Sims (2002). The model can be written in general state-space form as suggested by Sims (2002). This form is defined as

$$\widetilde{\Gamma_0}\widetilde{Y_t} = C + \widetilde{\Gamma_1}\widetilde{Y_{t-1}} + \widetilde{\Gamma_2}\epsilon_t + \widetilde{\Gamma_3}\zeta_t \tag{14}$$

where

$$\widetilde{Y_t} = [x_t, \pi_t, i_t, r_t^n, \mu_t, v_{1,t}, v_{2,t}, \ldots, v_{L,t}, E_t x_{t+1}, E_t \pi_{t+1}]' \tag{15}$$

$$\widetilde{\epsilon_t} = [\varepsilon_t^n, \varepsilon_t^\mu, \varepsilon_t^{MP}]' \tag{16}$$

C defines a vector of constants of required dimensions. ζ_t defines the vector of expectational errors (e.g. $\zeta_t^\pi = \pi_t - E_{t-1}\pi_t$) of required dimensions. Using standard techniques to solve the model with rational expectations (e.g. Sims (2002)) and the parameter values in Table 1, the solution to the system under rational expectations is

$$\widetilde{Y_t} = \widetilde{C} + \xi_1 \widetilde{Y_t} + \xi_2 \epsilon_t \tag{17}$$

where the matrices $\widetilde{C}$, ξ_1, and ξ_2 are defined in Appendix A.[5]

Adaptive Learning–In order to evaluate the expectations in equations (1) and (3) under adaptive learning, agents act as econometricians by forming a model based on variables that appear

[5]Discussion of the parameter values can be found in Table 1 in Section 4.1.

in the rational expectations solution and estimating its coefficients. This model is labeled the "Perceived Law of Motion" (PLM) and is based on the minimum state variable (MSV) solution that exists under rational expectations. The PLM is defined as

$$Y_t = a_t + b_t v_t + c_t w_t + d_t v_{1,t-1} + \varepsilon_t \tag{18}$$

where

$$Y_t = [x_t, \pi_t, i_t]' \tag{19}$$

$$v_t = [v_{1,t}, v_{2,t}, ..., v_{L,t}]' \tag{20}$$

The vector $w_t = [r_t^n, \mu_t]'$ is defined by

$$w_t = \widetilde{\phi} w_{t-1} + \bar{\varepsilon}_t \tag{21}$$

where

$$\widetilde{\phi} = \begin{bmatrix} \rho_n & 0 \\ 0 & \rho_\mu \end{bmatrix} \tag{22}$$

$$\bar{\varepsilon}_t = [\varepsilon_t^n, \varepsilon_t^\mu]' \tag{23}$$

By rewriting equations (6) – (9), the vector v_t is defined by

$$v_t = \Phi v_{t-1} + \eta_t \tag{24}$$

where

$$\eta_t = [\varepsilon_{1,t}^R, \ldots, \varepsilon_{L,t}^R]' \tag{25}$$

and Φ is an L x L matrix defined as

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{26}$$

(27)

a_t, b_t, c_t, and d_t are unknown coefficient matrices of appropriate dimensions that agents estimate and learn about over time.[6]

[6] Since this paper restricts attention to fundamentals solutions and Y_{t-1} does not appear in equations (1), (3), and (5), the PLM does not contain Y_{t-1}.

The addition of $v_{1,t-1}$ is a necessary component of the PLM. Recall that $v_{1,t-1}$ defines the central bank's forward guidance agents know in period $t-1$ that affects the monetary policy rule in period t, that is, the last term in equation (5). $v_{1,t-1}$ also is present in the rational expectations solution shown in Appendix A. In addition, the v_t term in the PLM does not contain $v_{1,t-1}$ by definition of equation (24). Hence, the inclusion of $v_{1,t-1}$ in the PLM is necessary in order for the PLM to be based on the MSV solution that exists under rational expectations.

An important component of adaptive learning models regards the information available to agents when they form expectations. Agents are assumed to know the values of the regressors in the PLM and previous period's coefficient estimates. The *i.i.d.* monetary policy shock is assumed to be unobserved.[7] Furthermore, the following is the timeline of events

1. At the beginning of period t, v_t, and w_t are observed by the agents and added to their information set.

2. Agents use v_t, w_t, and $v_{1,t-1}$ as well as previous period's estimates, ϕ_{t-1}, to form expectations about the future.

