Stochastic Volatility and Long-Run Risk in Endowment and Production Economies

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Abstract

I study the asset pricing implications of long run risk and stochastic volatility in general equilibrium. I analyse the Bansal and Yaron (2004) mechanism to generate additional risk premia for these factors and argue that it is specific to the endowment economy analysis. In a continuous time Cox-Ingersoll-Ross production economy state variables are endogenously correlated with consumption and are priced through this channel. Elasticity of intertemporal substitution will play only a small role for risk premia. In an application I also show that time varying conditional variance of production technology returns is a channel through which one can generate long-run risk in consumption.

Keywords: Long-run Risk, Variance Risk, Production, Asset Pricing, Recursive Utility, Continuous Time

JEL Classification: G12, E21, E30

1 Introduction

Recent strand of literature pioneered by Bansal and Yaron (2004) highlighted the importance of time varying risks of fundamentals - in particular stochastic conditional expected growth (long-run risk) and stochastic conditional variance (stochastic volatility) - for asset prices. Long-run risk combined with recursive preferences generated significant additional equity risk premium in endowment economy framework compared to standard C-CAPM with time additive preferences. It therefore contributed to the explanation of asset pricing puzzles. Another motivation for this line of research stemmed from the fact that there is significant evidence that "pure bets" on the state variables risks, such as stochastic volatility, are priced. For instance, a number of studies find evidence of negative variance.
risk premium in equity indices based on the data from the options market\(^1\).

In endowment economies with time additive preferences risks are priced insofar they are correlated with consumption growth. This is true even for risks affecting conditional expected consumption growth or its variance for which the agent has a hedging demand as it affects his investment opportunity set. Assuming exogenous correlations may seem ad hoc for a theorist. It is also difficult to measure empirically. Therefore the literature relied on Bansal and Yaron (2004) mechanism to generate risk premia for these factors, namely introducing Epstein Zin preferences and imposing restrictions on the values of relative risk aversion and elasticity of intertemporal substitution. In this paper I inspect the mechanism and argue that it is specific to the endowment economy analysis. In an economy with endogenous consumption wealth consumption ratio can adjust through wealth only to the changes in state variables. A particularly "smooth" production technology I will assume actually imply that all the changes to the wealth-consumption ratio due to state variables are reflected in the optimal consumption rather than wealth. State variables risk will be endogenously correlated to realised consumption risk and be priced through this trivial channel, often ignored. Elasticity of intertemporal substitution will play a negligible role for risk premia but will affect quantities.

I work with a Cox-Ingersoll-Ross (Cox, Ingersoll and Ross, 1985a) production economy with recursive preferences. A simple continuous time production technology allows me to find approximate closed form solutions to the model while allowing for very general specification of state variables affecting the fundamentals. I can easily analyse the effects of the preference parameters, namely risk aversion and elasticity of intertemporal substitution, on the risk premia and quantities. Continuous time specification also allows us to use portfolio choice tools and intuitions more complicated in discrete time. For illustrative purpose I present results for a comparable continuous time endowment economy.

One particular case I examine in more detail is stochastic volatility. In addition to confirming general results on the pricing of risk I show that time varying conditional variance of production technology returns is a channel through which one can generate long-run risk in consumption a la Bansal and Yaron (2004).

My paper is close in spirit to Tallarini (2000), Kaltenbrunner and Lochstoer (2007), Backus, Routledge and Zin (2007) and Croce (2008). The authors endogenise consumption within a standard real business cycle framework with recursive preferences and study asset pricing implications. Kaltenbrunner and Lochstoer (2007) and Croce (2008) generate consumption dynamics, and in particular the long-run risk component, assumed in

\(^1\)Variance risk premium is defined as the difference between the expectation of the quadratic variation process of returns under the real-world probability measure \(P\) and risk neutral probability measure \(Q\), i.e. \(E^P_t (QV_{t,t+\Delta}) - E^Q_t (QV_{t,t+\Delta})\).
reduced form by Bansal and Yaron (2004). I work in continuous time and with simpler assumptions on production technology which allow me to obtain closed form solutions and inspect the mechanisms behind.

Ai (2007) studies a Cox-Ingersoll-Ross production economy with imperfect information and learning about long-run risk in productivity.

Bollerslev, Gibson and Zhou (2005), Todorov (2007) and Bollerslev, Tauchen and Zhou (2008) provide evidence on the pricing of stock market variance risk using model free implied volatilities obtained from options markets. They find significant negative variance risk premium.

A series of papers uses Bansal and Yaron (2004) mechanism to explain variance risk premium, progressively increasing the level of sophistication of the conditional variance dynamics. Tauchen (2005), Bollerslev, Tauchen and Zhou (2008), Eraker (2008), Drechsler and Yaron (2008) progressively increase the level of sophistication of the conditional variance dynamics, introducing volatility of volatility and jumps with time-varying intensity. I will show how my endowment economy specification accommodate these features. Drechsler (2008) adds another layer of complexity by introducing Knightian uncertainty.

A main focus of my paper is the effect of the preferences parameters, relative risk aversion and elasticity of intertemporal substitution, on quantities and prices in equilibrium. Chacko and Viceira (2004) and Bhamra and Uppal (2005) solve portfolio-consumption choice problem with stochastic investment opportunity set in two particular setups. Chacko and Viceira (2004) find an approximate solution in a particular stochastic volatility specification. Bhamra and Uppal (2005) solve a three period model. To our knowledge general results in the case of time varying investment opportunity set are not yet available. Their results suggest that consumption and portfolio decisions depend on both risk aversion and elasticity of intertemporal substitution. However, the sign of the intertemporal hedging component in the optimal portfolio depends only on the size of the risk aversion relative to unity, while only the magnitude of the hedging portfolio depends on the elasticity of intertemporal substitution. The role of the parameters is reversed for the optimal consumption rule. My equilibrium results are consistent with the intuition provided by these papers.

Approximate analytical solution to my model is based on the method used by Campbell and Viceira (1999,2002), Campbell et al. (2004) and Chacko and Viceira (2004). The authors study various portfolio choice problems with time-varying investment opportunity set. I show how the Bellman equation linearisation can be applied to affine general equilibrium models.

The idea that all state variables affecting the fundamentals can be studied in a general
affine framework has been explored by Eraker and Shaliastovich (2008) for discrete time endowment economies.

The paper is organised as follows. In section 2 I present an affine representative agent economy with recursive preferences and describe the solution method. In section 3 I derive asset pricing results in endowment and production economies. Section 4 applies general results to a more specific model with stochastic volatility. Section 5 presents a calibration exercise for the production economy with stochastic volatility. Finally, section 6 concludes.

2 Representative Agent Equilibrium with Stochastic Differential Utility

2.1 Preferences

In this work representative agents preferences are described by a recursive utility function which generalises standard time separable power utility by allowing the separation of relative risk aversion from the elasticity of intertemporal substitution. It is the continuous time equivalent of the recursive utility parameterisation derived by Epstein and Zin (1989, 1991). We adopt Duffie and Epstein (1992a) parameterisation which is used as well in in Chacko and Viceira (2004)

\[ J = E_t \left( \int_t^\infty f(C_s, J_s)ds \right) \]

where \( f(C, J) \) is the normalised aggregator of current consumption and continuation utility that takes the form

\[ f(C, J) = \frac{\beta}{1 - \frac{1}{\psi}} (1 - \gamma) J \left( \left( \frac{C}{((1 - \gamma)J)^{\frac{1}{1-\gamma}}} \right)^{1 - \frac{1}{\psi}} - 1 \right) \]

The parameter \( \gamma \) controls agents risk aversion and \( \psi \) his elasticity of intertemporal substitution.

