A Central Bank's Escape from an Indeterminacy Trap*

Nelson Lind†

UC San Diego

September 11, 2014

[PRELIMINARY]

Abstract

A primary objective of monetary policy is to rule out non-fundamental sources of fluctuations due to indeterminacy of equilibrium. I demonstrate how a central bank can fail to achieve this policy objective if it does not account for indeterminacy when assessing structural identification. In standard models, parameter values which support determinacy can be observationally equivalent to parameter values supporting indeterminacy. As a result, a central bank can rationally believe that equilibrium is determinate, while its policy actually supports multiple equilibria. In this scenario, the central bank forgoes welfare gains from ruling out non-fundamental sources of business cycle fluctuations. To escape this indeterminacy trap, the central bank must adopt a policy rule which robustly ensures determinacy across observationally equivalent points in both the determinacy and indeterminacy regions.

*I thank Ivana Komunjer, Valerie Ramey, and Johannes Wieland for helpful comments and suggestions.

† *nrlind@ucsd.com*, econweb.ucsd.edu/~nrlind, Dept. of Economics, University of California, San Diego, 9500 Gilman Dr MC 0508, La Jolla, CA 92093-0508.

1 Introduction

A central bank fails to effectively guide economic outcomes when its policy allows for multiple equilibria. Optimal policies first aim to ensure only one equilibrium exists. Achieving this objective requires a policymaker to (1) identify the structure of an economy and (2) analyze whether prospective policies lead to a unique equilibrium.

Since equilibrium indeterminacy is a concern for policy design, it must also be a concern for any identification analysis which informs policy. Indeterminacy leaves room for self-fulfilling phenomena to generate non-fundamental sources of volatility. Since fundamental and non-fundamental sources of business cycle fluctuations can generate identical macroeconomic dynamics, a central bank which fails to account for indeterminacy implicitly ignores an important form of identification failure, and can end up with false inference about the structure of the economy.

This issue can lead to a policy trap – the central bank assumes that only fundamental sources of volatility drive economic fluctuations, its implied statistical model cannot be rejected by historical data, but its policy rule actually allows non-fundamental shocks to influence the business cycle. Since the fluctuations it observes are statistically indistinguishable from the fluctuations it expects, policy continues as usual. The central bank is stuck in an indeterminacy trap.

This phenomenon occurs in models with standard frictions. I illustrate the trap using a simple neoclassical economy with firm working capital. This model allows for both supply and demand shocks as potential sources of fluctuations in inflation, output, and interest rates. Points in the indeterminacy region where only supply shocks and non-fundamental shocks drive the business cycle end up being observationally equivalent to points in the determinacy region where both supply and demands shocks generate fluctuations. From a policy point of view, the distinction is vital. Demand shocks due to exogenous fundamentals can never be fully eliminated, while "demand shocks" due to self-fulfilling beliefs can be entirely dampened through policy.

To escape the indeterminacy trap, a central bank must design a new policy rule which is robust to this type of identification failure. It must account for economic structures which are associated with indeterminacy under its current policy rule and then adopt a new policy rule which is robust against these alternative scenarios.

This form of observational equivalence comes from considering boundary points in the parameter space – points where a potential fundamental shock drops out of the model. Specifically, increases in the degree of indeterminacy increase the number of state variables underlying economic fluctuations. When coupled with parameter restrictions which eliminate state variables, it is possible to get observational equivalence since the number of latent states remains constant.

For instance, indeterminacy in the neoclassical economy allows for self-fulfilling shocks which introduce an additional latent state. This additional source of macroeconomic dynamics influences the economy as if it were a demand shock. Now, exogenous demand shocks can be dropped from the model without altering the evolution of interest rates and output.

A global perspective is necessary to account for observational equivalence between the determinacy and indeterminacy regions of the parameter space. Removing the demand shock requires setting its standard deviation to zero. The resulting parameterization lies on the boundary of the parameter space and cannot remain local to the original parameterization. As a result, local identification results (such as in Iskrev (2010), Komunjer and Ng (2011), and Qu and Tkachenko (2010)) cannot detect this type of indeterminacy-induced identification failure[1]. Recent progress on global identification in dynamic stochastic general equilibrium models includes Qu and Tkachenko (2013), Morris (2013), and Kociecki and Kolasa (2014).

In the neoclassical example, the central bank can use a reduced form VAR(1) in output and interest rates to account for observational equivalence between the determinacy and indeterminacy regions. The technique used for this characterization comes from Morris (2013). The identified set then consists of a component associated with determinacy and a component associated with indeterminacy. Given knowledge of the historical policy rule and the coefficients from a reduced form VAR estimated on historical data, the central bank can escape the trap by designing a policy which is robust across the full identified set.

The problem of observational equivalence between points in the determinacy region and the indeterminacy region is well known. Benhabib and Farmer (1994), using standard metrics from the Real Business Cycle literature, first illustrate the issue by showing how models exhibiting indeterminacy can explain data as well as models with a unique equilibrium[2]. Beyer and Farmer (2004) show that models exhibiting determinacy can be statistically indistinguishable from models exhibiting indeterminacy. In this paper, I demonstrate why this form of identification failure is particularly important when designing policy. Ignoring indeterminacy during identification analysis can perpetuate sub-optimal policies.

Section 2 uses the Fisher equation to illustrate how indeterminacy can generate identification problems. Section 3 introduces a neoclassical economy where this lack of identification has consequences for policy. In particular, a policy trap can occur where the central bank allows for indeterminate outcomes but believes outcomes are determinate. Section 4 then illustrates how to design policy rules which rule out indeterminacy across the whole identified set. Section 5 concludes.

2 Lack of Global Identification Due to Indeterminacy

Economic structures supporting determinacy and structures supporting indeterminacy are often observationally equivalent. Benhabib and Farmer (1994) first made this point using the example of a growth model with increasing returns. However, this type of observational equivalence is quite generic.

[1] It is essential to understand the full identified set instead of looking for conditions under which there is a unique parameter point consistent with data. This observation points to adopting partial identification techniques (Horowitz and Manski (1995), Manski and Tamer (2002), Manski, Imbens and Manski (2004), and Chernozhukov et al. (2007)).

[2] Benhabib and Farmer (1999) review the related literature. For further examples see Benhabib and Farmer (2000) and Beyer and Farmer (2004).

Consider a Fisherian model of inflation determination:

$$r_t = i_t - \mathbb{E}_t \pi_{t+1}$$
$$i_t = \rho_i i_{t-1} + \phi \pi_t$$

The first equation is the Fisher equation. The exogenous real rate r_t is equal to the nominal interest rate net of expected inflation. The second equation specifies monetary policy. The central bank increases the nominal interest rate when inflation is above zero, and smoothes interest rates over time.

To get a bivariate model in the nominal and real interest rates, I combine the Fisher equation and the policy rule to get a single forward looking equation in the nominal rate, and specify an AR(1) process for the real rate:

$$\mathbb{E}_t i_{t+1} = (\rho_i + \phi) i_t - \phi r_t \tag{1}$$
$$r_t = \rho_r r_{t-1} + \varepsilon_t, \quad \mathbb{E}_{t-1}\varepsilon_t = 0, \quad \mathbb{E}_{t-1}[\varepsilon_t^2] = \sigma^2 \tag{2}$$

where $|\rho_r| < 1$ so that the real rate is a stationary exogenous process.

2.1 Determinacy and Indeterminacy

Since r_t is exogenous, the forward looking equation has a unique equilibrium if and only if it is unstable: $|\rho_i + \phi| \geq 1$. In this case, the method of undetermined coefficients implies that the nominal rate is linear in the real rate

$$i_t = \frac{\phi}{\rho_i + \phi - \rho_r} r_t$$

Using the AR(1) process for the real rate with this result implies an AR(1) process for the nominal interest rate:

$$i_t = \rho_r i_{t-1} + \frac{\phi}{\rho_i + \phi - \rho_r} \varepsilon_t \tag{3}$$

This equation specifies the data generating process for the nominal interest rate when the structural parameters satisfy $|\rho_i + \phi| \geq 1$.

This equation implies that, under determinacy, the observable data has a reduced form AR(1) representation. Alternative parameter values which imply indeterminacy will be observationally equivalent if they also have an AR(1) reduced form.