3. Y_t is realized.

4. In order to update their parameter estimates, agents compute a least squares regression of Y_t on 1, v_t, w, and $v_{1,t-1}$.

Agents update their estimates of a_t, b_t, c_t, and d_t by following the recursive least squares (RLS) formula

$$\phi_t = \phi_{t-1} + \tau_t R_t^{-1} z_t (Y_t - \phi_{t-1}^{'} z_t)' \tag{28}$$

$$R_t = R_{t-1} + \tau_t (z_t z_t^{'} - R_{t-1}) \tag{29}$$

where $\phi_t = (a_t, b_t, c_t, d_t)^{'}$ contains the PLM coefficients to be estimated. R_t defines the precision matrix of the regressors in the PLM $z_t \equiv [1, v_t, w_t, v_{1,t-1}]^{'}$. τ_t is known as the "gain" parameter and controls the response of ϕ_t to new information. The last expression in equation (28) defines the recent prediction error of the endogenous variables.

The gain parameter in equations (28) and (29) can either decrease over time or be fixed at certain values. In the decreasing gain or RLS case, $\tau_t = t^{-1}$ and past observations are equally weighted. An attractive feature of this case is that the rational expectations equilibrium (REE) can be a nested solution. Evans and Honkapohja (2001) explain that as $t \to \infty$ the coefficients in

[7] This is similar to Milani (2007a).

the PLM converge to the rational expectations coefficients with probability one. As is assumed in this current paper, the gain parameter can also be fixed at a certain value. Under this constant gain learning (CGL) alternative, the most recent observations play a larger role when updating agents' coefficients and expectations. Evans and Honkapohja (2001) describe that the coefficients in the PLM converge in distribution to their rational expectations values with a variance that is proportional to the constant gain coefficients.

Agents solve for $\hat{E}_t Y_{T+1}$ by using equation (18). For any $T \geq t$, their expectations infinite periods ahead are given by

$$\begin{aligned}\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} Y_{T+1} = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} a_{t-1} + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} b_{t-1} v_{T+1} \\ + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} c_{t-1} w_{T+1} + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} d_{t-1} v_{1,T}\end{aligned} \tag{30}$$

$$\begin{aligned}\hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} Y_{T+1} = \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} a_{t-1} + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} b_{t-1} v_{T+1} \\ + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} c_{t-1} w_{T+1} + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} d_{t-1} v_{1,T}\end{aligned} \tag{31}$$

By noting the geometric sums and expectations of v_t twelve periods ahead or greater equal the zero vector, equations (32) and (33) simplify to equal

$$\begin{aligned}\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} Y_{T+1} = (1-\beta)^{-1} a_{t-1} + b_{t-1} \Phi (I_L - \beta\Phi)^{-1} (I_L - (\beta\Phi)^{11}) v_t \\ + c_{t-1} (I_2 - \beta\widetilde{\phi})^{-1} \widetilde{\phi} w_t + d_{t-1} [1, \beta, \beta^2, \ldots, \beta^{11}] v_t\end{aligned} \tag{32}$$

$$\begin{aligned}\hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} Y_{T+1} = (1-\alpha\beta)^{-1} a_{t-1} + b_{t-1} \Phi (I_L - \alpha\beta\Phi)^{-1} (I_L - (\alpha\beta\Phi)^{11}) v_t \\ + c_{t-1} (I_2 - \alpha\beta\widetilde{\phi})^{-1} \widetilde{\phi} w_t + d_{t-1} [1, \alpha\beta, (\alpha\beta)^2, \ldots, (\alpha\beta)^{11}] v_t\end{aligned} \tag{33}$$

Equations (32) and (33) are substituted into equations (1) and (3) to give

$$Y_t = \Gamma_0(a_t, b_t, c_t, d_t) + \Gamma_1 Y_{t-1} + \Gamma_2(a_t, b_t, c_t, d_t) v_t + \Gamma_3(a_t, b_t, c_t, d_t) \widetilde{w_t} \tag{34}$$

where

$$\widetilde{w_t} = [w_t, \varepsilon_t^{MP}]' \tag{35}$$

Equation (34) is called the "Actual Law of Motion" (ALM).

