We study the relation between information quality and equity premium in a production economy. We show that the precision of the public information has a negative relation with the size of equity premium of the economy. This result contradicts those obtained in pure exchange economies, for example, Veronesi ([18]) and Yan ([21]). In pure exchange economies, there is only price adjustment in response to innovations of information. In production economies, both price and quantity respond to innovations of information, it is the quantity adjustment channel makes our result different from that obtained in pure exchange economies. It is argued by some author, that the equity premium has declined over the last few decades. The results in this paper would provide a potential explanation of the fact, i.e. it is because of more precise information about the fundamentals of the economy.

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1. Introduction

Information about fundamentals of the economy affect equilibrium asset prices and equity returns by changing investors’ expectation of future returns of the asset. This paper studies the relation between the precision of public information about the uncertain growth rate of the economy and the equity return. We consider a continuous time version of neoclassical stochastic growth model. The stochastic growth rate of the technology consists of two parts, the persistent part and the white noise part. In the incomplete information case, the persistent part of the growth rate is not observable to the investors. Investors can deduce and revise their beliefs about the persistent part of growth by observing the aggregate production process and another source of public information of the growth rate.

The main result is higher precision of the public information decreases equity premium, that is, imprecise information requires a higher equity premium. This is in sharp contrast to the results obtained in pure exchange economies, for example, Veronesi ([18]) and Yan ([21]). The reason for this seemingly paradoxical result is that in pure exchange economies consumers cannot adjust quantities in response to new information. As a consequence, only price changes. In production economies both price and quantity could adjust to innovations in information. It is the quantity adjustment channel that makes the difference.

Our result demonstrates that if one is only interested in pricing the asset in the CAPM framework taking the equilibrium consumption process as given, then the assumption of pure exchange economy maybe completely innocuous. However, if we are interested in comparing equilibrium asset pricing outcomes of different economies, then pure exchange economy and production economy can be vastly different. The reason for this is that pure exchange economies only respond to changes in fundamentals by price adjustment, while production
economies respond by both price adjustment and quantity adjustment. This intuition is very similar to those in the habit persistence asset pricing literature, for example, Boldrin, Christiano and Fisher ([1], [2], and [3]) and Jermman ([12]). The lesson we learned from that literature is that adding habit persistence to the standard pure exchange economy significantly increase the equity premium generated by the model, while the contribution of habit persistence in increasing the model predictions of equity premium in the standard real business cycle model is tiny. The reason is quantity adjustment channel in the real business cycle models may significantly outweigh the price adjustment channel.

At a technical level, we construct an operator method to show existence of solution to the H-J-B equation of the stochastic dynamic programming problem, while standard theorems on existence of partial differential equations fail to apply. Our method provides an algorithm to numerically solve the problem, and is potentially applicable to other asset pricing problems. It does not involve solving any optimization problem, but only requires iteratively solving numerical integration problems until convergence.

This paper builds on the literature on incomplete information and learning in financial markets. Early studies on incomplete information and asset pricing in partial equilibrium framework includes Williams ([20]), Merton ([16]), among others. Dothan and Feldman([6]), Detemple ([5]) and Gennotte ([11]) are the first to introduce the separation principle and examine optimal portfolio choice problem in the presence of incomplete information in a fully dynamic framework. Veronesi ([18]) and Yan ([21]) considers the relation between information quality and equity premium in pure exchange economies, while we consider production economies with a stochastic AK technology. Cagetti et al ([4]) study asset pricing implications of incomplete information in a continuous time version of neoclassical growth model.
They focus on the effect of model uncertainty and concerns for robustness, and they do not address the relation between information quality and equity premium. Absence of concern for robustness, in the standard continuous time version of the neoclassical growth model, the return to capital becomes locally risk free, for example, Merton ([15]). Therefore this model does not produce an equity premium on the claim of the representative firm. The model we consider here can be viewed as a continuous time embedding of the time to plan assumption as considered by Boldrin, Christiano and Fisher ([3]) in a stochastic AK economy. Equity premium come from the fact that investment decision must be made before the realization of the productivity shock.

The paper is organized as follows: section 2 describes the physical environment of the economy, and defines a proper notion of competitive equilibrium of the economy. Section 3 characterizes the equilibrium asset prices and allocation in a complete information stochastic AK economy. Section 4 studies the same model with incomplete information. We prove the main result of the paper, which states that the equity premium in the incomplete information economy is higher than the complete information case. We use numerical result to illustrate that equity premium is decreasing in information quality. Section 5 concludes.

2. The Economy

Consider an infinite horizon economy populated by a continuum of investors with identical CRRA utility function given by:

\[ E[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt] \]  

(1)
where \( \gamma = 1 \) stands for log utility.

Let \( \{\Omega, \mathcal{F}^*, P\} \) be a given complete probability space and \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \) be a right-continuous filtration that is augmented in the usual way. Let \( \{B_{Kt}, B_{et}\}_{t \geq 0} \) be a standard Brownian motion with respect to \( \mathcal{F} \). Let \( \{\theta_t\}_{t \geq 0} \) be a two state Markov chain adapted to \( \mathcal{F} \) with state space \( \Theta = \{\theta_H, \theta_L\} \), where \( \theta_H > \theta_L \). The infinitesimal generating matrix of \( \{\theta_t\}_{t \geq 0} \) is

\[
\Lambda = \begin{bmatrix}
-\lambda & \lambda \\
\mu & -\mu
\end{bmatrix}
\]

(2)

that is, the transition probability matrix between time \( t \) and \( t + \Delta \) is

\[
\begin{bmatrix}
1 - \lambda \Delta & \lambda \Delta \\
\mu \Delta & 1 - \mu \Delta
\end{bmatrix}
\]

Let \( \mathcal{G} \subseteq \mathcal{F} \) be the filtration that represents the evolution of agents’ information. If \( \mathcal{G} = \mathcal{F} \), we call the economy a complete information economy, and if \( \mathcal{G} \nsubseteq \mathcal{F} \), the economy is one with incomplete information.

An aggregate output process \( \{y_t\}_{t \geq 0} \) is feasible if it is \( \mathcal{G} \) adapted, and \( \exists \) a capital stock process \( \{K_t\}_{t \geq 0} \) such that

1) 
\[
dK_t = K_t[\theta_t dt + \sigma dB_{Kt}] - y_t, \quad K_0 = k_0
\]

(3)

2): \( K_t \geq 0, \forall t \geq 0 \).

Here \( \theta_t \) is the expected growth rate of the economy. Fluctuations in \( \theta \) can be viewed as business cycles, \( \theta_H \) represents a boom and \( \theta_L \) represents a recession. The realized growth
rate of the economy is determined by $\theta_t$ and a small noise represented by Brownian motion $B_{Kt}$.