When \( \psi \rightarrow 1 \) the normalised aggregator takes the form

\[ f(C, J) = \beta (1 - \gamma) J \left( \ln(C) - \frac{1}{1 - \gamma} \ln(1 - \gamma) \right) \]

The standard power utility special case can be obtained by setting \( \psi = \frac{1}{\gamma} \).

The separation of the elasticity of intertemporal substitution from the coefficient of relative risk aversion has an important implication for the agents preferences towards the
early resolution of uncertainty. In the power utility case investor is indifferent towards the timing of resolution of uncertainty, if \( \gamma > \frac{1}{\psi} \) investor prefers the early resolution of uncertainty, if \( \gamma < \frac{1}{\psi} \) investor prefers late resolution of uncertainty (see Epstein and Zin, 1989). Intuitively, with \( \gamma > \frac{1}{\psi} \) agents propensity to smooth consumption across states is greater than propensity to smooth consumption across time, while the opposite is true with \( \gamma < \frac{1}{\psi} \).

### 2.2 Optimality Conditions

In this section I discuss results on asset pricing in representative agent general equilibrium which can be derived in quite general setting. This exposition essentially follows Duffie and Epstein (1992) but encompasses main results such as Merton’s (1973) ICAPM and Breeden’s (1979) CCAPM and generalises them to stochastic differential utility parameterisation described above.

We assume complete markets. There are a variety of securities traded. There is riskless borrowing and lending.

There are \( n \) state variables following

\[
dX_t = a(X_t,t)dt + b(X_t,t)dB_t
\]

where \( a \) is \((n \times 1)\) vector and \( b \) is \((n \times d)\) matrix. \( N \) security prices follow

\[
dS_t = IS_t\alpha(X_t,t)dt + IS_t\beta(X_t,t)dB_t
\]

where \( \alpha \) is \((N \times 1)\) vector, \( \beta \) is \((N \times d)\) matrix and \( IS_t \) is \((N \times N)\) matrix with \( S_t^i \) as diagonal elements.

Agent maximises his lifetime utility by choosing his consumption \( C \) and his investment into securities - \((N \times 1)\) vector \( \phi \). The dynamics of his wealth are given by

\[
dW_t = (W_t\phi_t^T \lambda_t + W_t r_t - C_t) dt + W_t\phi_t^T \beta_t dB_t
\]

where \( \lambda \) is \((N \times 1)\) vector of expected total returns (taking into account potential dividends \( \delta \) on securities in excess of the riskless rate. The Bellman equation for the agents problem is

\[
Max_{C,\phi} \left( f(J,C) + J_t + J_w(w \phi^T \lambda + wr - C) + J_x a + \frac{1}{2} tr(\Sigma) \right) = 0 \tag{2}
\]

\(^2\)We will deal with infinite horizon problems. There is no boundary condition. At the same time we can assume that the solution does not depend on the particular point in time but just on the level of the state variables at that point so \( J_t = 0 \).
where

\[ \Sigma = \begin{pmatrix} b & b \\ w \phi^T \beta \\ w \phi^T \beta \end{pmatrix}^T \begin{pmatrix} J_{xx} & J_{xw} \\ J_{wx} & J_{ww} \end{pmatrix} \begin{pmatrix} b \\ w \phi^T \beta \end{pmatrix} \]

The first order conditions for consumption and investment are respectively

\[ f_C = J_w \quad (3) \]

\[ -\lambda = -\frac{J_{ww}w}{J_w} \beta \beta^T \phi - \frac{J_{ww}w}{J_{wx}} \beta b^T \quad (4) \]

Equation (4) is Merton’s (1973) ICAPM relationship.

If we further assume stochastic differential utility specification (1) and (??) the first order condition with respect to consumption becomes

\[ C_t = J_W^{-\psi} ((1 - \gamma)J)^{\frac{1 - \gamma \psi}{1 - \gamma}} \beta^\psi \quad (5) \]

Under this preferences assumption it can be shown \(^3\) that we have the following two factor model for excess expected returns

\[ \lambda = \left( -\frac{1 - \gamma}{1 - \psi} \right) \frac{1}{C} \beta \sigma_C^T + \frac{1 - \gamma \psi}{1 - \psi} \beta \beta^T \phi \quad (6) \]

This is an extension of Breeden (1979) result to the case of Epstein Zin stochastic differential utility. If \( \psi = \frac{1}{\gamma} \) we are back to the standard time additive power utility case and we have the result of Breeden (1979) where risk premia are proportionate to the covariance of security returns with aggregate consumption.

2.3 Technology, Market Clearing, Equilibrium

So far we haven’t made any assumptions on the technology available in the economy. The results here above are derived only from optimality condition for the representative agent portfolio-consumption choice problem. In equilibrium market clearing conditions and resource constraints have to be satisfied - given the prices agent has to invest in available technology only and consume the aggregate dividend. This will allow us to solve completely for equilibrium. In this paper we are going to consider two common types of technologies. In Lucas-tree type endowment economies optimal consumption has to be equal to the exogenous aggregate dividend and the wealth consists of the claim

\(^3\)We differentiate the first order condition \( f_C = J_w \) with respect to \( w \) and \( x \) to obtain \( J_{ww} = f_{CC}C_w + f_{CJ}J_w \) and \( J_{wx} = f_{CC}C_x + f_{CJ}J_x \) which can be substituted into (1) and exploit the homogeneity of the value function.
on the aggregate dividend flow. In Cox Ingersoll Ross-type production economies the agent can invest in a linear technology which determines the returns on invested wealth and chooses his consumption.

We In the endowment economy that the change in the first state variable describes the aggregate dividend growth

\[ \frac{dD_t}{D_t} = \iota dX_t \]  

where

\[ \iota = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \]

In equilibrium consumption is equal to the aggregate dividend

\[ C_t = D_t \]

The technology in our production economy is a particular case of Cox, Ingersoll and Ross (1985a). There is a single physical good which can be allocated to consumption or investment. Production possibilities consist of one linear activity. The transformation of an investment of an amount of good \( I_t \) is described by the first state variable

\[ \frac{dI_t}{I_t} = \iota dX_t \]

In equilibrium the agent has to choose the optimal consumption/investment and invest only in the available activity. In a production economy with linear technology and full depreciation returns on aggregate wealth are trivially determined in equilibrium by the technology.

\[ \frac{dW_t}{W_t} = \iota dX_t - \frac{C_t}{W_t} dt \]

Equilibrium can therefore be defined as the solution to the Bellman equation (2) with the above constraints.

### 2.4 Solution

In the following part of the section I am going to use simplified notation, for instance dropping the time subscripts.

For tractability I will impose more structure on the problem by assuming affine dynamics for the state variables
\[ a(x, t) = K_0 + K_1 x, \quad (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \]

\[ (b(x, t)b(x, t)^T)_{ij} = (L_0)_{ij} + (L_1)_{ij} x, \quad (L_0, L_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n} \]

In the CIR production economy optimal consumption is given by the first order condition (5) and as optimal portfolio is constrained to be the investment in the available production technology the wealth dynamics are given by

\[ dw = (w^1 a - c^1) dt + wb dB_t \]

The homogeneity of the value function (see the discussion in Duffie and Epstein, 1992) suggests the solution of the form

\[ J = \frac{w^{1-\gamma}}{1-\gamma} g(x) \quad (8) \]

Let’s first consider the case when \( \psi = 1 \). Optimal consumption implied by (5) combined with (8) is

\[ c = \beta w \]

The problem is reduced to the following ordinary differential equation

\[ w^{1-\gamma} g \left[ \beta (\ln \beta - \frac{1}{1-\gamma} \ln g) + (\alpha a - \beta) + \frac{1}{1-\gamma} g^{-1} g_x a + \frac{1}{2} tr(\Sigma) \right] = 0 \]

where

\[ \Sigma = \begin{pmatrix} b \\ lb \end{pmatrix}^T \begin{pmatrix} \frac{1}{1-\gamma} g^{-1} g_{xx} & g^{-1} g_x^T \\ g^{-1} g_x & -\gamma \end{pmatrix} \begin{pmatrix} b \\ lb \end{pmatrix} \quad (9) \]