4 Results

4.1 Calibration

The calibrated values for the model's parameters in large part are conventional and shown in Table 1. β is set to equal 0.99 which is a common value found in the literature. The parameter representing the intertemporal elasticity of substitution is fixed at one. This value has been assumed *a priori* in Smets and Wouters (2003). κ is set to equal 0.1. This number roughly corresponds to a high degree of price stickiness, α, found in empirical work by Klenow and Malin (2011), a value of ω found in Giannoni and Woodford (2004), and a value of θ found in the literature (e.g. Gertler and Karadi [2011]). Monetary policy positively responds to the output gap, and positively adjusts at more than a one-to-one rate to the inflation rate. The structural disturbances are not assumed to exhibit high persistence. The autoregressive coefficients for the demand and cost-push shocks are 0.5 and 0.2, respectively. The distribution of the white noise shocks is not assumed to be highly dispersed. There also is no covariance between the structural shocks.

The current paper examines results for the CGL case. In regards to choosing the CGL parameter $\bar{\tau}$, this paper uses 0.03. This result is close to the results used in the literature, such as Orphanides and Williams (2005), Milani (2007a), and Evans, Honkapohja, and Mitra (2012). For robustness, the current methodology also examines the results under different values of $\bar{\tau}$.

The value for the length of the forward guidance horizon L is chosen to match time-contingent forward guidance by the Federal Reserve. This is based off the FOMC September 2012 statement: "...the Committee also decided today to keep the target range for the federal funds rate at 0 to 1/4 percent and currently anticipates that exceptionally low levels for the federal funds rate are likely to be warranted at least through mid-2015." This statement was the first FOMC statement to use time-contingent forward guidance language. By taking "mid-2015" to be at most the end of the third quarter of 2015, the number of quarters from September 2012 to "mid-2015" is twelve. Thus, $L = 12$.

4.2 Impulse Response Functions

In this section, impulse responses of the output gap and inflation rate to a negative one-unit monetary policy shock and forward guidance shocks under different expectation assumptions are examined in Figures 1 and 2.[8] The forward guidance shocks are the anticipated shocks found in equations (6) - (9). Since equation (34) exhibits a nonlinear structure, standard linear techniques

[8] A projection facility is utilized to ensure beliefs are not explosive.

Table 1: **Parameter Values**

	Description	**Value**
σ	IES	1
β	Discount Factor	0.99
κ	Function of Price Stickiness	0.1
α	Price Stickiness	0.75
χ_π	Feedback Inflation	1.4
χ_x	Feedback Output Gap	0.125
ρ_n	Autoregressive Demand	0.5
ρ_μ	Autoregressive Cost-Push	0.2
σ_n^2	Demand Shock	0.01
σ_μ^2	Cost-Push Shock	0.01
σ_i^2	M.P Shock	0.001
$\sigma_{1,i}^2$	1 Period Ahead FG Shock	0.001
$\sigma_{2,i}^2$	2 Period Ahead FG Shock	0.001
$\sigma_{3,i}^2$	3 Period Ahead FG Shock	0.001
$\sigma_{4,i}^2$	4 Period Ahead FG Shock	0.001
$\sigma_{5,i}^2$	5 Period Ahead FG Shock	0.001
$\sigma_{6,i}^2$	6 Period Ahead FG Shock	0.001
$\sigma_{7,i}^2$	7 Period Ahead FG Shock	0.001
$\sigma_{8,i}^2$	8 Period Ahead FG Shock	0.001
$\sigma_{9,i}^2$	9 Period Ahead FG Shock	0.001
$\sigma_{10,i}^2$	10 Period Ahead FG Shock	0.001
$\sigma_{11,i}^2$	11 Period Ahead FG Shock	0.001
$\sigma_{12,i}^2$	12 Period Ahead FG Shock	0.001
L	FG Horizon	12
$\bar{\tau}$	CGL	0.03

Note: FG stands for forward guidance.

to compute impulse responses under adaptive learning do not apply. To remedy this situation, this paper follows Eusepi and Preston (2011) by proceeding in the following manner. In order to calculate the impulse responses of an endogenous variable to a particular shock, the model is simulated twice for 10,000+K periods, where K is the impulse response function horizon. The impulse responses are calculated starting at period 10,001 to ensure the model has converged to its stationary distribution. In the first simulation, time period 10,001 includes a negative one-unit shock. The K-period impulse response function is given by the difference between the first and second simulations over the final K periods. The process is then repeated for 5,000 simulations and the mean impulse response across the 5,000 simulations is calculated to arrive at the final impulse response trajectory. The impulse response function horizon is chosen to be twenty periods, that is, $K = 20$.