There is an exogenous signal process $\{e_t\}$, where $e_t$ is related to $\theta_t$ through:

$$de_t = \theta_t dt + \sigma dB_{et}, \quad e_0 = 0 \quad (4)$$

We consider an economy with a representative firm operating the technology as described by equation (3). The asset market structure is as follows: there is a stock, which is the claim to the representative firm. Since the production of the firm is the only source of consumption good in this economy, this effectively completes the market. We denote the stock price process $\{P_t\}$. The instantaneous dividend flow paid by the stock equals the output of the firm: $\{y_t\}$. Suppose the stock price is described by the stochastic differential equations of the following form (which will be verified by the equilibrium conditions):

$$dP_t = P_t[\mu_{Pt}dt + \sigma_{Pt}dW_t]$$

for some $\{W_t\}$, a standard Brownian motion with respect to the filtration $\{\mathcal{G}_t\}_{t \geq 0}$. The processes $\{\mu_{Pt}, \sigma_{Pt}, W_t\}$ are to be determined by equilibrium conditions. There is also a bond traded in the market. Bond allows agents to borrow and lend from each other at a locally risk-free interest rate, and the total supply of the bond is zero. Price of the bond is denoted by $b_t$, if one denote the locally risk-free interest rate by $r_t$, bond price satisfies the stochastic differential equation:

$$db_t = b_tr_t dt$$
Given the stock price process, the dividend process, one can define the cumulative return process of the stock:

\[
\frac{dR_t}{R_t} = (\mu_{Pt} + y_t)dt + \sigma_{Pt}dW_t
\]

We denote \(\mu_{Rt} = \mu_{Pt} + y_t\), hence \(\mu_{Rt} - r_t\) is the instantaneous risk premium on the stock.

**Definition: Competative Equilibrium:**

A competitive equilibrium of the economy consists of asset price processes \(\{P_t, b_t\}_{t \geq 0}\), a dividend process \(\{y_t\}\), a state price process \(\{\xi_t\}\). Consumption and portfolio holding strategy \(\{c^*_t, \alpha^*_t\}\) such that:

1) Household’s problem:

\[
\{c^*_t, \alpha^*_t\} \in \arg \max \{c_t, \alpha_t\} E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] \\
dw_t = w_t \left[ \frac{\alpha_t}{R_t} dR_t + \frac{1 - \alpha_t}{b_t} db_t \right] - c_t dt \\
w_0 = k_0, \ w_t \geq 0, \ c_t \geq 0
\]

2) Firm’s problem:

\[
\{y^*_t\} \in \arg \max E \left[ \int_0^\infty \xi_t y_t dt - K_0 \right] \\
dK_t = K_t \left[ \theta_t dt + \sigma dB_{Kt} \right] - y_t dt \\
K_0 = k_0, \ K_t \geq 0
\]

3) Market Clearing:

\(\forall t \geq 0, \ c_t = y_t, \ \alpha_t = 1\)
4) No arbitrage:
\[ P_t = \frac{1}{\xi_t} E \left[ \int_t^\infty \xi_{t+s} y_{t+s} ds | G_t \right] \]

5) Measurability: \( \{ P_t, b_t, \xi_t, y_t, c_t, \alpha_t \}_{t \geq 0} \) are \( G \)-adapted.

The first three requirements are standard in the definition of competitive equilibrium. However, I used two different price systems in household and firm’s problem. This is because, the sequential specification of household’s budget constraint allow us to pin down the equity return and premium explicitly, while the firm’s problem is more conveniently modeled using state price processes. Therefore the fourth condition requires that the two price systems are consistent with each other. The last condition requires that all equilibrium quantity and prices to be measurable with respect to agents’ information filtration.

**3. The Complete Information Economy**

In this section, we consider the equilibrium allocation and asset prices in a complete information economy, i.e. \( \mathcal{F} = \mathcal{G} \). The strategy of solving the equilibrium allocation and asset prices is standard. I first solve the social planner’s problem for the allocation, and use the allocation to calculate equilibrium asset prices. The following conditions guarantees that the discounted utility of the representative agent is finite for any feasible consumption-saving plan and is assumed throughout:

**Assumption 1:**
\[ \rho - (1 - \gamma) \theta + \frac{1}{2} \gamma (1 - \gamma) \sigma^2 > 0 \]  \( (5) \)

for \( \theta = \theta_H, \theta_L \).
The social planner’s problem is:

$$\max_{c_t \geq 0} E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]$$

s.t.  
$$dK_t = K_t[\theta_t dt + \sigma_K dB_K] - c_t dt$$

$$\theta_t \sim \Lambda, \quad K_0, \theta_0 \text{ given.}$$

Under assumption 1, define the value function to be:

$$V(K, \theta) = \max_{c_t \geq 0} E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]$$  \hspace{1cm} (6)

s.t.  
$$dK_t = K_t[\theta_t dt + \sigma_K dB_K] - c_t dt, \quad K_0 = K$$  \hspace{1cm} (7)

$$\theta_t \sim \Lambda, \quad \theta_0 = \theta$$

The following theorem characterizes the solution to the social planner’s problem for the complete information case:

**Theorem 3.1: Social Planner’s Problem, Complete Information Case**

1)  

$$V(K, \theta) = H(\theta) \frac{K^{1-\gamma}}{1-\gamma}$$

where $H(\theta_H)$ and $H(\theta_L)$ are given by the unique solution to the following set of equations:

$$\gamma H(\theta_H)^{1-\frac{1}{\gamma}} + \lambda [H(\theta_L) - H(\theta_H)]$$

$$+ [(1 - \gamma)\theta_H - \frac{1}{2} \gamma(1 - \gamma)\sigma^2 - \rho] H(\theta_H) = 0$$  \hspace{1cm} (8)
\[ \gamma H(\theta_L)^{1-\frac{1}{\gamma}} + \lambda [H(\theta_H) - H(\theta_L)] + [(1 - \gamma)\theta_L - \frac{1}{2}\gamma(1 - \gamma)\sigma^2 - \rho]H(\theta_L) = 0 \]  

(9)

2) The optimal consumption is given by:

\[ c(K, \theta) = H(\theta)^{-\frac{1}{\gamma}}K \]  

(10)

for \( \theta = \theta_H, \theta_L \).

3) \( H(\theta_H) > H(\theta_L) \) if \( \gamma < 1 \) and \( H(\theta_H) < H(\theta_L) \) if \( \gamma > 1 \).