We look for the solution of the form

\[ g = \exp^{A+Bx} \]

The identity we obtain substituting the solution guess has to be verified for all \( x \). Regrouping the terms in each element of \( x \) gives us a a system of \( n + 1 \) equations for \( A \) and \( B \). As detailed in the appendix equations for the elements of \( B \) form a system of quadratic equations which in general has to be solved numerically. Given \( B \), the last equation is linear in \( A \).
In the case where \( \psi \neq 1 \) we can proceed in the same way. Let's denote the wealth-consumption ratio, which is the function of state variables, by \( h \), such that \( g = h^{\frac{1}{1-\psi}} \). Optimal consumption is

\[
c = w\beta^\psi g^{\frac{1}{1-\psi}} = w\beta^\psi h^{-1}
\]

And analogously \( g \) is characterised by the following differential equation where \( \Sigma \) is the same as in (9)

\[
w^{1-\gamma}g \left[ \frac{\psi \beta^\psi g^{\frac{1-\psi}{\psi-1}} - \psi \beta}{\psi - 1} + \left( \alpha - \beta^\psi g^{\frac{1-\psi}{\psi-1}} \right) + \frac{1}{1-\gamma} g^{-1} g_x a + \frac{1}{2} tr(\Sigma) \right] = 0
\]

In this case the terms \( g^{\frac{1-\psi}{\psi-1}} \) do not allow us to assume a solution exponential-affine in state variables. As suggested by the portfolio choice literature (see for example Campbell and Viceira, 2002 and Chacko and Viceira, 2004) we can linearise the term \( \beta^\psi g^{\frac{1-\psi}{\psi-1}} = \beta^\psi h^{-1} \), in other words the consumption-wealth ratio, around the unconditional (endogenous) mean of log consumption-wealth ratio. Indeed

\[
\beta^\psi h^{-1} = \frac{c}{w} = \exp(\ln\frac{c}{w})
\]

Taking the first order Taylor expansion of \( \exp(\ln\frac{c}{w}) \) around \( (\ln\frac{c}{w})^* = E(\ln\frac{c}{w}) \)

\[
\exp(\ln\frac{c}{w}) \approx \exp(\ln\frac{c}{w})^* + \exp(\ln\frac{c}{w})^*(\ln\frac{c}{w} - (\ln\frac{c}{w})^*)
\]

\[
\beta^\psi h^{-1} \approx h_0 + h_1 \ln \beta^\psi h^{-1}
\]

where \( h_1 = (\ln\frac{c}{w})^* > 0 \) and \( h_0 = h_1(1 + \ln(h_1)) \).

Substituting the approximation we obtain a differential equation in \( h \) which can be solved by analogy with the case \( \psi = 1 \) assuming

\[
h = \exp^{A+Bx}
\]

Endowment economy equilibrium can be solved for in the same way. Notice that given exogenous consumption process, we can obtain wealth dynamics from (10). Furthermore, we can easily verify that if \( \ln c \) has affine dynamics, implied \( \ln w \) dynamics will also be affine with the conjectured \( h = \exp^{A+Bx} \)
\[
\frac{dw}{w} = \left( \frac{\alpha + h^{-1}h_x a + tr \left[ bb^T i^T (h^{-1}h_x) + \frac{1}{2} bb^T (h^{-1}h_{xx}) \right] }{b} \right) dt + (\nu + h^{-1}h_x) b dB_t
\]

\[
\frac{dw}{w} = \left( \frac{\alpha + Ba + tr \left[ bb^T i^T B + \frac{1}{2} bb^T (B B^T) \right] }{b} \right) dt + (\nu + B) b dB_t
\]

The linearisation of the Bellman equation we use is similar to the Campbell-Shiller linearisation of the budget constraint standard in discrete time models. The coefficients \( h_0 \) and \( h_1 \) have to be solved for numerically. Alternatively, we could linearise consumption-wealth ratio around \( \log \beta \), its value when \( \psi = 1 \). Campbell and Viceira (2002) argue that setting the loglinearisation parameter \( h_1 \) to \( \beta \) comes at a cost to the accuracy of the approximation.

Classic perturbation methods as presented in Judd (1998) are based on approximating the solution to some general problem around a known particular case. Kogan and Uppal (2000) discuss the approximation of optimal consumption and portfolio policies in the context of expected CRRA utility \( (\psi = \frac{1}{\gamma}) \). They notice that for a large number of problems closed form solution can be found in the particular case \( \gamma = 1 \), i.e. log-utility. Approximate analytical solution for the general case can be obtained by expanding it around the exact solution with \( \gamma = 1 \). CRRA utility is a particular case of our utility specification and we are able to find the closed form solution for the case \( (\psi = \gamma = 1) \). More generally, we are able to find exact solution to our problem in the case \( \psi = 1 \) (and arbitrary \( \gamma \)) and adapt the solution methods proposed by the authors to stochastic differential utility case.

3 Pricing of Risk

3.1 Inspecting the Bansal and Yaron (2004) mechanism

As we know from Breeden (1979) with time-additive expected utility risk premia are proportionate to the covariance with the consumption risk. The risks of the state variables which affect the expected consumption growth (long run risks) and its variance (stochastic volatility) is not priced to the extent they are uncorrelated with realised consumption. When assuming exogenous consumption dynamics in endowment economy analysis the covariance of state variables and realised consumption risks has to be justified theoretically and/or measured empirically, both of which proved difficult. Bansal and Yaron (2004) introduced Epstein Zin preferences in order to generate risk premia for state variables uncorrelated with realised consumption. Inspecting their mechanism we would like
to argue it is specific to endowment economy analysis.

To understand the pricing of state variables risks I will examine their demand and the supply. Let \( \eta \) denote the vector of risk premia for each Brownian risk uncorrelated with realised consumption. We can rewrite the portfolio FOC of the portfolio-consumption choice problem (4) to get the myopic and hedging demand components of the agent for these risks. The asset in positive net supply in the endowment economy is the claim on the aggregate dividend stream, the price of this claim is also the agents wealth. Therefore, the supply of the exposure to risk factors is the sensitivity of the aggregate wealth to these factors.

\[
\text{Myopic Demand} + \text{Hedging Demand} = \text{Supply}
\]

\[
- \frac{J_w}{J_{ww}} \eta - \frac{J_{wx}}{J_{ww}} - \frac{1}{w} \frac{\partial w}{\partial x}
\]

In the general expected utility case the consumption FOC is

\[
J_w - u_c = 0
\]

By implicit differentiation we get

\[
\frac{\partial w}{\partial x} = - \frac{J_{wx}}{J_{ww}}
\]

The supply of the risks factor exposure is always equal to the hedging demand, thus market clear for \( \eta = 0 \). This result doesn’t hold for arbitrary normalised aggregator \( f(c, J) \), as the first order condition is \( J_w - f_C = 0 \) and \( f_C \) can depend on \( w \). In this case market would clear for \( \eta \neq 0 \).