The ZLB on interest rates is also enforced in this section. Forward guidance has recently

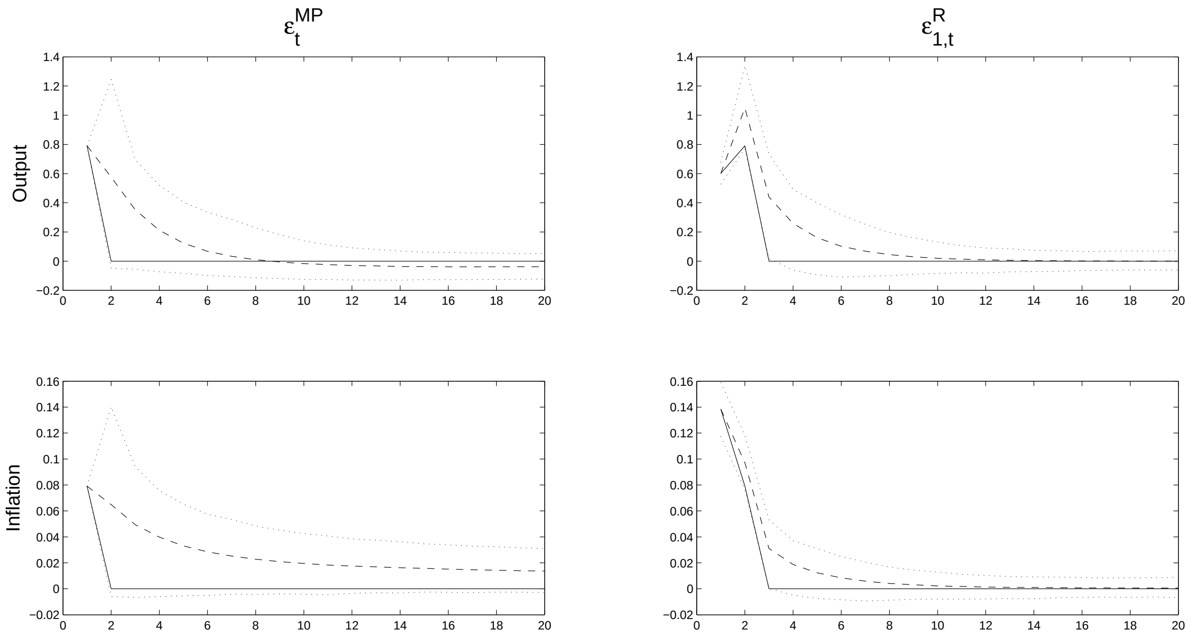

Figure 1: Impulse Responses of Endogenous Variables to Anticipated Shocks

Note: Solid Line: Rational Expectations; Dashed Line: CGL; Dotted Line: 95% Confidence Band

gained attractiveness and attention due to interest rates effectively reaching the ZLB because of the 2007-2009 global financial recession. Thus, it seems natural to model the ZLB on nominal interest rates when simulating forward guidance. Specifically, equations (1) and (5) are modified and result in the following

$$\hat{x}_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}[(1-\beta)\hat{x}_{T+1} - \sigma(i_T - i^* - \hat{\pi}_{T+1}) + \hat{r}_T^n] \tag{36}$$

$$i_t = max\{i^* + \chi_\pi \hat{\pi}_t + \chi_x \hat{x}_t + \varepsilon_t^{MP} + \sum_{l=1}^{L} \varepsilon_{l,t-l}^R, 0\} \tag{37}$$

where $i^* = r^* + \pi^*$ is the steady-state nominal interest rate. The " ˆ " symbol over the variables denotes log deviations from steady state.[9]

Impact–As seen in Figures 1 and 2, the initial response of the macroeconomic variables is approximately the same under both adaptive learning and rational expectations. This result is not surprising since Evans and Honkapohja (2001) state that CGL coefficients converge to a Normal distribution centered around its rational expectations counterparts. Thus, as noted in Eusepi and Preston (2011), the initial impact under adaptive learning could be greater or less than the initial

[9]In a zero steady-state inflation rate, $\pi^* = 0$. The model implied steady-state real interest rate $r^* = \beta^{-1} - 1$.