**Proof:** See Appendix A.

The above theorem says that the value function is homogeneous of degree \( 1 - \gamma \) in \( K \). \( H(\theta_H) \) and \( H(\theta_L) \) is given by a system of nonlinear equations, i.e. (8) and (9). Second part of the theorem says optimal control is stationary and markov. Further, it is homogeneous of degree 1 in \( K \). The third part of the theorem is also intuitive, it is equivalent to saying that the value function is strictly increasing in \( \theta \). I will denote

\[ \hat{c}(\theta) = \frac{c(K, \theta)}{K} = H(\theta)^{-\frac{1}{\gamma}} \]

\( \theta_t \) is the growth rate of the technology in this economy, \( \hat{c}(\theta_t) \) is the consumption rate, and \( \theta_t - \hat{c}(\theta_t) \) is the saving rate. The above theorem implies the following properties of the equilibrium consumption and saving rate of the economy:

**Corollary 3.1:** \( \hat{c}(\theta_H) > \hat{c}(\theta_L) \) if \( \gamma > 1 \), and \( \hat{c}(\theta_H) < \hat{c}(\theta_L) \) if \( \gamma < 1 \).

**Proof:** Straightforward from part 3) of theorem 3.1.
Corollary 3.2:

\[ \theta_H - \mathcal{c}(\theta_H) > \theta_L - \mathcal{c}(\theta_L) \]  

Proof: See Appendix A.

Corollary 2.1 says in the case of \( \gamma > 1 \), the optimal consumption policy requires consuming a higher fraction of \( K_t \) when the state is high, and a lower proportion of \( K_t \) when the state is low. Intuitively, this can be explained by the inter-temporal income effect and substitution effect. Note that \( \gamma \) is the inverse of inter-temporal elasticity of substitution of the representative agent. Higher \( \gamma \) corresponds to low willingness to substitute today’s consumption with tomorrow’s. For the representative consumer, if \( \theta \) is high, income effect creates an incentive to consume more today, while substitution effect creates an incentive to save more. In the case of \( \gamma = 1 \), i.e. log utility, the two effect cancels each other exactly. When \( \gamma > 1 \), income effect dominates, therefore the optimal policy is to consume more if the state is high. If \( \gamma < 1 \), we have the exact opposite case.

Corollary 2.2 says saving rate is always higher in booms than in recessions. If \( \gamma < 1 \), this is obvious, because the optimal consumption policy is such that agent always consume a lower fraction of total stock of capital when state is high. If \( \gamma > 1 \), Corollary 2.2 implies that although the optimal consumption policy respond to the state of the economy, it does not respond one for one; as a result, the saving rate is still higher in booms than in recessions.

With the allocation of the economy solved, we can now solve for the equilibrium price system of the economy:

Theorem 3.2: Equilibrium Price
The price of the stock is given by:

\[
\forall t, \quad dP_t = P_t[(\theta_t - H(\theta_t)^{-\frac{1}{2}})dt + \sigma_K dB_{Kt}], \quad P_0 = K_0
\] (12)

and the cumulative return process of the stock is:

\[
\frac{dR_t}{R_t} = \theta_t dt + \sigma_K dB_{Kt}
\] (13)

The proof is similar to that of theorem 4.4 and is omitted here. The above result is rather standard, i.e. the date \( t \) price of the claim to the firm in terms of date \( t \) consumption good is equal to the capital stock of the firm. This is a direct implication of the technology of the economy: one unit of consumption good can be transformed into one unit of capital stock perfectly elastically.

**Theorem 3.3: Equity Premium, Complete Information Economy**

The equilibrium equity premium of the above economy is given by:

\[
\mu_{rt} - r_t = \gamma \sigma^2
\] (14)

The proof of this theorem is similar to that of theorem 4.5 and is therefore omitted here. The above result can be seen in the following way. Equity premium of the stock is given by:

\[
\mu_{rt} - r_t = -cov_t[\frac{d\xi_t}{\xi_t}, \frac{dR_t}{R_t}]
\] (15)
In our case, \( \xi_t = e^{-\rho t}c_t^{-\gamma} = e^{-\rho t}H(\theta_t)K_t^{-\gamma} \). Using Ito’s formula, we have:

\[
d\xi_t = A[\xi_t]dt - \xi_t \gamma \sigma_K dB_t
\]

In the above expression, \( A[\xi_t] \) denote the drift term of \( d\xi_t \). Note in this case, \( \theta_t \) is independent of the innovations of \( K_t \), therefore innovations of \( \xi_t \) is correlated with innovations of \( K_t \) only through the term \( K_t^{-\gamma} \). We have:

\[
\mu_{Rt} - r_t = -\text{cov}_t \left[ \frac{d[e^{-\rho t}K_t^{-\gamma}]}{e^{-\rho t}K_t^{-\gamma}}, \frac{dK_t}{K_t} \right] = \gamma \sigma_K^2
\]

Having completely characterized the price and allocation of the complete information economy, I now turn to the incomplete information economy.

**4. Incomplete Information Economy**

Now consider the case in which \( \{\theta_t\}_{t \geq 0} \) is not observable. Again, we first consider the social planner’s problem to solve for the allocation of the economy. The social planner’s problem in this case is:

\[
\max_{c_t \geq 0, \alpha_t \in \Phi} E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] \quad (16)
\]

s.t. \( dK_t = K_t[\theta_t dt + \sigma_K dB_t] - c_t dt, \quad K_0 = k_0 \)

\[
\theta_t \sim \Lambda
\]

\[
dc_t = \theta_t dt + dB_{ct}, \quad c_0 = 0
\]
where $\mathcal{G}_t = \sigma\{K_s, e_s : 0 \leq s \leq t\}$ is the filtration generated by the $\{K_t, e_t\}_{t \geq 0}$ processes. Note in the optimization problem, the control is required to be adapted to $\mathcal{G}$, that is, consumption policy can only depend on the observables, $\{K_t, e_t\}_{t \geq 0}$.

The solution to the optimal control problem in (16) relies on a separation principle described for example in ([9]), and ([8]). The basic procedure is as follows, first solve a statistical inference problem and deduce the conditional distribution of $\theta_t$ given $\mathcal{G}_t$. Next, solve a dynamic programming problem using the conditional distribution of $\theta_t$ as a state variable. The assumption that $\{\theta_t\}$ is a two state Markov chain implies that the conditional distribution of $\theta_t$ is summarized by one state variable, i.e. the conditional probability of $\theta_t$ being at the high state. In principle, one can deal with the case where $\{\theta_t\}$ is a finite state Markov chain, in which case the conditional distribution of $\theta_t$ given data is a finite dimensional vector. The fact that the second stage optimization problem has a recursive structure and thus can be solved by dynamic programming techniques relies on the conditional distribution of finite dimensional vector of conditional probabilities given data being Markov. In this case, $\theta_t$ is a two state Markov chain, therefore $\pi_t = P(\theta_t = \theta_H|\mathcal{G}_t)$ serve as a sufficient statistic of the conditional distribution of $\theta_t$. It is convenient to denote $m(\pi)$ to be the conditional mean of $\theta$ given $\pi$, i.e.

$$m(\pi) = \pi \theta_H + (1 - \pi) \theta_L = E[\theta|\pi]$$

**Lemma 4.1: The Filtering Problem:**

1) The stochastic differential equation that governs the law of motion of $\pi_t$ is given
by:

\[ d\pi_t = [\mu - (\lambda + \mu)\pi_t]dt \] (17)

\[ +\pi_t(1 - \pi_t)(\theta_H - \theta_L)(\sigma^{-1}d\tilde{B}_{Kt} + \sigma_e^{-1}d\tilde{B}_{et}) \]

where

\[ d\tilde{B}_{Kt} = \frac{1}{K_t\sigma}[dK_t - (K_t m(\pi_t) - c_t)dt] \] (18)

\[ d\tilde{B}_{et} = \frac{1}{\sigma_e}[de_t - m(\pi_t)dt] \] (19)

2) \{\tilde{B}_{Kt}, \tilde{B}_{et}\} are independent standard Brownian motions with respect to \( G \).