For more intuition lets consider Epstein-Zin preferences which nest as a particular case time-additive CRRA utility. In the endowment economy the changes to the consumption-wealth ratio happen through wealth adjustments as consumption is given exogenously. The sensitivity of the changes in the consumption-wealth ratio to the changes in state variables is determined by the elasticity of intertemporal substitution. The hedging demand is determined by agents relative risk aversion (see Liu and Pan, 2003, for an example with stochastic volatility). If agents propensity to smooth consumption across time (the inverse of the elasticity of intertemporal substitution) is equal to the propensity to smooth across states (relative risk aversion) then hedging is exactly accomplished by the variation of the wealth. In the framework with exogenously given consumption Epstein-Zin preferences allow for a wedge between the hedging demand and the sensitivity of the changes of the wealth-consumption ratio.
3.2 Risk Premia in Cox Ingersoll Ross production economy

As explained in the previous section after substituting the solution form guess into agents problem in production economy case, we obtain a system of quadratic equations for the elements of $B$.

$$(\psi - 1)^{-2} (1 - \gamma) B (L_1)_{\bullet k} B +$$

$$(\psi - 1)^{-1} (B [(K_1)_{\bullet k} + (1 - \gamma) (L_1)_{\bullet k}] - h_1 B_k) +$$

$$((K_1)_{1k} - \gamma (L_1)_{11k}) = 0, \quad \forall k \in 1..n$$

In general we will have to solve this system numerically. We can notice by observing the equations (11) that, if we keep the linearisation coefficient $h_1$ constant, the solution $B$ is proportionate to $\psi - 1$. As consumption dynamics can be derived by applying Ito lemma to the first order condition for consumption (10)

$$dc/c = (\mu a - Ba + tr \left[ -bb^T \mu^T B + \frac{1}{2} bb^T (B^T B) \right] ) dt + (\sigma - B)bdB_t$$

the risk premia for state variables can be obtained from Epstein-Zin 2-factor pricing formula (6)

$$\eta = \gamma \mu b^T + \frac{1 - \gamma}{1 - \psi} Bbb^T$$

Conditional on $h_1$ the elasticity of intertemporal substitution does not affect risk premia. We will argue in the empirical section that this effect is small and does not change the sign of the second term of the premia, only their magnitude. If we chose to set the linearisation coefficient $h_1$ to $\beta$, the elasticity of intertemporal substitution would have no effect on the premia. The first term on the right hand side of (12) can be viewed as CAPM term, the second as ICAPM term.

This result can be explained as follows. Regardless of production or consumption specification, the wealth consumption ratio, which is the function of the state variable, varies in a similar way. In particular it is increasing or decreasing in state variables depending on the value of $\psi$ relative to unity. In the production economy, as consumption is not constrained by the exogenously given dividend process but wealth evolution is constrained by production technology, wealth-consumption ratio adjusts through consumption and not through wealth. Hedging demand for state variables risks is not automatically (fully or partially) compensated by the movements of the wealth. On the other hand, consumption is determined endogenously and is affected by $\psi$. As consumption adjusts to the shocks in state variables, it becomes endogenously correlated with them. Risk premia are explained in a more obvious way by the correlation between the state variables and
realised consumption risk.

The portfolio-consumption choice literature (Chacko and Viceira, 2004, Bhamra and Uppal, 2005) provide us with more intuition. Consumption and portfolio decisions depend on both risk aversion and elasticity of intertemporal substitution. However, the sign of the intertemporal hedging component in the optimal portfolio depends only on the size of the risk aversion relative to unity, while only the magnitude of the hedging portfolio depends on the elasticity of intertemporal substitution. The role of the parameters is reversed for the optimal consumption rule. Equilibrium consumption is constrained to be equal to the aggregate dividend, which in the endowment economy is given exogenously. The parameters of consumption dynamics do not depend on agents preferences. In equilibrium consumption choice has to be optimal for the representative agent. Asset prices have to adjust in a way that the agent optimally consumes the endowment. The effect of $\psi$ on the consumption choice feeds back into the asset prices and therefore both $\psi$ and $\gamma$ affect risk premia.

It is important to see that endowment and production specifications are not two completely separate models. In both specifications first order conditions (3) and (4) are verified. For example, assume an endowment economy model. If we use the wealth dynamics obtained in equilibrium as the technology for the production economy we will obtain initial exogenous consumption process as optimal consumption. Therefore empirical asset pricing studies can be seen as tests of first order conditions on jointly asset pricing data and consumption or production (wealth) data. It is a matter of choice of the empirical researcher which time series can be studied more accurately. Research efforts have been concentrated on testing asset pricing model with consumption data because while some measures of aggregate consumptions are available, aggregate wealth which includes such components as human capital is not observable. For a theorist the choice between production and endowment specification depends on whether he is more comfortable with making assumptions about exogenous consumption or endogenous production processes. If one wants to investigate theoretically the effects of the preference parameters, it is more intuitive to assume that production technology rather than exogenous consumption does not depend on preference parameters. A line of criticism against endowment economy analysis is that it doesn’t take into consideration the effects of preference parameters on consumption dynamics.

The results on the pricing of risk in the production economy depend on the very smooth production technology we assume. With linear production technology, full depreciation and no capital adjustment costs all the changes in the wealth consumption ratio due to state variables shocks is instantaneous and pass through consumption adjustment. This is not necessarily true with more elaborate production technology - both
wealth and consumption will adjust. One can view the two specifications presented in my paper - Lucas-tree endowment economy and Cox-Ingersoll-Ross production economy - as two extreme cases of production economy with capital adjustment costs and capital depreciation. In the former case the capital adjustment cost is infinite and there is no depreciation, in the latter the capital adjustment cost is equal to 0 and depreciation is full. More realistic macro-asset pricing model based on real business cycle literature will not take these two parameters to the extreme. Intuitively, results (with finite capital adjustment cost and partial depreciation) will be between the two cases I consider. In particular, since capital is not immediately and perfectly adjustable, \( \psi \) will have an effect on risk premia.

Consider now the endowment economy where dividend growth dynamics (instead of production) is described by (7). It is possible to show that risk premia for state variables are

\[
\eta = \gamma \bar{bb}^T + \frac{1 - \gamma \psi}{1 - \psi} \bar{bb}^T
\]

where again conditional on the linearisation coefficients \( \bar{B} \) is proportionate to \( 1 - \psi \). The expression generalises Bansal and Yaron (2004). The first term on the right hand side corresponds to the CRRA risk premium, the second term is non-zero if and only if \( \frac{1}{\psi} \neq \gamma \), in other terms if and only if the agent is not indifferent to the timing of the resolution of uncertainty.

### 3.3 Jumps

In the recent literature rare events or jumps have been advanced as an important empirical feature of the financial data and a promising explanation for asset pricing puzzles in equity and derivatives markets. It is very easy to incorporate Poisson jumps with affine jump intensity into our framework. The exercise would be an interesting extension. Empirically it will significantly improve the performance of our model. Theoretically, however, the results for continuous processes have an equivalent with jumps. The realisation of the jump risk is similar to the realisation of the Brownian risk and the time varying drift and diffusion to the time varying jump intensity.

### 4 Stochastic Volatility in General Equilibrium

#### 4.1 Endowment Economy

The aggregate dividend process in the economy is exogenous and assumed to follow a diffusion process. We assume that the volatility of the dividend growth is stochastic and
follows a Cox-Ingersoll-Ross process

\[ \frac{dD_t}{D_t} = \mu dt + \sigma_t dB^D_t \]
\[ d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \sigma_t \omega dB_t^\omega \]
\[ dB_t^D dB_t^\omega = \rho dt \]

In equilibrium consumption is equal to the aggregate dividend

\[ C_t = D_t \]

Appendices B and C discuss technical details of derivation of closed form expressions for \( A_3 \) and \( B_3 \) in the value function \( J_t = \frac{W_{t-1}}{1-\gamma} \exp \left( -\frac{1-\gamma}{1-\psi} (A_3 + B_3 \sigma_t^2) \right) \). In particular, for the range of parameters used for calibration of the model the expression for \( B_3 \) is given by

\[ B_3 = \frac{\psi(1-\gamma)\omega \rho - h_1 - \kappa) + \sqrt{\psi^2((1-\gamma)\omega \rho - h_1 - \kappa)^2 + \omega^2 \psi \gamma (1-\gamma)}}{\omega^2 (1-\gamma) \psi} \]

As a first benchmark we can analyse the equilibrium in an endowment economy with time additive CRRA utility. In this case the price of the asset paying aggregate dividend (representative agents wealth) can be found in exact closed form up to an integral. It can be shown that the price is given by the following expression where \( A_0(s) \) and \( B_0(s) \) are (nonlinear) functions of time only (see Appendix A).

\[ W_t = E_t \int_0^\infty e^{-\beta s} \left( \frac{C_t}{C_{t+s}} \right)^\gamma C_{t+s} ds = C_t F(\sigma_t^2) = C_t \int_0^\infty e^{-\beta s + A_0(s) + B_0(s) \sigma_t^2} ds \]

With time additive utility Breeden (1979) results hold. If stochastic variance is not correlated with the consumption (\( \rho = 0 \)) it carries no premium. Moreover, the relative risk aversion greater than unity \( \gamma > 1 \) implies that the wealth-consumption (price-dividend) ratio is increasing in volatility. Keeping exogenous correlation of consumption with volatility equal to zero (\( \rho = 0 \)), this also means that returns on aggregate wealth are positively correlated with volatility - we obtain the opposite of the "leverage effect".