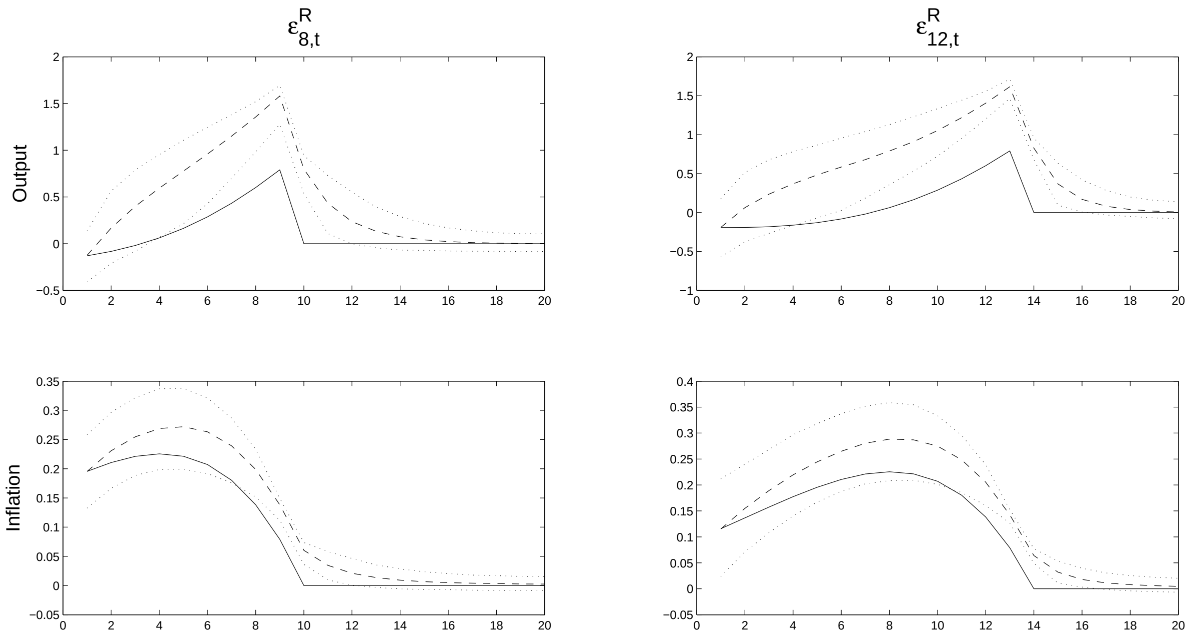

Figure 2: Impulse Responses of Endogenous Variables to Anticipated Shocks

Note: Solid Line: Rational Expectations; Dashed Line: CGL; Dotted Line: 95% Confidence Band

impact under rational expectations.

After Impact–Figures 1 and 2 display the impulse responses after the forward guidance announcement is known to agents. From the household's perspective, they must optimally allocate consumption across time based on their expectations of future variables. Since they know that the interest rate will increase in the future, a household changes their optimal consumption across time and decreases current consumption. In addition, firms know they may not be able to change their price in the future regardless of the state of the economy. Thus, they take into account expectations of future variables as seen in equation (3). When the central bank announces that the interest rate will increase in the future, a firm knows that future output gap and inflation rate will be affected, and thus, this action affects current pricing decision.

After the initial impact of a forward guidance announcement, the impulse response trajectories of rational expectations and adaptive learning proceed in different paths. This result is shown in Figures 1 and 2 by impulse responses displaying more persistence in the periods after news of a forward guidance commitment under adaptive learning than under rational expectations. Adaptive learning agents understand the form of the laws of motion for the endogenous variables, but they misvalue the effect forward guidance has on their expectations of future values of the macroeconomic variables. They learn about the coefficients each period, and thus, their beliefs about future values of the endogenous variables change each period as new information arrives. However, rational expectations agents know precisely the equilibrium probability distribution and how the anticipated changes in monetary policy will affect the endogenous variables at later dates. Thus, rational expectations agents' expectations display less persistence than their adaptive learning counterparts.