3) \( \pi_t \) has a unique invariant distribution. The density of the invariant distribution, denoted by \( p \), is given by:

\[ \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}[x^2(1 - x)^2p(x)] - \frac{\partial}{\partial x}[(\mu - (\lambda + \mu)x)p(x)] = 0 \] (20)

with the boundary condition \( p(0) = p(1) = 0 \).

**Proof:** For the first two parts of the lemma, see Lipster and Shiryayev ([14]), theorem 9.1, page 333. For part 3), see Kunita ([13]).

Given the law of motion of \( \{\pi_t\}_{t \geq 0} \), the second stage optimization problem can be
formulated as:

$$\max_{c_t \geq 0} E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]$$

s.t. $dK_t = K_t[m(\pi_t)dt + \sigma_K \tilde{B}_{Kt}] - c_t dt$

$$d\pi_t = [\mu - (\lambda + \mu)\pi_t]dt$$

$$+ \pi_t(1 - \pi_t)(\theta_H - \theta_L)(\sigma^{-1}d\tilde{B}_{Kt} + \sigma^{-1}d\tilde{B}_{et})$$

$$K_t \geq 0, \forall t \geq 0$$

$$K_0 = K, \pi_0 = \pi$$

Under assumption 1, define:

$$V(K, \pi) = \max_{c_t \geq 0} E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]$$

s.t. $dK_t = K_t[m(\pi_t)dt + \sigma_K \tilde{B}_{Kt}] - c_t dt$

$$d\pi_t = [\mu - (\lambda + \mu)\pi_t]dt$$

$$+ \pi_t(1 - \pi_t)(\theta_H - \theta_L)(\sigma^{-1}d\tilde{B}_{Kt} + \sigma^{-1}d\tilde{B}_{et})$$

$$K_t \geq 0, \forall t \geq 0$$

$$K_0 = K, \pi_0 = \pi$$

And we conjecture $V(K, \pi)$ is of the form:

$$V(K, \pi) = H(\pi) \frac{K^{1-\gamma}}{1-\gamma}$$

which is to be verified later. The $H(\pi)$ function is characterized by the theorem below:
Theorem 4.1: Social Planner’s Problem, Incomplete Information case.

Suppose the following differential equation (23) has a smooth solution on $[0, 1]$,

$$
\gamma H(\pi)^{1-\frac{\gamma}{\gamma}} + [(1 - \gamma)m(\pi) - \frac{1}{2}\gamma(1 - \gamma)\sigma^2 - \rho]H(\pi)
$$

$$
+ \left[\mu - (\lambda + \mu)\pi + (1 - \gamma)\pi(1 - \pi)(\theta_H - \theta_L)\right]H'(\pi)
$$

$$
\frac{1}{2}\pi^2(1 - \pi)^2(\theta_H - \theta_L)^2(\sigma_K^{-2} + \sigma_e^{-2})H''(\pi) = 0
$$

that satisfies

$$
\forall \pi \in [0, 1], \lim_{T \to \infty} e^{-\rho T}E_\pi[H(\pi_T)^{1-\gamma}] = 0
$$

where $E_\pi$ denote the conditional expectation is taken with respect to the initial condition $\pi_0 = \pi$, then

1) The value function of the dynamic programming problem (21) exists and is $C^2$. The conjecture in (22) is true, and the $H$ function is the unique function defined on $[0, 1]$ that satisfies (23) and the transversality condition (24).

2) The optimal consumption policy function is given by:

$$
c(K, \pi) = H(\pi)^{-\frac{1}{\gamma}}K
$$

3) $H(\pi)$ is strictly increasing if $\gamma < 1$ and strictly decreasing if $\gamma > 1$.

**Proof:** See Appendix B.

Recall in the complete information case, the $H$ function is defined by a system of nonlinear equations. Here $H$ is defined by a differential equation. This is because in this case, the state variable $\pi$ is a diffusion process, in particular, it has continuous path. Therefore,
optimality pins down the local behavior of $H$. Once the existence of solution to (23) and (24) is established, the function defined in (22) must be the value function of the dynamic programming problem (21) by the verification theorem (for example, Fleming and Soner ([10])). Uniqueness of value function implies uniqueness of solution to (23) and (24).

Equation (25) is the optimal policy function of the agent. If $\gamma > 1$, then $H$ is strictly decreasing, this implies optimal consumption as a fraction of $K_t$ is strictly increasing in $\pi_t$. If $\gamma < 1$, $\frac{c_t}{K_t}$ is strictly decreasing in $\pi_t$. This can be explained by the substitution and income effect of an increase in $\pi_t$. The substitution effect make agents consume less and save more when there is an increase in $\pi_t$; while income effect make agents consume more and save less. When $\gamma > 1$, intertemporal elasticity of substitution is low, therefore income effect dominates, and $\frac{c_t}{K_t}$ is increasing in $\pi_t$. When $\gamma < 1$, substitution effect dominates therefore $\frac{c_t}{K_t}$ is decreasing in $\pi_t$.

The following theorem establishes the existence of solution to (23). The proof is presented here because it is interesting in its own right. First, standard existence theorem of solution to ODE’s usually requires Lipchitz conditions, which is not satisfied in this case. Second, the proof constructs the solution to (23) as the fixed point to certain mappings, it provided a natural way of proving further properties of the $H$ function. Finally, it provides a numerical algorithm of computing the $H$ function. Numerical solution to dynamic programming problems like (21) generally requires discretizing the state space and solving an optimization problem at each step, which can be quite involved. The algorithm provided here does not involve solving any optimization problem. The theorem considers only the case $\gamma > 1$. The case $\gamma < 1$ is more involved and is left to future research.

**Theorem 4.2: Existence of Solution.**
Suppose assumption 1 holds, and $\gamma > 1$, then (23) has a unique solution on $[0, 1]$.