To study the indeterminacy case, I follow Lubik and Schorfheide (2003) and re-write equations (1) and (2) in the Sims (2002) form. Moving equation (1) back one period, and incorporating the rational expectations forecast error $\eta_t \equiv i_t - \mathbb{E}_{t-1} i_t$ gives:

$$\begin{bmatrix} i_t \\ r_t \end{bmatrix} = \begin{bmatrix} \rho_i + \phi & -\phi \\ 0 & \rho_r \end{bmatrix} \begin{bmatrix} i_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \eta_t \tag{4}$$

This system specifies the evolution of nominal and real interest rates, driven by fundamental shocks and rational expectations forecast errors. Solving for equilibrium is then equivalent to solving for forecast errors which ensure the system remains stable.

When $|\rho_i + \phi| < 1$ the system is already stable. In this case, equilibrium is indeterminate since η_t can be any random variable satisfying the martingale difference condition $\mathbb{E}_{t-1}\eta_t = 0$. Let $\{\nu_t\}$ be any martingale difference sequence. Then there is a corresponding equilibrium for $\{i_t, r_t, \eta_t\}$ with $\eta_t = \nu_t$ and $(i_t, r_t)'$ evolving according to equation (4).

2.2 Observational Equivalence

When there is indeterminacy, the nominal interest rate has a state space representation given by the state equation (4) and the observation equation $i_t = (1, 0)(i_t, r_t)'$. This state space representation reduces to an AR(1) for the nominal rate when $\sigma^2 = 0$ so that $r_t = 0$ in all periods:

$$i_t = [\rho_i + \phi] i_{t-1} + \nu_t \tag{5}$$

with ν_t some martingale difference sequence. Any determinate equilibrium also has an AR(1) representation, which implies that parameter points in the determinacy region can be observationally equivalent to points in the indeterminacy region. In particular, when there are no fundamental shocks but there is indeterminacy, outcomes can appear identical to outcomes under determinacy with fundamental shocks.

Since the AR(1) reduced form provides a link between the determinacy and indeterminacy scenarios, we can characterize the full identified set using the coefficients from a linear regression of i_t on i_{t-1}:

$$i_t = \alpha i_{t-1} + u_t, \quad \mathbb{E}[u_t] = 0, \quad \mathbb{E}[u_t i_{t-1}] = 0, \quad \mathbb{E}[u_t^2] = s^2$$

The coefficients in this regression can be associated with a determinate equilibrium in equation (3) so that $|\rho_i + \phi| \geq 1$ and

$$\rho_r = \alpha, \quad \sigma^2 = \left(\frac{\rho_i + \phi - \rho_r}{\phi}\right)^2 s^2, \quad \rho_i = \text{free}, \quad \phi = \text{free} \tag{6}$$

Alternatively, α and s^2 can be associated with an indeterminate equilibrium as in equation (5) which implies that $|\rho_i + \phi| < 1$ and

$$\rho_r = \text{free}, \quad \sigma^2 = 0, \quad \rho_i = \text{free}, \quad \phi = \alpha - \rho_i \tag{7}$$

Together, these two conditions define the full identified set when the nominal interest rate follows an AR(1) process with reduced form parameters α and s^2.

For each point $(\phi, \rho_i, \rho_r, \sigma^2)$ in the indeterminacy region without fundamental shocks, there is at least one point in the determinacy region with fundamental shocks which generates an identical process for the nominal interest rate. This lack of global identification arises because indeterminacy allows the nominal rate to act as an additional state variable, while shutting down an exogenous shock removes a state variable.

This example illustrates how the possibility of indeterminacy can lead to identification failure. But, since the structural parameters ρ_r and σ_r^2 do not enter the determinacy condition of $|\rho_i + \phi| \geq 1$, this lack of identification is irrelevant for policy analysis. The central bank can trivially design a policy which ensures a unique equilibrium. However, this irrelevance does not hold generally.

3 An Indeterminacy Trap

This section examines a neoclassical economy where structural parameters enter the economy's determinacy condition in addition to policy parameters. It is now crucial to account for observationally equivalent structures between the region of the parameter space supporting determinacy and the region supporting indeterminacy when designing a policy rule. If the central bank fails to account for indeterminacy, they can end up in a policy trap where they believe that their policy supports a unique equilibrium, but it actually leads to indeterminacy.

I consider an economy with perfectly competitive product and labor markets, but where firms face working capital constraints[3]. The presence of working capital in production[4] generates a cost channel of monetary policy[5], which can make excessively activist policy de-stabilizing (see Surico (2008) and Christiano et al. (2010)). In contrast to the previous example, this economy's determinacy condition depends both on the policy rule parameters and the economy's structural parameters. In particular, the strength of money demand by firms and the household's elasticity of intertemporal substitution determine whether or not a given policy rule supports a unique equilibrium or supports multiple equilibria.

In the presence of a cost channel, whether or not a given policy rule supports indeterminacy depends on the structure of the economy. Even if the policy rule is known, each point in the indeterminacy region is always observationally equivalent to some point in the determinacy region. This equivalence can occur between parameterizations where (a) only supply shocks directly influence outcomes but policy allows for indeterminacy and non-fundamental "demand" shocks, and (b) both supply and demand shocks directly drive the business cycle and the policy rule implies a unique equilibrium.

In this example economy, the data observed by the central bank has a bi-variate VAR(1) reduced form representation. Like in the previous section, I use this reduced form to characterize the full identified set and show that every parameterization supporting indeterminacy is observationally equivalent to some parameterization supporting determinacy. When a central bank focuses on the determinacy region, they will implicitly ignore these alternative points. As a result, a central bank can end up in an indeterminacy trap where they believe their policy supports a unique equilibrium when in fact their policy leads to indeterminacy.

[3] Incorporating additional monetary non-neutralities, say due to sticky prices, does not change the key mechanism of the model, which operates through the presence of firm working capital. For clarity and transparency of the example, I focus on the case of flexible prices.

[4] Benhabib and Farmer (2000) get indeterminacy in a similar economy where money enters the production function.

[5] See Barth and Ramey (2002), Christiano et al. (2005), and Ravenna and Walsh (2006).

3.1 A Neoclassical Economy

The economy is populated by three agents: a representative household, a representative firm, and the central bank. The household consumes final output produced by the firm, and supplies labor to the firm. It also chooses an amount of equity to invest into the firm. This equity is used as working capital to facilitate production. As a result, firm marginal costs depend on the opportunity cost of nominal funds, which is the risk-free nominal interest rate. The central bank controls the nominal interest rate by manipulating the money supply through cash transfers to the household. Markets are perfectly competitive.

As in Basu (1995) and Christiano et al. (2010), the firm uses both intermediate goods and labor as inputs in production. This assumption generates a real rigidity whereby small changes in production costs generate large changes in output, which amplifies the cost channel. The firm's production is Cobb-Douglas and can be specialized to supply either the intermediate goods market or the final goods market: $Y_t + X_t^s = (X_t^d)^\alpha (A_t N_t^d)^{1-\alpha}$ where Y_t denotes final output, X_t^s intermediate input supply, X_t^d intermediate input demand, N_t^d labor demand, and A_t is a labor augmenting technology shock.

The firm's working capital constraint requires that a fraction ϑ of input costs must be backed by cash on hand[6]: $\vartheta(P_t X_t^d + W_t N_t^d) \leq K_t$ where P_t is the nominal price of output, and W_t is the nominal wage. Then for given working capital of K_t, the firm's profit maximization problem is

$$\begin{aligned} D(K_t; W_t, P_t) = \max_{Y_t, X_t^s, X_t^d, N_t^d} \; & P_t(Y_t + X_t^s) - P_t X_t^d - W_t N_t \\ \text{s.t.} \quad & Y_t + X_t^s \leq (X_t^d)^\alpha (A_t N_t^d)^{1-\alpha} \\ & \vartheta(P_t X_t + W_t N_t^d) \leq K_t \end{aligned}$$

where $D(K_t; W_t, P_t)$ is the firm's profit, paid to the household.