The approximate analytical solution to the general Epstein-Zin preferences case converges to the exact solution in the case of log utility (\( \gamma = \psi = 1 \)). It does not generally

\[ ^4 \text{The representative agent is more risk averse than the log investor. This is commonly argued to be empirically true for the representative agent.} \]
coincide with it in the case of time additive power utility \( (\gamma = \frac{1}{\psi}) \). However, with both the exact and the approximate solution we obtain that in time additive case with \( \rho = 0 \) we have positive correlation of asset returns and its volatility for \( \gamma > 1 \) and negative correlation of asset returns and its volatility for \( \gamma < 1 \).

In the endowment economy with Epstein-Zin preferences asset returns are negatively correlated with its volatility if \( B_3 < 0 \). The wealth-consumption ratio is decreasing in volatility only if \( \psi > 1 \). This condition ensures that substitution effect is greater than the income effect. The result is similar to Tauchen (2005) discrete time setting. The author argue that this implication of the model contributes to the economic debate on the value of the elasticity if intertemporal substitution relative to unity \( \psi > 1 \) vs \( \psi < 1 \). With CRRA preferences the elasticity of intertemporal substitution is tied to the relative risk aversion and \( \gamma > 1 \) implies automatically \( \psi = \frac{1}{\gamma} < 1 \).

The variance risk premium can be obtained from (6)

\[
\lambda^\sigma = \gamma \sigma_t^2 \omega \rho + \left( \frac{1 - \gamma \psi}{1 - \psi} \right) B_3 \sigma_t^2 \omega^2
\]

The first term will contribute to the negative variance risk premium to the extent that consumption growth is negatively correlated with its volatility. The second term contributes to the variance risk premium even in the case \( \rho = 0 \). It is equal to zero in the expected utility case \( \gamma = \frac{1}{\psi} \). The expression \( \left( \frac{1 - \gamma \psi}{1 - \psi} \right) B_3 \) is negative iff \( \frac{1}{\psi} < \gamma \). In other terms, very intuitively, in the endowment economy variance risk premium generated through the separation of relative risk aversion and elasticity of intertemporal substitution is negative if agent prefers early resolution of uncertainty.

### 4.2 Production Economy

The transformation of the investment of an amount of good \( I_t \) and the dynamics of the state variable in our production economy are governed by the following stochastic differential equations

\[
\begin{align*}
\frac{dI_t}{I_t} &= \mu dt + \sigma_t dB_t^I \\
\sigma_t^2 &= \kappa (\theta - \sigma_t^2) dt + \sigma_t \omega dB_t^\sigma \\
\frac{dB_t^I dB_t^\sigma}{\rho} &= \rho
\end{align*}
\]

I would like to draw the attention that while the parameters \( \mu, \kappa, \theta, \rho \) and the variable \( \sigma_t^2 \) have the same notation as in the previous section they now describe the dynamics of the investment technology.

Appendices B and C discuss technical details of derivation of closed form expressions
for $A_4$ and $B_4$ in the value function. In particular, $B_4$ is given by

$$B_4 = \frac{(1 - \gamma)\omega - h_1 - \kappa + \sqrt{(1 - \gamma)\omega - h_1 - \kappa)^2 + \omega^2 (1 - \gamma)\gamma}}{\omega (1 - \gamma)\gamma}$$

Using the results of the previous section as well as (3), (4) and (6) I can find equilibrium interest rate, consumption and premia for risk factors. The short riskless rate is given by

$$r_t = \mu - \gamma\sigma_i^2 - \frac{1 - \gamma}{1 - \psi} B_4 \sigma_i^2 \omega \rho$$

Risk premia for variance and investment returns are given respectively by

$$\lambda^\sigma = \gamma\sigma_i^2 \omega \rho + \frac{1 - \gamma}{1 - \psi} B_4 \sigma_i^2 \omega^2$$

$$\lambda^I = \gamma\sigma_i^2 + \frac{1 - \gamma}{1 - \psi} B_4 W_i \sigma_i^2 \omega \rho$$

Equation (13) determines the difference between the real world $P$ and risk neutral $Q$ dynamics of $\sigma_i^2$. The variance factor premium is composed of two terms - one proportional to the covariance with wealth (CAPM-type) and one arising from dynamic hedging. "Leverage effect" $\rho < 0$ will imply that the first term is negative $\gamma\sigma_i^2 \omega \rho < 0$ contributing to the negative variance risk premium. The sign of $\left(\frac{1 - \gamma}{1 - \psi}\right) B_4$ is negative when $\gamma > 1$ and positive when $\gamma < 1$. In the former case the magnitude of the negative premium increases. Given $h_1$ (which is determined in equilibrium and therefore may depend on $\psi$ itself) the elasticity of intertemporal substitution $\psi$ does not affect the premium. In particular, starting from an arbitrary case we can fix $\gamma$ and set $\psi = \frac{1}{\gamma}$ without changing (significantly) the pricing of risk. In the next section we are going to solve numerically for $h_1$ and argue that the impact of $\psi$ on $\lambda^\sigma$ is negligible. More generally, in the next section we present numerical results on the impact of $\gamma$ and $\psi$ on the variance risk premium.

In a production economy consumption is endogenously determined. Consumption dynamics are given by

$$dC_t/C_t = (\mu - B_4 \kappa (\theta - \sigma_i^2) - B_4 \sigma_i^2 \omega \rho + (B_4 \sigma_i \omega)^2) dt + \sigma_i dB_t^i - B_4 \omega dW_t^\sigma$$

(14)

Stochastic variance of the linear production technology has two interesting effects. The

---

5 $\rho$ is estimated to be in the region between $-0.7$ and $-0.6$ for major stock market indices.

6 Assuming that the conditions outlined in the previous subsection hold. We will argue in the next section that this is the case for the range of parameters for stock market dynamics we are interested in.
value of the elasticity of intertemporal substitution relative to one is crucial to sign both of them.

The variance of consumption is stochastic and is endogenously correlated with realised consumption risk. This is true even if wealth is uncorrelated with its volatility ($\rho = 0$). Optimal consumption has a "dynamic hedging" component through the $-B_4 \sigma_t \omega dB_t^\sigma$ term in (14). The sign of $B_4$ is determined by the sign of $1 - \psi$. $\psi > 1$ contributes to the positive correlation of consumption with volatility, $\psi < 1$ having the opposite effect.

Stochastic conditional variance of the production technology induces time-varying expected consumption growth or long-run risk in consumption. Expected consumption growth depends on the state variable $\sigma_t^2$ through $-B_4 \kappa (\theta - \sigma_t^2)$ term in the drift. With $\psi > 1$, higher (lower) volatility implies lower (higher) expected consumption growth. The effect is really due to the fact that wealth-consumption ratio adjusts to the changes in state variables entirely through consumption. Notice as well that in reasonable calibrations the remaining terms in the drift $-B_4 \sigma_t^2 \omega \rho + (B_4 \sigma_t \omega)^2$ would be very small.