**Proof:**

Define

$$\eta_H = \left\{ \frac{\gamma}{\rho - (1 - \gamma)\theta_H + \frac{1}{2}\gamma(1 - \gamma)\sigma^2} \right\}^\gamma$$

(26)

and

$$\eta_L = \left\{ \frac{\gamma}{\rho - (1 - \gamma)\theta_L + \frac{1}{2}\gamma(1 - \gamma)\sigma^2} \right\}^\gamma$$

(27)

Since $\gamma > 1$, we have $\eta_L > \eta_H$. Let $B[\eta_H, \eta_L]$ denote the set of continuous functions on $[0, 1]$ bounded above and below by $\eta_L$, and $\eta_H$, respectively. I will prove $\exists f \in B[\eta_H, \eta_L]$ that satisfies (23).

Define a mapping $T : B[\eta_H, \eta_L] \to B[\eta_H, \eta_L]$, such that

$$Tf(\pi) = \gamma \cdot E_\pi \left\{ \int_0^\infty e^{[-\rho - \frac{1}{2}\gamma(1-\gamma)\sigma^2]t + (1 - \gamma)\int_0^t m_s ds} f(\pi_t)^{1-\frac{1}{\gamma}} dt \right\}$$

(28)

where

$$d\pi_t = [\mu - (\lambda + \mu)\pi_t + (1 - \gamma)\pi_t(1 - \pi_t)(\theta_H - \theta_L)]dt + \pi_t(1 - \pi_t)(\theta_H - \theta_L)\sqrt{\sigma^2 + \sigma_e^2}dW_t$$

(29)

In the above definition, $\{W_t\}_{t \geq 0}$ is a standard Brownian motion defined on some probability space, and $E_\pi$ denote that the expectation is taken such that the initial condition of the stochastic differential equation (29) is $\pi_0 = \pi$. As above $m$ is given by (20), The Feynman-Kac formula implies that if $H$ is a fixed point of the mapping $T$, then $H$ satisfies (23). Hence we
need to prove $T$ has a fixed point.

First, $T$ is monotone. This is obvious, since $f_1 \geq f_2$ implies $f_1^{1-\frac{1}{\gamma}} \geq f_2^{1-\frac{1}{\gamma}}$, therefore

$$\gamma \cdot E_m \{ \int_0^\infty e^{-\rho-\frac{1}{2}\gamma(1-\gamma)\sigma^2}t+(1-\gamma)\int_0^t \theta ds f_1^{1-\frac{1}{\gamma}} dt \} $$

$$\geq \gamma \cdot E_m \{ \int_0^\infty e^{-\rho-\frac{1}{2}\gamma(1-\gamma)\sigma^2}t+(1-\gamma)\int_0^t \theta ds f_2^{1-\frac{1}{\gamma}} dt \} $$

Next, $T$ maps $B[\eta_H, \eta_L]$ into $B[\eta_H, \eta_L]$. Suppose $f \leq \eta_L$, then

$$Tf \leq \gamma E_m[ \int_0^\infty e^{-\rho-\frac{1}{2}\gamma(1-\gamma)\sigma^2}t+(1-\gamma)\int_0^t \theta ds \eta_L^{1-\frac{1}{\gamma}} dt ] $$

$$\leq \gamma E_m[ \int_0^\infty e^{-\rho-\frac{1}{2}\gamma(1-\gamma)\sigma^2}t+(1-\gamma)\int_0^t \theta_L ds \eta_L^{1-\frac{1}{\gamma}} dt ] $$

$$= \eta_L$$

The second line is true because $\gamma > 1$, the last line is true by definition of $\eta_L$. Similarly, $f \geq \eta_H$ implies $Tf \geq \eta_H$.

Since $T$ is monotone, one can construct the fixed point as follows: start with $f$ such that

$$\forall \pi \in [0, 1], \quad f(\pi) = \eta_L$$

Consider the sequence $\{T^nf\}_{n=0}^\infty$, since $f_1 \leq \eta_L$, we have $Tf \leq \eta_L = f$. Therefore $T^nf$ is a decreasing sequence of functions on $B[\eta_H, \eta_L]$. It must have a limit in $B[\eta_H, \eta_L]$, call the
limit $T^\infty f$. To prove $T^\infty f$ is the fixed point of $T$, Only need to show

$$
\lim_{n \to \infty} E[\int_0^\infty e^{[-\rho - \frac{1}{2}(1-\gamma)\sigma^2]}t + (1-\gamma) \int_0^t m_s ds (T^n f)^{1-\frac{1}{\gamma}}(\pi_t) dt] \\
E[\int_0^\infty e^{[-\rho - \frac{1}{2}(1-\gamma)\sigma^2]}t + (1-\gamma) \int_0^t m_s ds \lim_{n \to \infty} (T^n f)^{1-\frac{1}{\gamma}}(\pi_t) dt]
$$

This is true since

$$
e^{[-\rho - \frac{1}{2}(1-\gamma)\sigma^2]}t + (1-\gamma) \int_0^t m_s ds \int_0^\infty f_n^{1-\frac{1}{\gamma}}
$$

is dominated by the integrable function $e^{[-\rho - \frac{1}{2}(1-\gamma)\sigma^2]}t + (1-\gamma) \theta_L t^{1-\frac{1}{\gamma}} \eta_L$.

The above proof suggests a natural way of computing the $H$ function numerically, namely start with $f = \eta_L$, and iterate the integral in (28) until $T^n f$ converges.

**Theorem 4.3: Monotonicity of $H$.**

Under the assumption of Theorem 4.1, $H$ is strictly decreasing if $\gamma > 1$ and strictly increasing if $\gamma < 1$.

**Proof:** See Appendix B.

**Theorem 4.4 Equilibrium Price**

The stock price of the economy is given by:

$$dP_t = P_t [(m_t - H(\pi_t)^{-\frac{1}{\gamma}}) dt + \sigma_K d\tilde{B}_{Kt}], \quad P_0 = k_0$$

(30)

and the cumulative return process of the stock is given by:

$$\frac{dR_t}{R_t} = m(\pi_t) dt + \sigma_K d\tilde{B}_{Kt}$$

(31)
Again, the result is a standard one, it simply says $P_t = K_t$, i.e. the value of the firm at date $t$ in terms of date $t$ consumption good is equal to its capital stock. To understand equation (31), note in the equilibrium $c_t = y_t$, i.e consumption equal to output, which is the dividend paid by the stock. Therefore,

$$\frac{dR_t}{R_t} = \frac{1}{P_t}[dP_t + y_t dt] = m(\pi_t)dt + \sigma_K d\tilde{B}_t$$

(32)

**Theorem 4.5: Equity Premium**

The equity premium in the incomplete information economy is given by:

$$\mu_{Rt} - r_t = \gamma \sigma^2 - \pi_t(1 - \pi_t)(\theta_H - \theta_L)\frac{H'(\pi_t)}{H(\pi_t)}$$