Define the marginal return on equity investment as $R_t \equiv \frac{\partial}{\partial K_t} D(K_t; W_t, P_t)$. By the envelope theorem, R_t must be the shadow value of slackening the working capital constraint. As a result, the firm's intermediate input demand and labor demand is given by

$$\begin{aligned} X_t^d &= \frac{\alpha}{1 + \vartheta R_t}(Y_t + X_t^s) \\ N_t &= \frac{1-\alpha}{1+\vartheta R_t} \frac{Y_t + X_t^s}{W_t / P_t} \end{aligned}$$

Substituting these demand functions in the production function and clearing the intermediate input market ($X_t^s = X_t^d$) implies that the real wage and final output must satisfy

$$\frac{W_t}{P_t} = A_t (1 + \vartheta R_t)^{-\frac{1}{1-\alpha}} \tag{8}$$

$$Y_t = \left(\frac{\alpha}{1-\alpha}\right)^\alpha (1 - \alpha + \vartheta R_t)(1 + \vartheta R_t)^{-\frac{1}{1-\alpha}} A_t N_t^d \tag{9}$$

[6] This assumption is consistent with the large amount of cash held by corporations in the United States (Mulligan (1997) and Bover and Watson (2005)).

Real wages and output are both decreasing in the opportunity cost of equity. Since R_t represents the marginal rent paid to owners of equity and this capital price will have to give the same return as a risk-free bond, we can see that increases in nominal interest rates will increase the marginal cost of production, reducing input demand and total production. The presence of working capital generates a cost-channel where hikes in the nominal interest rate by the central bank contract aggregate supply.

The household derives utility from consumption and hours worked of $u(C_t, N_t^s; D_t) = D_t[C_t^{1-\tau}/(1-\tau) - (N_t^s)^{1+\psi}/(1+\psi)]$. The variable D_t is an preference shock which shifts consumption and labor supply preferences over time. τ is the inverse elasticity ofinter temporal substitution and ψ is the inverse Frisch labor supply elasticity.

The household saves using one period nominal bonds (B_t) and by investing in firm equity (K_t). Denote the price of a one period nominal bond by Q_t. The return on equity, R_t, is taken as given by the household when choosing its supply of equity to the firm. The household's inter temporal optimization problem, given discount factor of $\beta \in (0,1)$, is then

$$\max_{\{C_t, N_t^s, B_t, K_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t D_t \left[\frac{C_t^{1-\tau}}{1-\tau} - \frac{(N_t^s)^{1+\psi}}{1+\psi} \right]$$
$$\text{s.t.} \quad P_t C_t + Q_t B_t + K_t \leq R_t K_t + W_t N_t^s + B_{t-1} + K_{t-1} + T_t$$

where T_t denotes lump-sum transfers due to monetary injections by the central bank.

Denote the gross inflation rate by $\Pi_t = P_t/P_{t-1}$ then the household's first order conditions are

$$\frac{W_t}{P_t} = C_t^{\tau} (N_t^s)^{\psi} \tag{10}$$
$$Q_t = \beta \mathbb{E}_t \frac{D_{t+1}}{D_t} \left(\frac{C_{t+1}}{C_t} \right)^{-\tau} \frac{1}{\Pi_{t+1}} \tag{11}$$
$$R_t = 1 - Q_t \tag{12}$$

The household equates the return across savings instruments so that the firm's cost of capital is equal to the discount on a one period nominal bond.

After (a) eliminating the real wage from equations (8) and (10), (b) eliminating labor supply and demand using market clearing ($N_t^s = N_t^d$) and the output condition (9), (c) eliminating R_t by using condition (12), and (d) eliminating consumption using market clearing ($C_t = Y_t$), I log-linearize the equilibrium system around a constant inflation steady state to end up with three equilibrium conditions:

$$y_t = s_t - \delta i_t \tag{13}$$
$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\tau}(i_t - \mathbb{E}_t \pi_{t+1} - r_t) \tag{14}$$
$$r_t = d_t - \mathbb{E}_t d_{t+1} \tag{15}$$

Here, lower case variables represent log deviations from steady state, i_t is the deviation of $\ln Q_t^{-1}$ from steady state, the variable r_t represents the effective rate of time preference due to preference shocks, and the variable $s_t = \frac{1+\psi}{\tau+\psi} a_t$ is an aggregate supply shock due to changes in technology. The elasticity $\delta \equiv \frac{1+\psi}{\tau+\psi} \frac{1}{1-\alpha} \frac{\vartheta\beta/\Pi}{1+\vartheta(1-\beta/\Pi)} - \frac{\psi}{\tau+\psi} \frac{\vartheta\beta/\Pi}{1-\alpha+\vartheta(1-\beta/\Pi)} > 0$ summarizes how firm working capital generates a cost channel.

The first equation is an aggregate supply curve, vertical in (π_t, y_t)-space[7]. Because of firm working capital (when $\delta > 0$), output is decreasing in the level of nominal interest rates. When the opportunity cost of holding cash is high, production costs increase, and output falls. The variable s_t represents the first best level of output, which is the level of output that would prevail in the absence of firm working capital constraints (as $\delta \to 0$).

The second and third equations together come from the household's consumption Euler equation. When current demand shocks are high relative to expected future demand shocks it is as if agents are more impatient, and so the effective rate of time preference is high. A high rate of time preference translates into higher consumption and therefore higher aggregate demand.

To close the model, I specify the central bank's interest rate policy. The only friction in the economy comes from the firm's money demand. Therefore, the optimal policy is to satiate money demand and achieve very low and stable nominal interest rates. To implement a near-optimal outcome, the central bank chooses a low interest rate target and seeks to stabilize the nominal interest rate around this target.

Since shocks to supply and demand are not directly observable, the central bank approximates the optimal allocation by focusing on a simple interest rate rule:

$$i_t = \rho_i i_{t-1} + \phi[\pi_t + \omega(y_t - y_{t-1})] \tag{16}$$

The interest rate rule reacts to inflation and output growth, while smoothing interest rate adjustments.

Finally, to economize on notation, I combine equations (13), (14), (15), and (16) into a single forward looking equation in the nominal interest rate. Given that supply and demand shocks follow exogenous AR(1) processes:

$$s_t = \rho_s s_{t-1} + \varepsilon_{s,t}, \quad \mathbb{E}_{t-1}\varepsilon_{s,t} = 0, \quad |\rho_s| < 1 \tag{17}$$

$$d_t = \rho_d d_{t-1} + \varepsilon_{d,t}, \quad \mathbb{E}_{t-1}\varepsilon_{d,t} = 0, \quad |\rho_d| < 1 \tag{18}$$

I eliminate expected inflation, the effective rate of time preference, and output to get:

$$\gamma_{i,0}\mathbb{E}_t i_{t+1} = \gamma_{i,1} i_t - (1-\rho_d)d_t + (1-\rho_s)(\tau-\omega)s_t \tag{19}$$

[7]With real effects from money surprises as in the New Classical model, the aggregate supply curve is no longer vertical, but the results are identical with a slight change in the equilibrium innovation structure. With sticky prices as in the New Keynesian model (Woodford (2003)), changes in expected inflation shift the non-vertical aggregate supply curve (New Keynesian Phillips Curve). As a result, equilibrium implies a VAR reduced form in $(i_t, \pi_t, y_t)'$ instead of simply $(i_t, y_t)'$, which complicates the analysis without changing the key results.

where

$$\gamma_{i,0} \equiv \phi^{-1} - \delta(\tau - \omega)$$
$$\gamma_{i,1} \equiv 1 + \rho_i \phi^{-1} - \delta(\tau - \omega)$$

Equations (17), (18), and (19) define a linear rational expectations system in $(i_t, s_t, d_t)'$. Given the equilibrium nominal interest rate and the exogenous supply and demand shocks, the evolution of y_t and π_t can be recovered using equations (13) and (16).