After Shock Realized–The impulse response graphs of rational expectations and adaptive learning do not follow the same path after the shock is realized upon the economy. The impulse responses with rational expectations agents converge quicker to zero percentage deviation from the unshocked series. Rational expectations agents understand that the shock will not occur in the future and they quickly adjust their expectations. However, the impulse responses under adaptive learning exhibit more persistence than the impulse responses under rational expectations. This outcome is present because the dynamics of the impulse responses under adaptive learning are driven by adjustments in the beliefs of the agents. Adaptive learning agents revise their estimates of the parameters of the economy each period, while rational expectations agents fully understand the model's parameters. The impulse responses of a conventional monetary policy shock shown in the first column of Figure 1 also display the same difference in persistence.

The results coincide with the literature on adaptive learning. The outcomes match Eggertsson (2008) who found that temporary policy shifts do not have as large of an effect on the economy as permanent policy shifts under the assumption of rational expectations. The persistence results also coincide with Milani (2007a) who found that a DSGE model with constant-gain learning generates persistence in the macroeconomic variables.

To summarize, agents forming expectations through the adaptive learning process misvalue the effects forward guidance has on their expectations. The output gap and inflation rate exhibit more persistence under adaptive learning than under rational expectations. After the shock has been realized, rational expectations agents quickly adjust their expectations to the knowledge that the shock is gone, while adaptive learning agents' beliefs are more persistent. These results are attributed to rational expectations agents precisely understanding the effects forward guidance has on their expectations, while adaptive learning agents revise their beliefs every period.

5 Conclusion

In order to combat the effects of the recent financial recession, central banks have instituted forward guidance. Because the effectiveness of forward guidance hinges on how expectations respond to forward guidance, it is of interest to investigate the link between expectation assumptions and forward guidance. The standard benchmark assumption is that agents form expectations based on the rational expectations hypothesis. However, if agents form expectations using an imperfect information model, the results are different.

This paper presents an infinite horizon DSGE model with forward guidance and compares the results under two types of expectation assumptions. Under the assumption of rational expectations, Evans and Honkapohja (2001) state agents are assumed to know the distribution of the structural shocks, correct form of the model, and parameters of the model. A main difference of adaptive learning is that agents estimate the parameters of the model over time and do not know the beliefs of other agents in the economy. The results show that adaptive learning agents misvalue the effect forward guidance has on their expectations. The responses of the macroeconomic variables to a one-unit forward guidance shock display more persistent behavior under adaptive learning than under rational expectations. This result can be attributed to rational expectations agents precisely understanding the effects forward guidance has on their expectations, while adaptive learning agents revise their beliefs every period.

There are other modifications to the model presented in this paper that are worth noting. For instance, this paper allows agents to know the end date of forward guidance. Another type of forward guidance policy allows the central bank to link the expiration date of forward guidance to economic conditions. For instance, the unemployment rate is a criterion that the Federal Reserve currently uses to link to its forward guidance policy. The RLS formula also could be altered to include a gain parameter that changes based on the recent forecast errors as discussed in Milani (2007b) and Marcet and Nicolini (2003). This formation of the gain parameter allows agents to better track structural breaks in the economy. In addition, agents can be assumed to have heterogeneous expectations as in Branch and McGough (2009). Branch (2004) uses survey data and shows evidence that respondents have heterogeneous expectations. Overall, the role of expectations formation is especially crucial to understand the effects of forward guidance.

References

Bank of England (2013). Monetary policy trade-offs and forward guidance. Technical report.

Branch, W. A. (2004). The theory of rationally heterogeneous expectations: Evidence from survey data on inflation expectations. *The Economic Journal*, 114(497):592–621.

Branch, W. A. and McGough, B. (2009). A new keynesian model with heterogeneous expectations. *Journal of Economic Dynamics and Control*, 33(5):1036–1051.

Branch, W. A. and McGough, B. (2011). Business cycle amplification with heterogeneous expectations. *Economic Theory*, 47(2-3):395–421.