**Proof:** Appendix B.

It is clear from theorem 4.3 and theorem 4.4 that the equity premium in the incomplete information economy is higher than that of the complete information if and only if $\gamma > 1$. This result is in sharp contrast with that obtained in pure exchange economies, for example Veronesi ([18]). In pure exchange economies, equity premium in the incomplete information economy is higher than that of complete information economy if and only if $\gamma < 1$. This result is actually robust to different specifications of the information structure, for example the case where $\{\theta_t\}_{t\geq 0}$ is an Ornstein-Ulenbeck process, or the case where investors have advanced information as was studied in Pikovski and Karatzas([17])\(^1\). In the incomplete information case, equity premium is a function of $\pi_t$. The density of the stationary distribution of $\pi_t$ is

\(^1\)Proof of results in this case is available from author upon request.
given by (20), therefore the long-run average equity premium, which will be denoted by $\mu^*$ of the economy is given by:

$$\mu^* = \gamma \sigma^2 K - \int_0^1 \pi (1 - \pi)(\theta_H - \theta_L) \frac{H'(\pi)}{H(\pi)} p(\pi) d\pi$$

where $p(\cdot)$ is given by (20). From theorem 4.5 and the differentiability of the $H$ function with respect to $\sigma_e$, we know

$$\frac{\partial}{\partial \sigma_e} \mu^* \bigg|_{\sigma_e=0} \geq 0 \quad (33)$$

that is, the equity premium is locally non-increasing with respect to information quality at $\sigma_e = 0$. We conjecture (33) is true for all values of $\sigma_e$, however we are not able to prove it analytically. The numerical results illustrate that this is true for the parameter values chosen.

To compare the results of pure exchange economy with that of production economy, let us first review briefly the results in the pure exchange economy studies by Veronesi ([18]). Consider an economy in which representative consumer have CRRA preference as given in (1). The dividend process of the stock is

$$dD_t = D_t[\theta_t dt + \sigma dB_{Dt}]$$

where $\{\theta_t\}$ follows a two state Markov chain as was described in (2). In the complete information case, the price of the stock is given by:

$$P_t = C^*(\theta_t)D_t$$
where

\[ C^*(\theta_H) = \frac{\mu + \lambda + \phi_L}{(\lambda + \phi_H)(\mu + \phi_L) - \mu \lambda} \]  
(34)

\[ C^*(\theta_L) = \frac{\mu + \lambda + \phi_H}{(\lambda + \phi_H)(\mu + \phi_L) - \mu \lambda} \]  
(35)

and where

\[ \phi_H = (\gamma - 1)\theta_H + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 + \rho \]  
(36)

\[ \phi_L = (\gamma - 1)\theta_L + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 + \rho \]  
(37)

In the case \( \{\theta_t\} \) is not observable, agents deduce the conditional distribution of \( \theta_t \) based on the observation of \( D_t \) and \( e_t \), whose law of motion is given in (4). Stock price is given by

\[ P_t = E[C^*(\theta_t)|\mathcal{F}^{D,e}_t]D_t = C(\pi_t)D_t \]

where I denote \( C(\pi_t) = E[C^*(\theta_t)|\mathcal{F}^{D,e}_t] \).

Note that the state price of this economy is

\[ \xi_t = e^{-\rho_t}D_t^{-\gamma} \]

Using (15), the risk premium in pure exchange economy, complete information case is therefore given by:

\[ \mu_{Rt} - r_t = -\text{cov}_t\left[ \frac{d\xi_t}{\xi_t}, \frac{dR_t}{R_t} \right] = -\text{cov}_t\left[ \frac{d[e^{-\rho_t}D_t^{-\gamma}]}{e^{-\rho_t}D_t^{-\gamma}}, \frac{d[C^*(\theta_t)D_t]}{C^*(\theta_t)D_t} \right] \]  
(38)

In the complete information case, \( \theta_t \) is known, and is independent of the innovations in \( D_t \),
therefore

\[ \mu_{Rt} - r_t = -\text{cov}_t \left[ \frac{d[e^{-\rho t} D_t^{-\gamma}]}{e^{-\rho t} D_t^{-\gamma}}, \frac{d[D_t]}{D_t} \right] = \gamma \sigma^2 \]

In case of incomplete information, we have:

\[ \mu_{Rt} - r_t = -\text{cov}_t \left[ \frac{d\xi_t}{\xi_t}, \frac{dR_t}{R_t} \right] = -\text{cov}_t \left[ \frac{d[e^{-\rho t} D_t^{-\gamma}]}{e^{-\rho t} D_t^{-\gamma}}, \frac{d[C(\pi_t) D_t]}{C(\theta_t) D_t} \right] \]

the innovation in \( \pi_t \) is positively correlated with that of \( \frac{dD_t}{D_t} \). When \( \gamma > 1 \), inter-temporal elasticity of substitution is low, therefore hedging demand is high when agent expect future consumption is low. Hedging demand drive up the stock price when the state \( \pi_t \) is low. This implies \( C(\cdot) \) is a decreasing function. Therefore the presence of incomplete information dampens the negative correlation in equation (38), which implies that equity premium is low in the incomplete information case. Here incomplete information affect equity premium completely through price adjustment.

In the production economy we considered in this paper, the equation (15) is still true. However, the presence of incomplete information changes investors optimal consumption policy, therefore changes the pricing kernel \( \xi_t \). It also affect the price and dividend paid off by the stock, however, the cumulative return process of the stock is not affected by the information structure (see (32)); It is completely determined by technology. Of course, the conditional distribution of the cumulative return process does change with information structure. Note in the production economy, we have (see theorem 3.1):

\[ \xi_t = e^{-\rho t} C_t^{-\gamma} = e^{-\rho t} H(\pi_t) K_t^{-\gamma} \]
Intuition of the result in complete information case is already discussed in the remark after theorem 3.3. Here I only consider the case of incomplete information. We have:

$$\mu_{\text{det}} - r_t = -\text{cov}_t \left[ \frac{d\xi_t}{\xi_t}, \frac{dR_t}{R_t} \right] = -\text{cov}_t \left[ \frac{d[e^{-\rho t}H(\pi_t)K_t^{-\gamma}]}{e^{-\rho t}H(\pi_t)K_t^{-\gamma}}, \frac{dK_t}{K_t} \right]$$  \hspace{1cm} (39)

where the $H$ function is defined in (23). Again, $\pi_t$ and $\frac{dK_t}{K_t}$ are positively correlated. Also, as was explained in the remark after theorem 3.1, if $\gamma > 1$, substitution effect dominates, and $H(\pi_t)$ is a decreasing function. Hence the presence of incomplete information enhances the negative correlation between the two terms in (15). Therefore in incomplete information economies, the equity premium is larger if $\gamma > 1$. As information contained in $e_t$ gets less precise, agents rely more on the $\{K_t\}$ process to infer information about $\theta_t$, therefore $\pi_t$ and $\frac{dK_t}{K_t}$ are more positively correlated, we should expect the equity premium is higher. However, changes in information quality also change the shape of the $H$ function, which complicate the result. We demonstrate the relation between equity premium and information quality by numerical results. Note that in production economies, the relation between information quality and equity premium is driven by the way optimal consumption respond to innovations in information, i.e. the quantity adjustment channel.