3.2 Determinacy and Indeterminacy

The Blanchard and Kahn (1980) conditions, applied to equations (17), (18), and (19), imply that equilibrium is determinate if and only if the coefficient on i_t has magnitude greater than the coefficient on $\mathbb{E}_t i_{t+1}$:

$$|\gamma_{i,1}| \geq |\gamma_{i,0}| \quad \Leftrightarrow \quad |1 + \phi^{-1}\rho_i - \delta(\tau - \omega)| \geq |\phi^{-1} - \delta(\tau - \omega)| \tag{20}$$

This condition depends both on the policy parameters of (ρ_i, ϕ, ω) and the structural parameters (δ, τ). Note that as $\delta \to 0$, the determinacy condition reduces to the condition in the Fisherian model ($|\rho_i + \phi| \geq 1$). For small δ, the central bank must react sufficiently strongly to deviations of inflation and output growth from target. If the central bank believes that $\delta \approx 0$, they will always conclude that this type of modified Taylor principle ensures a unique equilibrium.

In contrast, when $\delta > 0$, almost any policy rule can be associated with indeterminacy, as summarized in the following proposition[8]:

Proposition 1 *For given policy parameters of (ρ_i, ϕ, ω), equilibrium is determinate if and only if $\phi \neq 0$ and one of:*

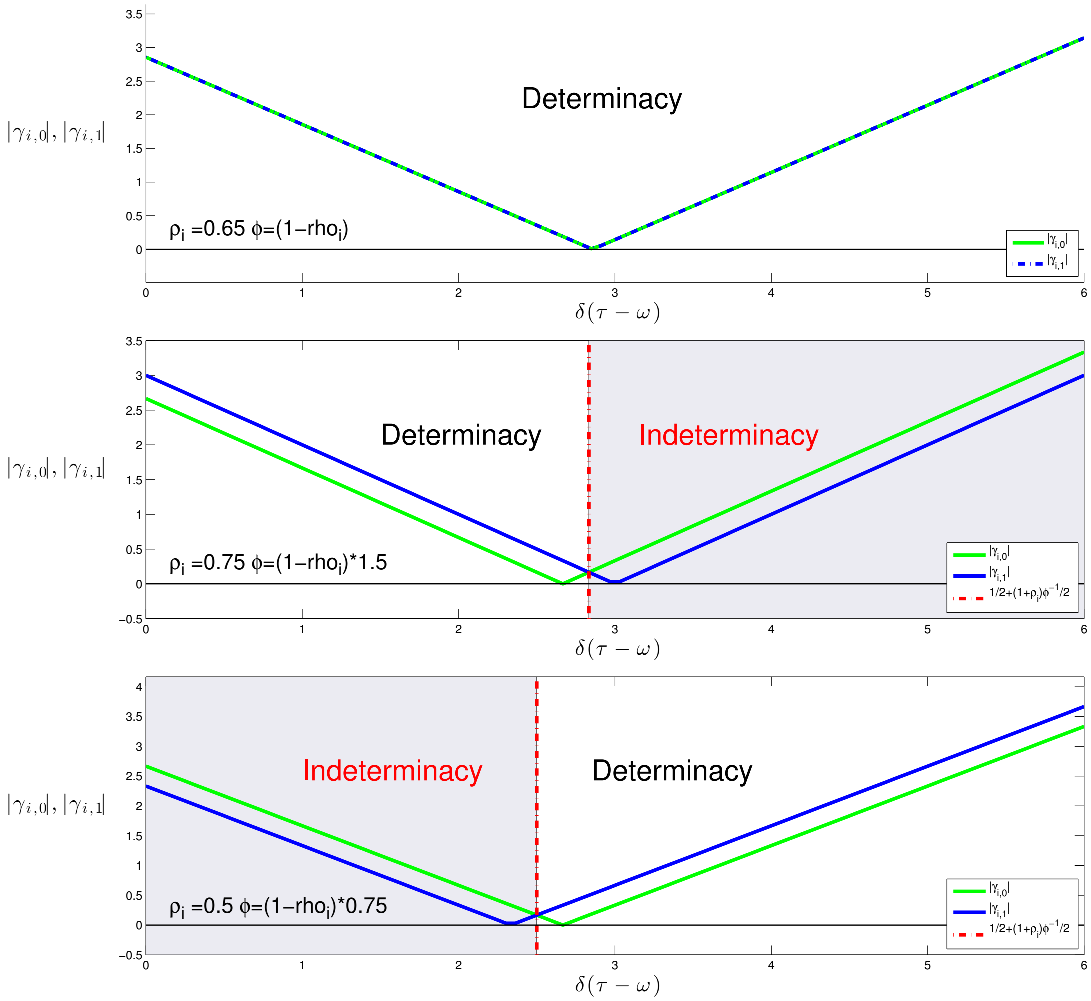
1. $\phi^{-1} = 1 + \phi^{-1}\rho_i$.
2. $1 + \rho_i\phi^{-1} > \phi^{-1}$ *and* $\delta(\tau - \omega) \leq \frac{1}{2} + \frac{1}{2}(1 + \rho_i)\phi^{-1}$.
3. $1 + \rho_i\phi^{-1} < \phi^{-1}$ *and* $\delta(\tau - \omega) \geq \frac{1}{2} + \frac{1}{2}(1 + \rho_i)\phi^{-1}$.

Proof. The result follows from condition (20) and working through cases for the signs of $\gamma_{i,0}$ and $\gamma_{i,1}$. See appendix A. ■

The proposition shows that, for almost any policy rule, there is always a cutoff in the parameter combination $\delta(\tau - \omega)$ which defines the boundary between the determinacy and indeterminacy regions. When $1 + \rho_i\phi^{-1} > \phi^{-1}$, $\delta(\tau - \omega)$ must be sufficiently small for determinacy to hold, and while $1 + \rho_i\phi^{-1} < \phi^{-1}$ it must be sufficiently large. Figure 1 depicts the three cases by plotting $\delta(\tau - \omega)$ against $|\gamma_{i,0}|$ and $|\gamma_{i,1}|$ for different parameter pairs (ρ_i, ϕ).

[8] See Surico (2008) for the first derivation of an analogous result.

Figure 1: **Determinacy Condition** (20)**: Cases (1-3) in Proposition 1**



Note that the boundary case with $1 + \rho_i \phi^{-1} = \phi^{-1} \Leftrightarrow \phi + \rho_i = 1$ always supports a unique equilibrium. This knife-edge situation theoretically allows a central bank to rule out indeterminacy without any knowledge of $\delta(\tau - \omega)$. However, any small deviation (or perceived deviation by economic agents) from this stringent rule will immediately split the parameter space into determinacy and indeterminacy regions. Due to this sensitivity, the central bank may not want to rely on this class of policies.

Unlike the Fisherian model in section 2, this economy's determinacy condition depends on the structural coefficients δ and τ (respectively, the elasticity of aggregate supply and the elasticity of aggregate demand to the nominal interest rate). As a result, it is essentially impossible to rule out indeterminacy without some information about the economy's structure. Almost any policy rule can support multiple equilibria. It is now essential that the central bank uses historical data to infer plausible values for these structural parameters when designing a policy rule.

3.3 Data Generating Process

In order to solve for the equilibrium interest rate and output, I convert the rational expectations system given in equations (17), (18), and (19) to the canonical structure analyzed in Sims (2002) and Lubik and Schorfheide (2003):

$$\begin{bmatrix} \gamma_{i,0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_t \\ s_t \\ d_t \end{bmatrix} = \begin{bmatrix} \gamma_{i,1} & (1-\rho_s)(\tau-\omega) & -(1-\rho_d) \\ 0 & \rho_s & 0 \\ 0 & 0 & \rho_d \end{bmatrix} \begin{bmatrix} i_{t-1} \\ s_{t-1} \\ d_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \eta_t \tag{21}$$

where $\eta_t \equiv i_t - \mathbb{E}_{t-1} i_t$.

When the system is inherently stable with $|\gamma_{i,1}| < |\gamma_{i,0}|$, η_t can be any martingale difference sequence. Equilibrium is indeterminate and we can specify the rational expectations forecast error as

$$\eta_t = \lambda_s \varepsilon_{s,t} + \lambda_d \varepsilon_{d,t} + \nu_t, \quad \mathbb{E}\nu_t^2 = \sigma_\nu^2$$

where $\{\nu_t\}$ is some martingale difference sequence which is uncorrelated with $\varepsilon_{s,t}$ and $\varepsilon_{d,t}$.