These results have two important implications. It provides theoretical justification for consumption correlation with its volatility. This is a testable empirical implication. Moreover the sign of the correlation depends on the sign of $1 - \psi$ and therefore its econometric estimate can contribute to the debate on $\psi \leq 1$. The results also suggest that this correlation should not be ignored in consumption-based asset pricing models. A series of recent papers (see for example Kaltenbrunner and Lochstoer, 2007, and Croce, 2008) endogenises consumption within standard real business cycle framework and tries to replicate Bansal and Yaron (2004) persistent expected consumption growth (long run risk). The mechanisms are progressive consumption adjustment to technology shocks and time varying expected technology growth (or long run productivity risk). In my setup with linear technology with constant expected return $\mu$, full depreciation and no adjustment costs both channels are closed. I am able to generate time-varying expected consumption growth through stochastic conditional variance of technology returns.

5 A Stylised Production Economy

I calibrate the CIR production economy model to parameters reproducing the dynamics of a stylised broad stock market index. The aim of the exercise is to look at the variance risk premium given realistic stock market dynamics. Assuming that stock market return is equal to the return on aggregate wealth is obviously subject to standard Roll critique.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\theta$</th>
<th>$\kappa$</th>
<th>$\omega$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.04</td>
<td>5</td>
<td>0.65</td>
<td>-0.65</td>
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</table>
I set $\beta = 0.05$ and look at a very reasonable range for RRA parameter $\gamma$ from 1 to 5. Equilibrium quantities and prices depend on the linearisation coefficient $h_1$ which is itself endogenous. I evaluate it numerically by taking an initial value for $h_1$ (for example $\beta$, the solution in the case of log utility) and using a recursive procedure on equations (??) and (??) until convergence. I am able to investigate the impact that the elasticity of intertemporal substitution $\psi$ has on risk premium through the equilibrium value of $h_1$. My numerical analysis shows that changing $\psi$ has a small effect on its magnitude and no effect on the sign of the premium. In particular the sign is not affected by setting $\psi > 1$ or $\psi < 1$.

The accuracy of the my approximation depends on the variability of the log consumption-wealth ratio around its mean. Chacko and Viceira (2004) report the standard deviations of the unconditional standard deviation of the log consumption-wealth ratio in the context of the approximate solution to their particular continuous time portfolio choice problem. In my closely related setting the unconditional variance of the log consumption-wealth ratio is equal to $\frac{1}{2} B^2 \theta^2 \frac{\psi^2}{\pi}$. We will use the same approximation technique Table 1 reports the unconditional standard deviation of the log consumption-wealth ratio $(100 \sqrt{\frac{1}{2} B^2 \theta^2 \frac{\psi^2}{\pi}})$ for the production economy model.

I present the results for the variance risk premium $\lambda = \frac{\gamma^2}{\sigma^2}$ for different $\gamma$ and $\psi$ as well as the breakdown into the component of the variance risk premium explained by correlation with the return on wealth - $\gamma \omega \rho$ - and the component due to "dynamic hedging" - $\left(\frac{1-\gamma}{\gamma}\right) B \omega^2$. The values for $\lambda$ can be compared to the estimated variance risk premium parameter in Bollerslev, Gibson and Zhou (2005) who use the reduced form Heston model to describe the dynamics of the S&P 500 index. However their parameter estimates differ from the ones of my stylised economy. The comparison suggests that the relative risk aversion coefficient $\gamma$ of around 4 is sufficient to generate variance risk premium as measured in the derivatives markets.

<table>
<thead>
<tr>
<th>$\gamma \backslash \psi$</th>
<th>1.33</th>
<th>1.10</th>
<th>1.00</th>
<th>0.90</th>
<th>0.75</th>
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</tr>
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<tr>
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<td>0.00</td>
<td>0.03</td>
<td>0.07</td>
<td>0.15</td>
</tr>
<tr>
<td>1.00</td>
<td>0.14</td>
<td>0.04</td>
<td>0.00</td>
<td>0.04</td>
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<td>0.20</td>
</tr>
<tr>
<td>2.00</td>
<td>0.30</td>
<td>0.09</td>
<td>0.00</td>
<td>0.09</td>
<td>0.22</td>
<td>0.45</td>
</tr>
<tr>
<td>3.00</td>
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<td>0.15</td>
<td>0.00</td>
<td>0.15</td>
<td>0.38</td>
<td>0.76</td>
</tr>
<tr>
<td>4.00</td>
<td>0.80</td>
<td>0.24</td>
<td>0.00</td>
<td>0.24</td>
<td>0.61</td>
<td>1.22</td>
</tr>
<tr>
<td>5.00</td>
<td>1.32</td>
<td>0.40</td>
<td>0.00</td>
<td>0.41</td>
<td>1.04</td>
<td>2.10</td>
</tr>
</tbody>
</table>

Table 1: Unconditional standard deviation of the log consumption-wealth ratio
Table 2: Variance risk premium and its components for EIS=0.9 and EIS=1.1

Higher risk aversion \( \gamma \) produces higher variance risk premium. The main explanatory factor of the risk premium is correlation of the variance with the returns on wealth - standard CAPM effect. The dynamic hedging term in the risk premium is important and its relative importance is increasing with \( \gamma \).

My model is consistent with the reduced form modeling of index dynamics and variance risk premium specification proposed in the option pricing literature by Heston (1993). Under the real-world probability measure \( P \) the variance follows a CIR process. Variance risk premium is proportionate to the variance level \( \lambda^\sigma = \lambda \sigma_t^2 \), hence the \( Q \) dynamics of variance can also be written as a CIR process

\[
d\sigma_t^2 = \kappa' (\theta' - \sigma_t^2) dt + \sigma_t \omega dB_t^Q
\]

with \( \kappa' = \kappa + \lambda \), \( \theta' = \frac{\kappa \theta}{\kappa + \lambda} \). By analogy with Cox-Ingersoll-Ross term structure of interest rates (Cox, Ingersoll and Ross, 1985b) I can easily derive the term structure of prices of variance swap contracts in the economy.

\[
E^Q \left( \int_t^T \sigma_s^2 ds \right) = \sigma_t^2 \left( 1 - e^{-\kappa' (T-t)} \right) \kappa' + \theta' \left( e^{-\kappa' (T-t)} + (T-t) \kappa' - 1 \right)
\]

One factor model cannot match accurately the entire term structure of variance swaps - for example if it matches the 3 month value it then predicts a very steep slope. A two factor stochastic volatility or jump component, with separate risk premia for each factor, can improve the fitting of the term structure. The model is easily extended in this direction provided we stay in the affine class.
6 Conclusion

In this paper I assumed a simplified and particularly tractable framework to examine the pricing of risk in a production economy with long-run risk and stochastic volatility. I show that while endowment economy framework is well adapted to formulate such problems as the equity premium puzzle, assuming exogenous consumption dynamics might be misleading for theoretical conclusions about the effects of risk aversion and elasticity of intertemporal substitution on risk premia. The mechanisms highlighted in the paper can help to understand the functioning of more realistic but less tractable models, particularly the ones solved with numerical methods. Interestingly, time varying conditional variance of the production technology creates long-run risk in the equilibrium consumption process. This suggests an interesting route to be explored in the context of real business cycle models with long run risk. Empirically we find that a very stylised model can generate realistic variance risk premium with the relative risk aversion parameter within a reasonable range of values.
References


A Time Additive Utility Case

We are looking at the case $\rho = 0$, which can be easily generalised.