**NUMERICAL RESULTS TO BE INSERTED HERE.**

5. Conclusion

We study the relation between the precision of public information and the size of equity premium in a simple version of the neoclass growth economy. Our result differs from those obtained in pure exchange economy settings. We identify the importance of quantity
adjustment channel in affecting the pricing kernel of the economy and therefore the equity premium of the economy, a mechanism that is absent in pure exchange economies. Our result provide a potential explanation of the declining equity premium over the last few decades, that is, it is because the public information about the fundamentals of the economy is more precise.

References


Appendix

A1. Appendix A. Proof of theorem 3.1 and theorem 3.2

1. Theorem 3.1

The H-J-B equation of the optimization problem in (6) is:

\[ V(K, \theta) = \max_{c \geq 0} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + A(c)V(K, \theta) \right\} \quad (A1) \]

where

\[ A(c)V(K, \theta) = \Lambda V(K, \theta) + \frac{1}{2}K^2 \sigma^2 V_{kk}(K, \theta) + (K \theta - c)V_k \]

(\text{A2})

Because the objective function is homogenous of degree $1 - \gamma$, and constraint is homogenous of degree 1, we know the function $V$ is homogenous of degree $1 - \gamma$. Therefore we write:

\[ V(K, \theta) = H(\theta) \frac{K^{1-\gamma}}{1-\gamma} \quad (A3) \]

Therefore (A1) is written as:

\[ \rho H(\theta) \frac{K^{1-\gamma}}{1-\gamma} = \max_{c \geq 0} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \Lambda H(\theta) \frac{K^{1-\gamma}}{1-\gamma} \right\} \]

\[ -\frac{1}{2} \gamma^2 H(\theta)K^{1-\gamma} + (K \theta - c)K^{-\gamma}H(\theta) \}

\[ \quad (A4) \]

Assuming interiority of solution (which can be verified later), and taking first order condition with respect to $c$, we have:

\[ c^{-\gamma} = K^{-\gamma}H(\theta) \]
or

\[ c = [H(\theta)]^{-\frac{1}{\gamma}} K \]  

(A5)

Plug (A5) back into equation (A4), cancelling terms that involves \( K^{1-\gamma} \) on both sides of the equation, and after some rearrangement, we obtain:

\[
\rho H(\theta) = \gamma [H(\theta)]^{1 - \frac{1}{\gamma}} + \Lambda H(\theta) - \frac{1}{2} \gamma (1 - \gamma) \sigma^2 H(\theta) + \theta H(\theta)(1 - \gamma)
\]

(A6)

Since \( \Lambda \) is a \( 2 \times 2 \) matrix, (A6) is a system of two nonlinear equations in two unknowns (that is, \( H(\theta_H) \), and \( H(\theta_L) \)).

(A6) is written as:

\[
\gamma H(\theta_H)^{1 - \frac{1}{\gamma}} + \lambda [H(\theta_L) - H(\theta_H)]
\]
\[+ [(1 - \gamma) \theta_H - \frac{1}{2} \gamma (1 - \gamma) \sigma^2 - \rho] H(\theta_H) = 0 \]

(A7)

\[
\gamma H(\theta_L)^{1 - \frac{1}{\gamma}} + \lambda [H(\theta_H) - H(\theta_L)]
\]
\[+ [(1 - \gamma) \theta_L - \frac{1}{2} \gamma (1 - \gamma) \sigma^2 - \rho] H(\theta_L) = 0 \]

(A8)

To see (A7) and (A8) admits a unique solution, write (A7) and (A8) as:

\[
\gamma H(\theta_H)^{\frac{1}{\gamma}} = \lambda (1 - \phi^{-1}) + \rho + \frac{1}{2} \gamma (1 - \gamma) \sigma^2 + (\gamma - 1) \theta_H
\]

(A9)

\[
\gamma H(\theta_L)^{\frac{1}{\gamma}} = \mu (1 - \phi) + \rho + \frac{1}{2} \gamma (1 - \gamma) \sigma^2 + (\gamma - 1) \theta_L
\]

(A10)
and define:

\[ \phi = \frac{H(\theta_H)}{H(\theta_L)} \]

Deviding both sides of (A9) and (A10), we have:

\[ \phi^{-\frac{1}{\gamma}} = \frac{\lambda(1 - \phi^{-1}) + \rho + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 + (\gamma - 1)\theta_H}{\mu(1 - \phi) + \rho + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 + (\gamma - 1)\theta_L} \]  

(A11)

It is enough to show the following:

1) If \( \gamma > 1 \), then (A11) has a unique solution on \((\phi_1, 1)\).

2) If \( \gamma < 1 \), then (A11) has a unique solution on \((1, \phi_2)\), where

\[ \phi_1 = \sqrt{\frac{[\lambda - \mu + (\gamma - 1)(\theta_H - \theta_L)]^2 + 4\lambda\mu - [\lambda - \mu + (\gamma - 1)(\theta_H - \theta_L)]}{2\mu}} \]

\[ \phi_2 = 1 + \frac{1}{\mu}[\rho + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 + (\gamma - 1)\theta_L] \]

Denote

\[ R(\phi) = \frac{\lambda(1 - \phi^{-1}) + \rho + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 + (\gamma - 1)\theta_H}{\mu(1 - \phi) + \rho + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 + (\gamma - 1)\theta_L} \]

Suppose \( \gamma > 1 \), LHS of (A11) is a strictly decreasing function of \( \phi \), and \( \phi^{-\frac{1}{\gamma}} \to \infty \) as \( \phi \to 0 \), \( \phi^{-\frac{1}{\gamma}} = 1 \) if \( \phi = 1 \). The RHS of (A11) is a strictly increasing function, \( R(\phi_1) = 1 \) and \( R(1) > 1 \). This establish both existence and uniqueness of solution for \( \gamma > 1 \). To prove (2), Note \( \phi^{-\frac{1}{\gamma}} \) is strictly decreasing function on \([1, \infty)\) with \( \phi^{-\frac{1}{\gamma}} \to 0 \) as \( \phi \to \infty \), and \( \phi^{-\frac{1}{\gamma}} = 1 \) if \( \phi = 1 \). Also, if \( \gamma < 1 \), \( R(\phi) \) is a strictly increasing function on \((1, \infty)\).Under assumption 1, \( 0 < R(1) < 1 \), and \( R(\phi) \to \infty \) as \( \phi \to \phi_2 \). This establishes the desired result.
The above argument also proved part 3) of theorem 3.1 and corollary 3.1:

\[ \gamma > 1 \Rightarrow H(\theta_H) < H(\theta_L) \quad (A12) \]

\[ \gamma = 1 \Rightarrow H(\theta_H) = H(\theta_L) \]

\[ \gamma < 1 \Rightarrow H(\theta_H) > H(\theta_L) \]