The determinacy region corresponds to when this system is unstable ($|\gamma_{i,1}| \geq |\gamma_{i,0}|$). In this case, the equilibrium nominal interest rate only depends on the aggregate supply and demand shocks, and the method of undetermined coefficients implies that:

$$i_t = \kappa_s s_t + \kappa_d d_t \tag{22}$$

where

$$\kappa_s \equiv \frac{(1-\rho_s)(\tau-\omega)}{\gamma_{i,0}\rho_s - \gamma_{i,1}} \quad \text{and} \quad \kappa_d \equiv -\frac{1-\rho_d}{\gamma_{i,0}\rho_d - \gamma_{i,1}}$$

Unless the nominal interest follows exactly this equation, system (21) would have to either explode or contain a unit root. This implies that the forecast error and the initial interest rate must be precisely

$$\eta_t = \kappa_s \varepsilon_{s,t} + \kappa_d \varepsilon_{d,t}$$
$$i_0 = \kappa_s s_0 + \kappa_d d_0$$

to ensure that the system remains stable.

From these results and using the aggregate supply condition (13), we can fully characterize the observable process for the nominal interest rate and output[9]:

Lemma 1 *The observable process for the nominal interest rate and output has the following state space*

[9] Since the realization of the inflation is determined via the policy rule as a known function of current and past interest rates and output, it provides no additional statistical information. Without loss of generality, I focus on inference based solely off of observing (i_t, y_t).

representation:

$$\begin{bmatrix} i_t \\ s_t \\ d_t \end{bmatrix} = \begin{bmatrix} \gamma_{i,1}/\gamma_{i,0} & (1-\rho_s)(\tau-\omega)/\gamma_{i,0} & -(1-\rho_d)/\gamma_{i,0} \\ 0 & \rho_s & 0 \\ 0 & 0 & \rho_d \end{bmatrix} \begin{bmatrix} i_{t-1} \\ s_{t-1} \\ d_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \eta_t \tag{23}$$

$$\begin{bmatrix} i_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\delta & 1 & 0 \end{bmatrix} \begin{bmatrix} i_t \\ s_t \\ d_t \end{bmatrix} \tag{24}$$

where

$$\eta_t = \begin{cases} \kappa_s \varepsilon_{s,t} + \kappa_d \varepsilon_{d,t} & \textit{if } |\gamma_{i,1}| \geq |\gamma_{i,0}| \\ \lambda_s \varepsilon^s_t + \lambda_d \varepsilon^d_t + \nu_t & \textit{if } |\gamma_{i,1}| < |\gamma_{i,0}| \end{cases} \quad \textit{and} \quad i_0 = \begin{cases} \kappa_s s_0 + \kappa_d d_0 & \textit{if } |\gamma_{i,1}| \geq |\gamma_{i,0}| \\ \textit{unrestricted} & \textit{if } |\gamma_{i,1}| < |\gamma_{i,0}| \end{cases}$$

This state space representation highlights that the dynamics of the observable data may depend on up to three state variables – the nominal interest rate, the supply shock, and the demand shock. When there is determinacy, the number of effective states becomes two since the rational expectations forecast error moves to ensure that the nominal interest rate only depends on supply and demand shocks.

3.4 Observational Equivalence and the Indeterminacy Trap

Under determinacy, the nominal rate and output exactly reveal the latent supply and demand shocks. The state space representation in lemma 1 reduces to a VAR(1):

$$\begin{bmatrix} i_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\delta & 1 \end{bmatrix} \begin{bmatrix} \kappa_s & \kappa_d \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \rho_s & 0 \\ 0 & \rho_d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/\kappa_d & -\kappa_s/\kappa_d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \begin{bmatrix} i_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -\delta & 1 \end{bmatrix} \begin{bmatrix} \kappa_s & \kappa_d \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \end{bmatrix} \tag{25}$$

Outcomes under indeterminacy can be observationally equivalent only if they also imply a VAR(1) for the nominal rate and output.

If there is indeterminacy, the data's state space representation collapses to a VAR(1) only if the number of underlying states can be reduced from three to two. For instance, this occurs if either the supply shock or the demand shock drops out from the system.

I focus on when the demand shock is absent[10]. Set $\sigma^2_d \equiv \mathbb{E}\varepsilon^2_{d,t} = 0$ so that $d_t = 0$ in all periods, and assume that there is indeterminacy so that $|\gamma_{i,1}| < |\gamma_{i,0}|$. Then the observed data has the following VAR(1) representation:

$$\begin{bmatrix} i_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\delta & 1 \end{bmatrix} \begin{bmatrix} \gamma_{i,1}/\gamma_{i,0} & (1-\rho_s)(\tau-\omega)/\gamma_{i,0} \\ 0 & \rho_s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \begin{bmatrix} i_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -\delta & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu_t \\ \varepsilon_{s,t} \end{bmatrix} \tag{26}$$

[10]Note that the aggregate supply curve (13) trivially reveals the parameter δ if the supply shock is always equal to zero.

This process is observationally equivalent to the VAR(1) in equation (25) if and only if it has the same coefficient matrix multiplying $(i_{t-1}, y_{t-1})'$ and has the same residual variance-covariance structure.

Matching coefficients and the residual variance-covariance matrices reveals that every point in the indeterminacy region without demand shocks is observationally equivalent to a point in the determinacy region with demand shocks. This result is summarized in proposition 2.

Let $\theta^S \equiv (\delta, \tau, \rho_s, \rho_d, \sigma_s^2, \sigma_d^2, \sigma_{s,d})$ denote the structural parameters, $\theta^P \equiv (\rho_i, \phi, \omega)$ the policy parameters, and $\lambda \equiv (\lambda_s, \lambda_d, \sigma_\nu^2)$ the equilibrium selection parameters. Then:

Proposition 2 *For each structure and policy pair (θ_0^S, θ_0^P) in the indeterminacy region where only supply shocks drive the business cycle and any equilibrium selection parameter λ_0, there exists a pair (θ_1^S, θ_1^P) in the determinacy region (where both supply and demand shocks influence the business cycle) which generates the exact same stochastic process for nominal interest rates and output. Moreover, this holds even if policy is identical between the two scenarios: $\theta_0^P = \theta_1^P$.*

Proof. The result follows from comparing the VAR(1) coefficients in equation (25) with the VAR(1) coefficients in equation (26). Alternatively, the full characterization of the identified set in appendix B provides an equivalent proof. ■

This proposition says that even with the knowledge of the current policy rule, a central bank can believe equilibrium is determinate when it is actually indeterminate. Each point in the indeterminacy region without demand shocks generates a VAR(1) which is consistent with a point in the determinacy region with demand shocks. As a result, whenever multiple equilibria exist, a central bank which believes that their policy rule generates determinacy will be able to rationalize their belief, even if equilibrium is actually indeterminate.

As a result, the central bank can end up in an indeterminacy trap. Suppose the central bank implements a policy of θ^P which interacts with the true structure of θ_0^S (with $\sigma_d^2 = 0$) to generate indeterminacy. The resulting equilibrium is given by the VAR(1) in equation (26). Due to proposition 2, there always exists an alternative parameterization, θ_1^S, which generates the exact same reduced form process but is associated with a determinate equilibrium, as in equation (25). Therefore, if the central bank maintains the assumption that outcomes are determinate during its identification analysis, they will incorrectly conclude that the economy has the structure of θ_1^S, and may continue implementing the status quo policy of θ^P. They are trapped supporting indeterminacy while believing that their policy supports a unique equilibrium.

Note that this scenario can occur for essentially every possible historical policy rule. By proposition 1, almost every policy rule allows for indeterminacy for some structure of the economy. A central bank cannot ex-ante rule out this indeterminacy trap scenario.

4 Escaping the Indeterminacy Trap

To avoid indeterminacy, the central bank must recognize the observational equivalence between points in the determinacy and indeterminacy regions, and then design a policy rule which ensures determinacy for all points in the identified set. I call a rule which can ensure determinacy across the full identified set a *robust policy rule*.