CRRA expected utility

$$U_t = E_t \left( \int_t^\infty e^{-\beta s} \frac{C_{t+s}^{1-\gamma}}{1 - \gamma} ds \right)$$

The price of the asset paying aggregate dividend is given by

$$W_t = E_t \left( \int_0^\infty e^{-\beta s} \left( \frac{C_t}{C_{t+s}} \right)^\gamma C_{t+s} ds \right) = C_t \int_0^\infty e^{-\beta s} E_t \left( C_{t+s}^{1-\gamma} \right) ds$$

Lest assume that

$$E_t \left( C_{t+s}^{1-\gamma} \right) = C_t^{1-\gamma} e^{A_0(s) + B_0(s) \sigma_t^2}$$

Expectations are martingales. We apply Ito lemma to $C_t^{1-\gamma} e^{A_0(s) + B_0(s) \sigma_t^2}$ and set the drift to zero. The zero drift condition can be written

$$(1 - \gamma) \mu + \frac{1}{2} \gamma (\gamma - 1) \sigma_t^2 - A'_0 - B'_0 \sigma_t^2 + B_0 \kappa (\theta - \sigma_t^2) + \frac{1}{2} B_0^2 \sigma_t^2 \omega^2 = 0$$

This equality has to hold for each $\sigma_t^2$. Regrouping the terms in $\sigma_t^2$ we obtain an ODE system

$$B'_0 = \frac{1}{2} B_0^2 \omega^2 - B_0 \kappa + \frac{1}{2} \gamma (\gamma - 1) = 0$$
$$A'_0 = B_0 \kappa \theta + (1 - \gamma) \mu$$

The first equation is a Riccati ODE with constant coefficients in $B_0(s)$. With $\kappa^2 - \omega^2 \gamma (\gamma - 1) > 0$ the solution is

$$B_0(s) = \frac{\kappa - \sqrt{\kappa^2 - \omega^2 \gamma (\gamma - 1)}}{\omega^2} \left( e^{\sqrt{\kappa^2 - \omega^2 \gamma (\gamma - 1)} s} - 1 \right)$$

Given $B_0(s)$, $A_0(s)$ can be easily obtained

$$A_0(s) = \frac{2 \kappa \theta}{\omega^2} \ln \left( \frac{2 \sqrt{\kappa^2 - \omega^2 \gamma (\gamma - 1)}}{\frac{2 \sqrt{\kappa^2 - \omega^2 \gamma (\gamma - 1)}}{\kappa + \sqrt{\kappa^2 - \omega^2 \gamma (\gamma - 1)}} + (1 - \gamma) \mu} \right)$$
See Appendix for the details of the last two results.

**B Exact Solution with $\psi = 1$**

When $\psi \to 1$, the normalised aggregator of current consumption and continuation utility that takes the form

$$f(C, J) = \beta(1 - \gamma)J \left( \ln(C) - \frac{1}{1 - \gamma} \ln((1 - \gamma)J) \right)$$

The FOC with respect to consumption

$$C = \beta(1 - \gamma)\frac{J}{J_W}$$

We assume the value function of the form

$$J = \frac{W^{1-\gamma}}{1-\gamma}F$$

$$F = e^{A_1 + B_1\sigma_t^2}$$

The FOC simplifies to

$$C = \beta W$$

Substituting the value function guess and the FOC to the endowment economy HJB equation we obtain

$$0 = \beta \left( \ln(\beta) - \frac{1}{1 - \gamma} \left( A_1 + B_1\sigma_t^2 \right) \right) + \mu - \frac{1}{2} \gamma \sigma_t^2$$

$$+ \frac{1}{1 - \gamma} B_1 \kappa (\theta - \sigma_t^2) + \frac{1}{2} \frac{1}{1 - \gamma} B_1^2 \sigma_t^2 \omega^2 + B_1 \sigma_t^2 \omega \rho$$

The equality has to hold for every $\sigma_t^2$ and therefore we can regroup the terms $\sigma_t^2$ to obtain a system of equations.

$$\omega^2 B_1^2 + B_1 \omega (1 - \gamma) - \beta \kappa - \gamma (1 - \gamma) = 0$$
$$-\beta A_1 + \mu (1 - \gamma) + B_1 \kappa \theta = 0$$

Assume the discriminant of the quadratic equation is positive. The equation has two real roots given by
\[ B_1 = \frac{-(\omega \rho(1 - \gamma) - \beta - \kappa) \pm \sqrt{(\omega \rho(1 - \gamma) - \beta - \kappa)^2 + \omega^2 \gamma(1 - \gamma)}}{\omega^2} \]

We choose the root which makes the solution converge to the known log-utility solution by setting the relative risk aversion parameter to unity.

In the same way, substituting the value function guess and the FOC to the production economy HJB equation we obtain

\[
0 = \beta \left( \ln(\beta) - \frac{1}{1-\gamma} (A_2 + B_2 \sigma_i^2) \right) + (\mu - \beta) - \frac{1}{2} \gamma \sigma_i^2
\]

\[
+ \frac{1}{1-\gamma} B_2 \kappa (\theta - \sigma_i^2) + \frac{1}{2} \frac{1}{1-\gamma} B_2^2 \sigma_i^2 \omega^2 + B_2 \sigma_i^2 \omega \rho
\]

The equality has to hold for every \( \sigma_i^2 \) and therefore we can regroup the terms \( \sigma_i^2 \) to obtain a system of equations.

\[
\omega^2 B_2^2 + B_2 2 (\omega \rho(1 - \gamma) - \beta - \kappa) - \gamma(1 - \gamma) = 0
\]

\[
-\beta A_2 + (\mu - \beta)(1 - \gamma) + B_2 \kappa \theta = 0
\]

We notice that the equations for \( B_2 \) and \( B_1 \) are the same, therefore \( B_2 = B_1 \).

C Approximate Analytical Solution

We approximate the consumption wealth ratio

\[ \beta^\psi H^{-1} \approx h_0 + h_1 (c_t - x_t) = h_0 + h_1 \ln \beta^\psi H^{-1} \]

where \( h_1 = \exp(c_t - w_t) > 0 \), \( h_0 = h_1 (1 + \ln(h_1)) \).

Substituting the approximation in the first term of equation the solution we notice that a natural candidate is the solution of the form \( H = \exp(A_t + B_t \sigma_t^2) \) and ODE (??) can be written
0 = -h_0 - h_1(-A_3 - B_3\sigma_t^2 + \psi \ln \beta) + \beta^\psi \\
+ (\mu + B_3\kappa(\theta - \sigma_t^2) + \frac{1}{2}B_3^2\sigma_t^2\omega^2 + B_3\sigma_t^2\omega\rho)(1 - \psi) \\
- \frac{\gamma}{2}(1 - \psi)(\sigma_t^2 + B_3\sigma_t^2\omega^2 + 2B_3\sigma_t^2\omega\rho) - B_3\kappa(\theta - \sigma_t^2) \\
+ \frac{1}{2}(1 - \psi)B_3^2\sigma_t^2\omega^2 - (1 - \gamma)(B_3\sigma_t^2\omega\rho + B_3^2\sigma_t^2\omega^2)

The equality has to hold for every $\sigma_t^2$ and therefore we can regroup the terms $\sigma_t^2$ to obtain a system of equations. The first is a quadratic equation in $B_3$ and, given $B_3$, the second is a linear equation in $A_3$.

$$B_3^2\omega^2 \frac{(1 - \gamma)\psi}{1 - \psi} + B_3\psi(-(1 - \gamma)\omega\rho + h_1 + \kappa) - \frac{\gamma}{2}(1 - \psi) = 0$$

$$-h_0 + h_1 A_3 - h_1 \psi \ln \beta + \beta^\psi + \mu(1 - \psi) - \psi B_3 \kappa \theta = 0$$

If the discriminant of the quadratic equation is positive, which is the case we will work with, the equation has two real roots. The discriminant is always positive when $\gamma < 1$.