Corollary 3.2 is straightforward if \( \gamma < 1 \). If \( \gamma > 1 \), by (A9) and (A10), we have:

\[ \theta_H - H(\theta_H)^{\frac{1}{\gamma}} = \frac{1}{\gamma}[\lambda(\phi^{-1} - 1) - \rho - \frac{1}{2} \gamma(1 - \gamma)\sigma^2 + \theta_H] \quad (A13) \]

\[ \theta_L - H(\theta_L)^{\frac{1}{\gamma}} = \frac{1}{\gamma}[\mu(\phi - 1) - \rho - \frac{1}{2} \gamma(1 - \gamma)\sigma^2 + \theta_L] \quad (A14) \]

Since \( \phi < 1 \), the result follows.

2. Theorem 3.2

The proof is similar to theorem 4.4 and is omitted here.

A2. Appendix B: Proof of theorem 4.1, 4.3 and 4.4:

Here are the outlines of proof of theorem 4.1, 4.3 and 4.4.

1. Theorem 4.1:

Consider the optimization problem (21). Suppose a smooth solution \( H \) to (23) exist and satisfies condition (24). Conjecture the value function is of the form (22), the H-J-B equation is:

\[ \rho V(K, \pi) = \max_{c \geq 0} \left\{ \frac{c^{1-\gamma}}{1 - \gamma} + A(c)V(K, \pi) \right\} \quad (A15) \]
where

\[ A(c)V(K, \pi) = \frac{1}{2}K^2\sigma^2 V_{KK} + \frac{1}{2}\sigma^2 V_{\pi\pi} \]

\[ + K\pi(1 - \pi)(\theta_H - \theta_L)V_{Km} \]

\[ + [Km(\pi) - c]V_K + [\mu - (\lambda + \mu)\pi]V_\pi \]

where

\[ \sigma^2 = \pi^2(1 - \pi)^2(\theta_H - \theta_L)^2[\sigma^{-2} + \sigma^{-2}] \]

Plug in the conjectured the value function, (23) implies \( V \) satisfies (A15) with the optimal policy function given by (25). Next, assumption 1 and (23) implies the transversality condition is satisfied. Finally, apply the verification theorem, \( V \) is indeed the value function of the original problem.

2. Theorem 4.3:

I only prove the case \( \gamma > 1 \). The case \( \gamma < 1 \) can be done in a similar fashion. To see \( H \) is decreasing, it is enough to show the \( T \) operator defined in (??) maps decreasing function into decreasing functions. Fix \((\omega, t)\), consider the function inside the integral:

\[ e^{\left[-\rho - \frac{1}{2}\gamma(1-\gamma)^2|t + (1-\gamma)\int_0^t m_s ds f(\pi_t)\right]^{1-\frac{1}{\gamma}}} \]  

(A16)

Note if \( f \) is decreasing, then so is \( f^{1-\frac{1}{\gamma}} \). Hence \( f(\pi_t)^{1-\frac{1}{\gamma}} \) is decreasing in initial condition \( \pi \), this is because diffusion process is monotone in initial condition. Also, since for each \( s \), \( m_s \) is increasing in \( \pi \),

\[ e^{(1-\gamma)\int_0^t m_s ds} \]  

(A17)
is decreasing in $m$. Finally, notice both (A17) and \( f(\pi_t)^{1-\frac{1}{\gamma}} \) are nonnegative, we conclude (A16) is decreasing in $gp$, and this holds for all $(\omega, t)$, as needed.

3. Theorem 4.4

To calculate the stock price, note in the equilibrium consumption equals dividend paid by the stock, we have:

\[
P_t \xi_t = E\left[ \int_t^\infty \xi_{t+s} c_{t+s} ds | \pi_t, K_t \right]
\]

Using $\xi_t = e^{-\rho t} c_t^{-\gamma}$:

\[
\frac{P_t}{c_t} = E\left[ \int_t^\infty e^{-\rho s} \frac{H(\pi_{t+s})^{-\frac{1}{\gamma}} K_{t+s}^{-\gamma}}{H(\pi_t)^{-\frac{1}{\gamma}} K_t^{-\gamma}} | 1-\gamma ds | \pi_t, K_t \right]
\]

after some rearrangement, we get:

\[
P_t = H(\pi_t)^{-1} K_t^{-\gamma} E\left[ \int_t^\infty e^{-\rho s} H(\pi_{t+s})^{1-\frac{1}{\gamma}} K_{t+s}^{1-\gamma} | 1-\gamma ds | \pi_t, K_t \right]
\]

(A18)

In the equilibrium,

\[
dK_t = K_t [m_t - H(m_t)^{-\frac{1}{\gamma}}] dt + \sigma d\tilde{B}_K
\]

Using Dynkin’s formula, one get:

\[
E\left[ \int_t^\infty e^{-\rho s} H(\pi_{t+s})^{1-\frac{1}{\gamma}} K_{t+s}^{1-\gamma} ds | \pi_t, K_t \right] = H(\pi_t) K_t^{1-\gamma}
\]

Combining with (A18), we get the desired result.
Theorem 4.5

The expected stock return is $m(t)$. To solve for the equity premium, it is enough to calculate the risk free. The state price process

$$\xi_t = e^{-\rho t} e^{-\gamma t} = H(\pi_t) K_t^{-\gamma}$$

we can compute the risk-free interest rate by:

$$r_t = -E_t \left[ \frac{d\xi_t}{\xi_t} \right]$$

Using Ito’s formula, we get:

$$r_t = \rho + \gamma [m(\pi_t) - H(\pi_t)^{-\gamma}] - \frac{1}{2} \gamma (1 + \gamma \sigma^2)$$

$$- \frac{H'(\pi_t)}{H(\pi_t)} [\mu - (\lambda + \mu) \pi_t] - \frac{1}{2} \frac{H''(\pi_t)}{H(\pi_t)} \sigma^2$$

$$+ \gamma \frac{H'(\pi_t)}{H(\pi_t)} \pi_t (1 - \pi_t) (\theta_H - \theta_L)$$

Using equation (23) and after some simplification, we have

$$r_t = m_t - \gamma \sigma^2 + \pi_t (1 - \pi_t) (\theta_H - \theta_L) \frac{H'(\pi_t)}{H(\pi_t)}$$

as needed.