The following lemma characterizes the whole identified set based on the coefficients from a reduced form VAR(1):

$$\begin{bmatrix} i_t \\ y_t \end{bmatrix} = A \begin{bmatrix} i_{t-1} \\ y_{t-1} \end{bmatrix} + u_t, \quad \mathbb{E}u_t = 0, \quad \mathbb{E}\begin{bmatrix} i_{t-1} \\ y_{t-1} \end{bmatrix} u_t' = 0, \quad \mathbb{E}u_t u_t' = \Omega\Omega'$$

where

$$A = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix}, \quad \Omega = \begin{bmatrix} \omega_{1,1} & 0 \\ \omega_{1,2} & \omega_{2,2} \end{bmatrix}$$

Lemma 2 *For given policy parameters $\theta^P = (\rho_i, \phi, \omega)$ and reduced form parameters A and Ω, with A a stable matrix,*

1. *If $(\alpha_{1,1} - \alpha_{2,2})^2 + 4\alpha_{1,2}\alpha_{2,1} < 0$ then the structural model is miss-specified and the identified set is empty.*

2. *If $(\alpha_{1,1} - \alpha_{2,2})^2 + 4\alpha_{1,2}\alpha_{2,1} \geq 0$ and $\alpha_{1,1} \neq \phi + \rho_i - \frac{\alpha_{1,2}\alpha_{2,1}}{1-\alpha_{2,2}}$, then indeterminacy is impossible and the identified set contains up to two points. For each $\delta > 0$ satisfying $\alpha_{1,2}\delta^2 + (\alpha_{2,2} - \alpha_{1,1})\delta - \alpha_{2,1} = 0$ we have a point $\theta^S = (\delta, \tau, \rho_s, \rho_d, \sigma_s^2, \sigma_d^2, \sigma_{s,d})$ in the determinacy region with*

$$\begin{aligned} \rho_s &= \alpha_{2,2} + \delta\alpha_{1,2} \\ \rho_d &= \alpha_{1,1} - \delta\alpha_{1,2} \\ \tau &= \omega + \frac{1}{\phi\delta(1-\rho_s)} \frac{\delta\alpha_{1,2}(\rho_s - \rho_i - \phi)}{\rho_s - \rho_d - \delta\alpha_{1,2}} \end{aligned}$$

   and

$$\begin{bmatrix} \sigma_d^2 & \sigma_{s,d} \\ \sigma_{s,d} & \sigma_s^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/\kappa_d & -\kappa_s/\kappa_d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \begin{bmatrix} \omega_{1,1} & 0 \\ \omega_{1,2} & \omega_{2,2} \end{bmatrix} \begin{bmatrix} \omega_{1,1} & \omega_{1,2} \\ 0 & \omega_{2,2} \end{bmatrix} \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/\kappa_d \\ 1 & -\kappa_s/\kappa_d \end{bmatrix}$$

3. *If $(\alpha_{1,1} - \alpha_{2,2})^2 + 4\alpha_{1,2}\alpha_{2,1} \geq 0$ and $\alpha_{1,1} = \phi + \rho_i - \frac{\alpha_{1,2}\alpha_{2,1}}{1-\alpha_{2,2}}$, then indeterminacy is possible. For each root $\delta > 0$ from case (2), there is the same point in the determinacy region and additionally a point*

$\theta^S = (\delta, \tau, \rho_s, \rho_d, \sigma_s^2, \sigma_d^2, \sigma_{s,d})$ in the indeterminacy region such that

$$\begin{aligned}
\rho_s &= \alpha_{2,2} + \delta\alpha_{1,2} \\
\tau &= \omega + \frac{\phi^{-1}\alpha_{1,2}}{1 - \rho_s + \delta\alpha_{1,2}} \\
\rho_d &= \mathit{free} \\
\sigma_s^2 &= (\delta\omega_{1,1} + \omega_{1,2})^2 + \omega_{2,2}^2 \\
\sigma_d^2 &= 0 \\
\sigma_{s,d} &= 0
\end{aligned}$$

with equilibrium selection parameters $\lambda = (\lambda_s, \lambda_d, \sigma_\nu^2)$ satisfying

$$\begin{aligned}
\lambda_s &= \frac{(\delta\omega_{1,1} + \omega_{1,2})\omega_{1,1}}{\sigma_s^2} \\
\lambda_d &= \mathit{free} \\
\sigma_\nu^2 &= \omega_{1,1}^2 - \lambda_s^2\sigma_s^2
\end{aligned}$$

Proof. See appendix B. ■

This lemma shows that when indeterminacy is plausible, there are up to four possible points in the identified set. A robust policy must ensure determinacy for all four of these points.

Lemma 2 also implies that within the determinacy region there is a lack of global identification. Even if past data had to come from a unique equilibrium, there can be two points in the identified set (see Morris (2013)). Even when historical policies are known to be robust, the central bank must still account for multiple points in the identified set when assessing any modification of policy.

A robust policy ensures that the determinacy condition (20) holds for every point in the identified set. Since only the structural parameters δ and τ are relevant for this condition, we can use the (up to) four identified possibilities for this parameter pair and proposition 1 to characterize the set of all robust policies. This final result is summarized as a proposition:

Proposition 3 *Given historical policy of $\tilde{\theta}^P = (\tilde{\rho}_i, \tilde{\phi}, \tilde{\omega})$ and reduced form coefficients A and Ω (A stable), suppose that past data may have been generated under indeterminacy, so that $(\alpha_{1,1} - \alpha_{2,2})^2 + 4\alpha_{1,2}\alpha_{2,1} \geq 0$ and $\alpha_{1,1} = \tilde{\phi} + \tilde{\rho}_i - \frac{\alpha_{1,2}\alpha_{2,1}}{1-\alpha_{2,2}}$. Then the policy parameters (ρ_i, ϕ, ω) define a robust policy, ruling out indeterminacy, if and only if $\phi \neq 0$ and one of:*

1. $1 + \rho_i\phi^{-1} = \phi^{-1}$

2. $1 + \rho_i\phi^{-1} > \phi^{-1}$ *and for each $\delta > 0$ such that $\alpha_{1,2}\delta^2 + (\alpha_{2,2} - \alpha_{1,1})\delta - \alpha_{2,1} = 0$ we have*

$$\delta(\tilde{\omega} - \omega) + \max\left\{ \frac{1}{\tilde{\phi}(1 - \alpha_{2,2} - \delta\alpha_{1,2})} \frac{\alpha_{2,2} + \delta\alpha_{1,2} - \tilde{\rho}_i - \tilde{\phi}}{\alpha_{2,2} - \alpha_{1,1} + \delta\alpha_{1,2}}, \frac{1}{\tilde{\phi}} \frac{\delta\alpha_{1,2}}{1 - \alpha_{2,2}} \right\} \leq \frac{1}{2} + \frac{1}{2}(1 + \rho_i)\phi^{-1}$$

3. $1+\rho_i\phi^{-1} < \phi^{-1}$ *and for each* $\delta > 0$ *such that* $\alpha_{1,2}\delta^2 + (\alpha_{2,2} - \alpha_{1,1})\delta - \alpha_{2,1} = 0$ *we have*

$$\delta(\tilde{\omega} - \omega) + \min\left\{\frac{1}{\tilde{\phi}(1-\alpha_{2,2}-\delta\alpha_{1,2})}\frac{\alpha_{2,2}+\delta\alpha_{1,2}-\tilde{\rho}_i-\tilde{\phi}}{\alpha_{2,2}-\alpha_{1,1}+\delta\alpha_{1,2}}, \frac{1}{\tilde{\phi}}\frac{\delta\alpha_{1,2}}{1-\alpha_{2,2}}\right\} \geq \frac{1}{2} + \frac{1}{2}(1+\rho_i)\phi^{-1}$$

Proof. The result follows from combining lemma 2 with proposition 1. ■

This characterization reveals that it is always possible, given the past history of the economy, to design a new policy which rules out indeterminacy. For instance, choose θ^P so that $1+\rho_i\phi^{-1} > \phi^{-1}$ with $\phi > 0$ and choose any value for ω. Then the past data provides up to two values for the term

$$\delta(\omega - \tilde{\omega}) + \max\left\{\frac{1}{\tilde{\phi}(1-\alpha_{2,2}-\delta\alpha_{1,2})}\frac{\alpha_{2,2}+\delta\alpha_{1,2}-\tilde{\rho}_i-\tilde{\phi}}{\alpha_{2,2}-\alpha_{1,1}+\delta\alpha_{1,2}}, \frac{1}{\tilde{\phi}}\frac{\delta\alpha_{1,2}}{1-\alpha_{2,2}}\right\}$$

corresponding to the two potential solutions for δ. Then, by making ϕ smaller and ρ_i larger, the term $\frac{1}{2}+\frac{1}{2}(1+\rho_i)\phi^{-1}$ can always be made larger than the largest of these two values. The inequality characterizing a robust policy can always be satisfied for a choice of sufficiently high interest rate smoothing and sufficiently weak reactions of policy to inflation and output growth.