If $\gamma > 1$ the condition for positive discriminant is likely to hold for $\gamma$ not too high, $\rho$ positive or not too high in absolute value, $\omega$ not too high, $\psi$ not too low.

The roots of the equation are

$$B_3 = \frac{\psi((1 - \gamma)\omega\rho - h_1 - \kappa) \pm \sqrt{\psi^2((1 - \gamma)\omega\rho - h_1 - \kappa)^2 + \omega^2\psi\gamma(1 - \gamma)}}{\omega^2(1 - \gamma)\psi}$$

We note that we can find the exact analytical solution to the above problem in special cases (for instance when $\gamma = \psi = 1$ we are in the well known case of the log-utility) and we pick the approximate solution which converges to the exact solution in these special cases. If $(1 - \gamma)\omega\rho - h_1 - \kappa < 0$ we choose

$$B_3 = \frac{\psi((1 - \gamma)\omega\rho - h_1 - \kappa) + \sqrt{\psi^2((1 - \gamma)\omega\rho - h_1 - \kappa)^2 + \omega^2\psi\gamma(1 - \gamma)}}{\omega^2(1 - \gamma)\psi}$$

We proceed in exactly the same way with the production economy HJB equation. Integrating the value function guess and the FOC for consumption we obtain
\[0 = -h_0 - h_1(-A_4 - B_4\sigma^2 + \psi \ln \beta) + \psi \beta + (\mu - \frac{1}{2}\gamma\sigma^2)(1 - \psi)\]

\[-B_4\kappa(\theta - \sigma^2) + \frac{1}{2}\sigma^2\omega^2 \left(\frac{1 - \gamma}{1 - \psi}\right) B^2 - (1 - \gamma)B_4\sigma^2\omega\rho\]

By regrouping the terms in \(\sigma^2\) we obtain a quadratic equation in \(B_4\) and given \(B_4\) a linear equation in \(A_4\).

\[\frac{1}{2}\sigma^2\omega^2 \left(\frac{1 - \gamma}{1 - \psi}\right) B^2 - ((1 - \gamma)\omega\rho - h_1 - \kappa)B_4 - \frac{1}{2}\gamma(1 - \psi) = 0\]

\[-h_0 + h_1A_4 - \psi h_1 \ln \beta + \psi \beta + \mu(1 - \psi) - B_4\kappa\theta = 0\]

We work with the case in which the discriminant of the quadratic equation is positive. We note that we can find the exact analytical solution to the above problem in special cases (for instance when \(\gamma = \psi = 1\) we are in the well known case of the log-utility) and we pick the approximate solution which converges to the exact solution in these special cases. If \((1 - \gamma)\omega\rho - h_1 - \kappa < 0\) we choose

\[B_4 = \frac{((1 - \gamma)\omega\rho - h_1 - \kappa) + \sqrt{((1 - \gamma)\omega\rho - h_1 - \kappa)^2 + \omega^2 (1 - \gamma)}}{\omega^2 \left(\frac{1 - \gamma}{1 - \psi}\right)}\]

\section{Riccati Equation with Constant Coefficients}

The general form of Riccati equations with constant coefficients is

\[\frac{\partial y}{\partial x} = ay^2 + by + c\] \hspace{1cm} (15)

The solution is

\[y = -\frac{1}{2} - \tan\left(\frac{1}{2}x\sqrt{-b^2 + 4ac} - \frac{1}{2}K\sqrt{-b^2 + 4ac}\right)\sqrt{-b^2 + 4ac} + b\]

Since this expression is difficult to manipulate we present the solution which holds if coefficients satisfy a certain condition.

Let’s apply the following transformation to \(y\)

\[y \equiv -\frac{u'}{ua}\]

\[-\frac{u''ua - u'^2a}{(ua)^2} = a\frac{u'^2}{(ua)^2} - b\frac{u'}{ua} + c\]
We obtain a second order linear homogeneous differential equation with constant coefficients

\[ u'' - bu' + acu = 0 \]

Lets further assume

\[ u = e^{\lambda x} \]

\[ e^{\lambda x} (\lambda^2 - b\lambda + ac) = 0 \]

The auxiliary equation

\[ \lambda^2 - b\lambda + ac = 0 \]

Lets assume two distinct real roots \( b^2 - 4ac > 0 \)

\[ \lambda = \frac{b \pm \sqrt{b^2 - 4ac}}{2} \]

The solution to the problem in terms of \( u \) is therefore

\[ u = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x} \]

\( y \) can be written as

\[ y = -\frac{\lambda_1 e^{\lambda_1 x} + \lambda_2 Ke^{\lambda_2 x}}{(e^{\lambda_1 x} + Ke^{\lambda_2 x})a} \]

\[ y(x) = -\frac{\frac{b + \sqrt{b^2 - 4ac}}{2} e^{\frac{b + \sqrt{b^2 - 4ac}}{2} x} + \frac{b - \sqrt{b^2 - 4ac}}{2} e^{\frac{b - \sqrt{b^2 - 4ac}}{2} x}}{\left(e^{\frac{b + \sqrt{b^2 - 4ac}}{2} x} + Ke^{\frac{b - \sqrt{b^2 - 4ac}}{2} x}\right)a} \]

\[ = -\frac{\frac{b + \sqrt{b^2 - 4ac}}{2} e^{\frac{\sqrt{b^2 - 4ac}}{2} x} + \frac{b - \sqrt{b^2 - 4ac}}{2} e^{-\frac{\sqrt{b^2 - 4ac}}{2} x}}{\left(e^{\frac{\sqrt{b^2 - 4ac}}{2} x} + Ke^{-\frac{\sqrt{b^2 - 4ac}}{2} x}\right)a} \]

Imposing the initial condition \( y(0) = 0 \)

\[ -\frac{\frac{b + \sqrt{b^2 - 4ac}}{2} + K\frac{b - \sqrt{b^2 - 4ac}}{2}}{(1 + K)a} = 0 \]
\[ K = \frac{-b + \sqrt{b^2 - 4ac}}{b - \sqrt{b^2 - 4ac}} \]

\[ y(x) = -\frac{\frac{b + \sqrt{b^2 - 4ac}}{2}}{\left(\frac{e^{\frac{b^2 - 4ac}{2ac}}}{e^{\frac{b^2 - 4ac}{2ac}}} - \frac{b + \sqrt{b^2 - 4ac}}{b - \sqrt{b^2 - 4ac}} \frac{e^{-\frac{b^2 - 4ac}{2ac}}}{e^{-\frac{b^2 - 4ac}{2ac}}}\right) a}
\]

\[ = -\frac{b + \sqrt{b^2 - 4ac}}{2a} \left(\frac{e^{\frac{b^2 - 4ac}{2ac}} - 1}{e^{\frac{b^2 - 4ac}{2ac}} - \frac{b + \sqrt{b^2 - 4ac}}{b - \sqrt{b^2 - 4ac}}}\right)
\]

\[ = \frac{e^{\frac{b^2 - 4ac}{2ac}} - 1}{2e^{\frac{b^2 - 4ac}{2ac}} + (-b + \sqrt{b^2 - 4ac})(e^{\frac{b^2 - 4ac}{2ac}} - 1)}
\]

In affine models literature the Riccati equation of the type (15) is often coupled with
the problem of the following form

\[ \frac{dz}{dx} = ly + m \]

It can be verified that the solution is given by

\[ z = \frac{l}{a} \ln \left(\frac{-2\sqrt{b^2 - 4ac}(-b + \sqrt{b^2 - 4ac})x/2}{2\sqrt{b^2 - 4ac} + (-b + \sqrt{b^2 - 4ac})(e^{\frac{b^2 - 4ac}{2ac}} - 1)}\right) + mx + C \]