Calvo, G. (1983). Staggered prices in a utility-maximizing framework. *Journal of monetary Economics*, 12(3):383–398.

Carlstrom, C., Fuerst, T., and Paustian, M. (2012). Inflation and output in new keynesian models with a transient interest rate peg. Retrieved from: http://www.bankofengland.co.uk/research/documents/workingpapers/2012/wp459.pdf.

Chang, B. Y. and Feunou, B. (2013). Measuring uncertainty in monetary policy using implied volatility and realized volatility. *Bank of Canada, manuscript.*

Chen, H., Cúrdia, V., and Ferrero, A. (2012). The macroeconomic effects of large-scale asset purchase programmes. *The Economic Journal*, 122(564):F289–F315.

Del Negro, M., Giannoni, M., and Patterson, C. (2012). The forward guidance puzzle. *FRB of New York Staff Report*, (574).

Eggertsson, G. B. (2008). Great expectations and the end of the depression. *The American Economic Review*, 98(4):1476–1516.

Eggertsson, G. B. and Woodford, M. (2003). The zero bound on interest rates and optimal monetary policy. *Brookings papers on economic activity*, (1):139–211.

Eusepi, S. and Preston, B. (2010). Central bank communication and expectations stabilization. *American Economic Journal: Macroeconomics*, 2(3):235–271.

Eusepi, S. and Preston, B. (2011). Expectations, learning, and business cycle fluctuations. *American Economic Review*, 101(6):2844–72.

Evans, G. W. and Honkapohja, S. (2001). *Learning and expectations in macroeconomics*. Princeton University Press.

Evans, G. W., Honkapohja, S., and Mitra, K. (2009). Anticipated fiscal policy and adaptive learning. *Journal of Monetary Economics*, 56(7):930–953.

Evans, G. W., Honkapohja, S., and Mitra, K. (2012). Fiscal policy and learning. Retrieved from: http://research-repository.st-andrews.ac.uk/bitstream/10023/2260/1/mitracdmawp1202.pdf.

Evans, G. W., Honkapohja, S., and Mitra, K. (2013). Notes on agents behavioural rules under adaptive learning and studies of monetary policy. *Macroeconomics at the Service of Public Policy*, page 63.

Gagnon, J., Raskin, M., Remache, J., and Sack, B. (2010). Large-scale asset purchases by the federal reserve: did they work? Technical report, Staff Report, Federal Reserve Bank of New York.

Gertler, M. and Karadi, P. (2011). A model of unconventional monetary policy. *Journal of Monetary Economics*, 58(1):17–34.

Giannoni, M. and Woodford, M. (2004). Optimal inflation-targeting rules. In *The Inflation-Targeting Debate*, pages 93–172. University of Chicago Press.

Honkapohja, S. and Mitra, K. (2005). Performance of inflation targeting based on constant interest rate projections. *Journal of Economic Dynamics and Control*, 29(11):1867–1892.

Klenow, P. J. and Malin, B. A. (2011). Microeconomic evidence on price-setting. *Handbook of Monetary Economics*, 3:231–284.

Laséen, S. and Svensson, L. E. (2011). Anticipated alternative policy rate paths in policy simulations. *International Journal of Central Banking*.

Magnus, X. and Neudecker, H. (1988). Matrix differential calculus. *New York*.

Marcet, A. and Nicolini, J. P. (2003). Recurrent hyperinflations and learning. *The American Economic Review*, 93(5):1476–1498.

Marcet, A. and Sargent, T. J. (1989). Convergence of least squares learning mechanisms in self-referential linear stochastic models. *Journal of Economic theory*, 48(2):337–368.

Milani, F. (2007a). Expectations, learning and macroeconomic persistence. *Journal of monetary Economics*, 54(7):2065–2082.

Milani, F. (2007b). Learning and time-varying macroeconomic volatility. Technical report, mimeo, University of California, Irvine.

Milani, F. and Rajbhandari, A. (2012). Expectation formation and monetary dsge models: Beyond the rational expectations paradigm. *Advances in Econometrics*, 28:253–288.

Orphanides, A. and Williams, J. C. (2005). The decline of activist stabilization policy: Natural rate misperceptions, learning, and expectations. *Journal of Economic dynamics and control*, 29(11):1927–1950.