Corollary 1 *For any known historical policy* $\tilde{\theta}^P$*, unknown true structure* θ^S*, and chosen output growth weight* ω*, there exists a sufficiently large value for* ρ_i *and small value for* $\phi > 0$ *satisfying the generalized taylor principle,* $\phi + \rho_i > 1$*, such that* $\theta^P = (\rho_i, \phi, \omega)$ *is a robust policy. Knowledge of the full set of identified values for* δ *and* τ *is necessary to choose appropriate parameter values.*

The central bank can always engineer an escape from the indeterminacy trap, provided they account for the full identified set. Crucially, they must account for observational equivalence between points in both the determinacy and indeterminacy regions of the parameter space.

For instance, if $\omega = 0$ then the rule

$$i_t = i_{t-1} + \phi\pi_t \Leftrightarrow i_t = i_{t-1} + \phi(p_t - p_{t-1})$$

will imply determinacy if ϕ is not too large. This type of rule makes the price level p_t stationary – a policy of price level targeting. However, a switch to price level targeting will not be robust if the central bank reacts too strongly to changes in the economy. A sufficient upper bound on the reaction coefficient can be determined from the data using proposition 3.

5 Conclusion

This paper illustrates the potential consequences of ignoring indeterminacy when using dynamic stochastic general equilibrium models to interpret historical data and inform policy. Through the example of a neoclassical model with a cost channel, I show that a central bank can end up in a trap if they implicitly assume that a unique equilibrium generated past outcomes. Because the assumption of determinacy can always be rationalized by the data, they can end up believing that their policy implements a unique equilibrium, but in reality it leaves outcomes indeterminate. In this scenario, the central bank foregoes welfare gains from ruling out non-fundamental sources of business cycle fluctuations.

In order to escape this indeterminacy trap, the central bank must perform a complete and global analysis of the set of economic structures which are consistent with historical data. In the neoclassical example, up to four observationally equivalent parameterizations exist and they straddle both the determinacy and indeterminacy regions. This result enables a complete characterization of all robust policies – those which always support a unique equilibrium and rule out non-fundamental business cycle fluctuations. In this economy, a policy rule with a sufficient degree of interest rate smoothing and a sufficiently small but positive reaction to inflation and output growth provides a class of robust policies. As a particular example, price level targeting with not too strong policy reaction will always lead to determinacy. However, calculating a sufficient upper bound on the policy reaction coefficient requires knowledge of all economic structures in the identified set.

This paper points to the need for global identification conditions and partial identification techniques for dynamic stochastic general equilibrium models. Policy makers need to be able to account for the full range of economic scenarios which are consistent with historical data. Especially when using these models to inform policy, a central bank must account for statistical indistinguishability between non-fundamental and fundamental sources of business cycles.

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A Proof of Proposition 1

First, if $\phi = 0$ then π_t appears nowhere in the model apart from through its step-ahead expectation. In this case there is always indeterminacy in inflation.

Assume $\phi \neq 0$. By the Blanchard and Kahn (1980) conditions, a policy $\theta^P = (\rho_i, \phi, \omega)$ implies determinacy if and only if condition (20) is satisfied: $|1 + \phi^{-1}\rho_i - \delta(\tau - \omega)| \geq |\phi^{-1} - \delta(\tau - \omega)|$. First, note that if $\phi^{-1} = 1 + \phi^{-1}\rho_i$, then this condition is always satisfied.

Now suppose that $\phi^{-1} \neq 1 + \phi^{-1}\rho_i$ and consider four cases:

1. Suppose that $1 + \phi^{-1}\rho_i \geq \delta(\tau - \omega)$ and $\phi^{-1} \geq \delta(\tau - \omega)$. Then equilibrium is determinate if and only if $1 + \phi^{-1}\rho_i - \delta(\tau - \omega) \geq \phi^{-1} - \delta(\tau - \omega) \Leftrightarrow 1 + \phi^{-1}\rho_i \geq \phi^{-1}$. But since this condition is always satisfied in this case, we can conclude that equilibrium is always determinate.

2. Suppose that $1 + \phi^{-1}\rho_i \geq \delta(\tau - \omega)$ and $\phi^{-1} < \delta(\tau - \omega)$. Then equilibrium is determinate if and only if $1 + \phi^{-1}\rho_i - \delta(\tau - \omega) \geq \delta(\tau - \omega) - \phi^{-1} \Leftrightarrow \frac{1}{2} + \frac{1}{2}(1 + \rho_i)\phi^{-1} \geq \delta(\tau - \omega)$.

3. Suppose that $1 + \phi^{-1}\rho_i < \delta(\tau - \omega)$ and $\phi^{-1} < \delta(\tau - \omega)$. Then equilibrium is determinate if and only if $1 + \phi^{-1}\rho_i - \delta(\tau - \omega) \leq \phi^{-1} - \delta(\tau - \omega) \Leftrightarrow 1 + \phi^{-1}\rho_i \leq \phi^{-1}$. This condition is always satisfied so equilibrium is determinate.

4. Suppose that $1 + \phi^{-1}\rho_i < \delta(\tau - \omega)$ and $\phi^{-1} \geq \delta(\tau - \omega)$. Then equilibrium is determinate if and only if $\delta(\tau - \omega) - 1 - \phi^{-1}\rho_i \geq \phi^{-1} - \delta(\tau - \omega) \Leftrightarrow \frac{1}{2} + \frac{1}{2}(1 + \rho_i)\phi^{-1} \leq \delta(\tau - \omega)$.

Combining these conditions gives the result.

B Identified Set

The approach in this section uses techniques introduced by Morris (2013). At the parameter values considered, whether or not there is determinacy or indeterminacy, the nominal interest rate and output follows a bivariate VAR(1). Denote a regression of $(i_t, y_t)'$ on $(i_{t-1}, y_{t-1})'$ by

$$\begin{bmatrix} i_t \\ y_t \end{bmatrix} = A \begin{bmatrix} i_{t-1} \\ y_{t-1} \end{bmatrix} + u_t, \quad \mathbb{E}u_t = 0, \quad \mathbb{E}\begin{bmatrix} i_{t-1} \\ y_{t-1} \end{bmatrix} u_t' = 0, \quad \mathbb{E}u_t u_t' = \Omega\Omega' \tag{27}$$

where

$$A = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix}, \quad \Omega = \begin{bmatrix} \omega_{1,1} & 0 \\ \omega_{1,2} & \omega_{2,2} \end{bmatrix}$$

We can back out the identified set from the reduced form coefficients of this VAR(1). Recall, the policy parameter vector $\theta^P = (\rho_i, \phi, \omega)$ is chosen by the central bank and therefore is assumed to be known for purposes of structural identification. When this vector is unknown, the following identified set becomes a manifold with three additional free dimensions.

B.1 Determinacy with Demand Shocks: $|\gamma_{i,1}| \geq |\gamma_{i,0}|$ and $\sigma_d > 0$

Since the nominal interest rate and output evolve according to the VAR(1) in equation (25), matching coefficients on $(i_{t-1}, y_{t-1})'$ gives four conditions:

$$\alpha_{1,1} = \rho_d + \delta\kappa_s(\rho_s - \rho_d)$$
$$\alpha_{1,2} = \kappa_s(\rho_s - \rho_d)$$
$$\alpha_{2,1} = \delta(\rho_s - \rho_d) - \delta^2\kappa_s(\rho_s - \rho_d)$$
$$\alpha_{2,2} = \rho_s - \delta\kappa_s(\rho_s - \rho_d)$$

Substitute the second equation into the other equations to eliminate $\kappa_s(\rho_s - \rho_d)$ terms. Then difference the first and last equations to get

$$\rho_s - \rho_d = \alpha_{2,2} - \alpha_{1,1} + 2\delta\alpha_{1,2}$$

Use this result in the third equation to get a quadratic equation in δ:

$$0 = \alpha_{1,2}\delta^2 + (\alpha_{2,2} - \alpha_{1,1})\delta - \alpha_{2,1} \implies \delta = \frac{\alpha_{1,1} - \alpha_{2,2} \pm \sqrt{(\alpha_{1,1} - \alpha_{2,2})^2 + 4\alpha_{1,2}\alpha_{2,1}}}{2\alpha_{1,2}}$$

Note that the model requires that $\delta > 0$, so for correct specification it must be the case that the reduced form coefficients imply that there is at least one strictly positive solution for δ.