Preston, B. (2005). Learning about monetary policy rules when long-horizon expectations matter. *International Journal of Central Banking*, 1(2):81–126.

Sims, C. A. (2002). Solving linear rational expectations models. *Computational economics*, 20(1):1–20.

Smets, F. and Wouters, R. (2003). An estimated dynamic stochastic general equilibrium model of the euro area. *Journal of the European economic association*, 1(5):1123–1175.

Smets, F. and Wouters, R. (2007). Shocks and frictions in us business cycles: A bayesian dsge approach. *The American Economic Review*, 97(3):586–606.

Swanson, E. T. and Williams, J. C. (2012). Measuring the effect of the zero lower bound on medium-and longer-term interest rates. *Federal Reserve Bank of San Francisco Working Paper*, 2012-02.

Williams, N. (2003). Adaptive learning and business cycles. *Manuscript, Princeton University.*

Woodford, M. (2005). Central bank communication and policy effectiveness. In *The Greenspan Era: Lessons for the Future*, pages 399–474. Federal Reserve Bank of St. Louis.

Woodford, M. (2012). Methods of policy accommodation at the interest-rate lower bound. In *Jackson Hole symposium, August, Federal Reserve Bank of Kansas City.*

Appendix

A Rational Expectations Solution

By following Sims (2002), the model consisting of equations (2), (4), (5), (6) − (9), (12), and (13) can be solved to yield the solution

$$\widetilde{Y}_t = \widetilde{C} + \xi_1 \widetilde{Y}_t + \xi_2 \epsilon_t \tag{A.1}$$

where

$$\widetilde{C} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]' \tag{A.2}$$

$$\xi_1 = \begin{bmatrix}
0 & 0 & 0 & 0.57 & -0.35 & -0.77 & -0.55 & -0.35 & -0.18 & -0.03 & 0.08 & 0.16 & 0.21 & 0.23 & 0.24 & 0.23 & 0.21 & 0 & 0 \\
0 & 0 & 0 & 0.11 & 0.21 & -0.08 & -0.13 & -0.16 & -0.18 & -0.18 & -0.17 & -0.16 & -0.13 & -0.11 & -0.08 & -0.06 & -0.04 & 0 & 0 \\
0 & 0 & 0 & 0.27 & 0.32 & 0.77 & -0.30 & -0.33 & -0.34 & -0.32 & -0.29 & -0.25 & -0.21 & -0.16 & -0.11 & -0.07 & -0.03 & 0 & 0 \\
0 & 0 & 0 & 0.50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.29 & -0.07 & 0 & -0.77 & -0.55 & -0.35 & -0.18 & -0.03 & 0.08 & 0.16 & 0.21 & 0.23 & 0.24 & 0.23 & 0 & 0 \\
0 & 0 & 0 & 0.06 & 0.04 & 0 & -0.08 & -0.13 & -0.16 & -0.18 & -0.18 & -0.17 & -0.16 & -0.13 & -0.11 & -0.08 & -0.06 & 0 & 0
\end{bmatrix} \tag{A.3}$$

$$
\xi_2 = \begin{bmatrix}
1.15 & -1.78 & -0.77 & -0.55 & -0.35 & -0.18 & -0.03 & 0.08 & 0.16 & 0.21 & 0.23 & 0.24 & 0.23 & 0.21 & 0.18 \\
0.23 & 1.03 & -0.08 & -0.13 & -0.16 & -0.18 & -0.18 & -0.17 & -0.16 & -0.13 & -0.12 & -0.08 & -0.06 & -0.04 & -0.02 \\
0.54 & 1.59 & 0.77 & -0.30 & -0.33 & -0.34 & -0.32 & -0.29 & -0.25 & -0.21 & -0.16 & -0.11 & -0.07 & -0.04 & -0.01 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0.57 & -0.35 & 0 & -0.77 & -0.54 & -0.35 & -0.1 & -0.03 & 0.08 & 0.16 & 0.21 & 0.23 & 0.24 & 0.23 & 0.21 \\
0.11 & 0.21 & 0 & -0.08 & -0.13 & -0.16 & -0.18 & -0.18 & -0.17 & -0.16 & -0.13 & -0.12 & -0.08 & -0.06 & -0.04
\end{bmatrix} \tag{A.4}
$$