With this result, we can go back to the first and last equations to get

$$\rho_s = \alpha_{2,2} + \delta\alpha_{1,2}$$
$$\rho_d = \alpha_{1,1} - \delta\alpha_{1,2}$$

and then the second equation implies that

$$\kappa_s = \frac{\alpha_{1,2}}{\rho_s - \rho_d}$$

Therefore, there are at most two values for $(\delta, \rho_s, \rho_d, \kappa_s)$ which are consistent with given reduced form coefficients of $(\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2})$.

To determine $\kappa_d, \gamma_{i,0}, \gamma_{i,1}$, and τ,, use:

$$\kappa_s = (1 - \rho_s)(\tau - \omega)/(\gamma_{i,0}\rho_s - \gamma_{i,1})$$
$$\kappa_d = -(1 - \rho_d)/(\gamma_{i,0}\rho_d - \gamma_{i,1})$$
$$\gamma_{i,0} = \phi^{-1} - \delta(\tau - \omega)$$
$$\gamma_{i,1} = 1 + \phi^{-1}\rho_i - \delta(\tau - \omega)$$

Combining the first and last two equations gives an expression for τ given $\theta^P = (\rho_i, \phi, \omega)$ and $(\delta, \rho_s, \kappa_s)$:

$$\tau = \omega + \frac{\kappa_s(\rho_s \phi^{-1} - 1 - \phi^{-1}\rho_i)}{(1-\kappa_s\delta)(1-\rho_s)}$$

Now, the variance-covariance structure of fundamental innovations can be recovered from the residual covariance structure as

$$\begin{bmatrix} \sigma_d^2 & \sigma_{s,d} \\ \sigma_{s,d} & \sigma_s^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/\kappa_d & -\kappa_s/\kappa_d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \begin{bmatrix} \omega_{1,1} & 0 \\ \omega_{1,2} & \omega_{2,2} \end{bmatrix} \begin{bmatrix} \omega_{1,1} & \omega_{1,2} \\ 0 & \omega_{2,2} \end{bmatrix} \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/\kappa_d \\ 1 & -\kappa_s/\kappa_d \end{bmatrix}$$

In summary, the part of the identified set for the full deep parameter vector of $\theta^S = (\delta, \tau, \rho_s, \rho_d, \sigma_s^2, \sigma_d^2, \sigma_{s,d})$ consists of up to two points corresponding to the two possible solutions for δ.

B.2 Indeterminacy and No Demand Shocks: $|\gamma_{i,1}| < |\gamma_{i,0}|$ and $\sigma_d = 0$

When equilibrium is indeterminate and there are no demand shocks which directly influence the economy, the nominal interest rate and output evolve according to the VAR(1) in equation (26). Matching coefficients on $(i_{t-1}, y_{t-1})'$ with the reduced form VAR(1) in equation (27) gives four conditions:

$$\begin{aligned}
\alpha_{1,1} &= \gamma_{i,1}/\gamma_{i,0} + \delta(1-\rho_s)(\tau-\omega)/\gamma_{i,0} \\
\alpha_{1,2} &= (1-\rho_s)(\tau-\omega)/\gamma_{i,0} \\
\alpha_{2,1} &= \delta(\rho_s - \gamma_{i,1}/\gamma_{i,0}) - \delta^2(1-\rho_s)(\tau-\omega)/\gamma_{i,0} \\
\alpha_{2,2} &= \rho_s - \delta(1-\rho_s)(\tau-\omega)/\gamma_{i,0}
\end{aligned}$$

Substitute the second equation into the other equations to eliminate $(1-\rho_s)(\tau-\omega)/\gamma_{i,0}$ terms and difference the first and last equations to get

$$\rho_s - \gamma_{i,1}/\gamma_{i,0} = \alpha_{2,2} - \alpha_{1,1} + 2\delta\alpha_{1,2}$$

Use this result in the third equation to get a quadratic equation in δ:

$$0 = \alpha_{1,2}\delta^2 + (\alpha_{2,2} - \alpha_{1,1})\delta - \alpha_{2,1} \implies \delta = \frac{\alpha_{1,1} - \alpha_{2,2} \pm \sqrt{(\alpha_{1,1}-\alpha_{2,2})^2 + 4\alpha_{1,2}\alpha_{2,1}}}{2\alpha_{1,2}}$$

The same equation arises in the case of determinacy. The identified values for δ are identical between the determinacy region and the indeterminacy region.

Then

$$\rho_s = \alpha_{2,2} + \delta\alpha_{1,2}$$
$$\gamma_{i,1}/\gamma_{i,0} = \alpha_{1,1} - \delta\alpha_{1,2}$$
$$(\tau - \omega)/\gamma_{i,0} = \frac{\alpha_{1,2}}{1-\rho_s}$$

The first equation is identical to the case where there is determinacy – the same value for ρ_s is identified regardless of whether or not there is determinacy. The second and third equations are similar to the determinacy case, but have their left hand sides modified. By comparing to the determinacy case, we can see that if the central bank assumes there is determinacy when there is actually indeterminacy, they will conclude that the persistence of demand shocks is equal to $\gamma_{i,1}/\gamma_{i,0}$ instead of ρ_d and the elasticity of the nominal interest rate to the supply shock is $(\tau - \omega)/\gamma_{i,0}$ instead of κ_s.

From the structure of the coefficient $\gamma_{i,0}$ and $\gamma_{i,1}$, we can use this result to back out a value for τ:

$$\tau = \omega + \frac{\phi^{-1} - (1 + \phi^{-1}\rho_i)\gamma_{i,0}/\gamma_{i,1}}{(1 - \gamma_{i,0}/\gamma_{i,1})\delta}$$

In turn, the third equation then implies a parameter restriction which must hold when there is indeterminacy:

$$\frac{\tau - \omega}{\phi^{-1} - \delta(\tau - \omega)} = \frac{\alpha_{1,2}}{1-\rho_s}$$

The central bank can use this restriction to test the null hypothesis of indeterminacy.

Matching the variance-covariance structure of the residual gives three (non-redundant) conditions:

$$\omega_{1,1}^2 = \lambda_s^2\sigma_s^2 + \sigma_\nu^2$$
$$(\delta\omega_{1,1} + \omega_{1,2})\omega_{1,1} = \lambda_s\sigma_s^2$$
$$(\delta\omega_{1,1} + \omega_{1,2})^2 + \omega_{2,2}^2 = \sigma_s^2$$

The third equation identifies σ_s^2 given that δ is already pinned down. The structural parameters σ_d^2 and $\sigma_{s,d}$ must both equal zero when demand shocks are absent. The first and second equations place restrictions on the equilibrium selection parameter vector $\lambda = (\lambda_s, \lambda_d, \sigma_\nu^2)'$. Though λ_d is unidentified, λ_s, and σ_ν^2 are identified. The value of ρ_d is not identified because it becomes a nuisance parameter in the indeterminacy region.

In summary, corresponding to each value of δ identified from the data (and identical between the determinacy and indeterminacy regions) and modulo λ_d and ρ_d, there is a unique identified value for the structural parameter vector of $\theta^S = (\delta, \tau, \rho_s, \cdot, \sigma_s^2, \sigma_d^2, \sigma_{s,d})$ (where the entry corresponding to ρ_d is left empty since it is a nuisance parameter in the indeterminacy region) with $\sigma_d^2 = 0$ and $\sigma_{s,d} = 0$.