OPTIMAL CHOICE OF MONETARY INSTRUMENTS IN AN ECONOMY WITH REAL AND LIQUIDITY SHOCKS

JOYDEEP BHATTACHARYA AND RAJESH SINGH

Iowa State University

Abstract. Poole (QJE, 1970) using a IS-LM model presented the first formal treatment of the classic question: how should a monetary authority decide whether to use the money stock or the interest rate as the policy instrument. We update the seminal work of Poole in a microfounded flexible-price general equilibrium model of money using explicit welfare criteria. Specifically, we study the optimal choice of monetary policy instruments in a overlapping-generations economy where limited communication and stochastic relocation creates an endogenous transactions role for fiat money. The economy is hit with real endowment shocks and liquidity shocks. Overall, our results suggest that the central insight of Poole survives: when the shocks are real (nominal), welfare is higher under money growth (inflation rate) targeting. Expansionary policies may be optimal. Deviations from optimal policies create fairly large welfare losses.

1. Introduction

The optimal conduct of monetary policy, whether to target the money growth rate or the inflation rate, has survived as one of the most contentious issues in monetary economics. Popular until recently, Milton Friedman’s (1960) “mechanical monetarism” advised central banks to stop setting interest rates and instead set the money growth rate permanently at the estimated growth rate of the real economy. Since the 80s, however, the dominant paradigm in the practice of monetary policy shifted, bringing with it a renewed “dedication to price stability” via the direct control of inflation via interest rate targeting.¹

Poole (1970) presented the first formal treatment of the larger question: how should a monetary authority decide whether to use the money stock or the interest rate as the policy instrument. The debate at the time, as summarized by Poole, took the following shape: while some argued that “monetary policy should set the money stock while letting the interest rate fluctuate as it will”, others believed that monetary authorities should...
“push interest rates up in times of boom, and down in times of recession, while the money supply is allowed to fluctuate as it will.”

Using a stochastic IS-LM model, with reduction in variability of aggregate output as the yardstick, Poole reached the conclusion that if “disturbances originated primarily in the IS function that summarized the real sectors of the economy [...], the money stock is the proper control instrument. But if the LM function, representing the monetary sector, is the source of the disturbances, the interest rate is the proper control variable” (Poole and Lieberman, 1972). The bottom line advice was clear and extremely influential: when the shocks are real in nature, fix the money supply; if the shocks are monetary, fix the interest rate.

This paper takes up Poole’s “instrument problem” within the context of a modern “optimal policy-making framework” as described by Stern and Miller (2004).

The setting is a two-period lived pure-exchange overlapping generations model in the tradition of Townsend (1987) and Champ, Smith, and Williamson (1997) where limited communication and stochastic relocation create an endogenous transactions role for fiat money. More specifically, at the end of each period some fraction of agents is relocated (the “movers”) to a location different from the one they were born in and the only asset they can use to communicate with their past is fiat money. This allows money to be dominated in rate of return. The other asset is a linear storage technology with a fixed real return. The “stochastic relocations” act like shocks to agents’ portfolio preferences and, in particular, trigger liquidations of some assets at potential losses. They have the same consequences as “liquidity preference shocks” in Diamond and Dybvig (1983), and motivate a role for banks that take deposits, hold cash reserves, and make other less liquid investments. Depending on agents’ risk aversion, the banks’ cash reserves are sensitive/insensitive to the real return on money.

We study two variants of this model, one in which there are real shocks (the young-age good endowment of the agents is stochastic), and one where the fraction of agents relocating is itself random (liquidity preference or monetary shocks). In either case, banks can promise a real return to only the non-movers. For the movers, the banks can promise an amount of money (paid out of the bank’s reserve holdings) but not the real return on it. To see this, consider the case of endowment shocks. Here, the bank this period cares about next period’s endowment because the latter will potentially influence that period’s money demand, hence the price level and thus the return on money between this period and the next. But next period’s money demand depends on the following period’s endowment, and

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2Poole (2000) looks back at this debate: “To a considerable extent, then, the argument over the monetary aggregates 30 years ago was really an argument over the importance of the goal of low inflation and the responsibility of the central bank for the realized rate of inflation on the average over a period of several years. The debate arose from the fact that in that era many – perhaps most – economists believed that inflation was substantially independent of money growth and that central banks could not control inflation. Cost-push inflation was the result of rising labor and material costs. Demand-pull inflation was a consequence of excess aggregate demand. Money growth, many argued, played at best a limited role in creating, or controlling, excess aggregate demand. Fiscal policy was king.”

3Stern and Miller (2004) argue that questions regarding optimal monetary policy are best conducted in dynamic, stochastic general equilibrium models of money that incorporate a rationale for why money is held even when dominated in return by assets of similar risk profile. Poole (1970) satisfies all these desiderata except for the return dominance issue and the fact that his criterion for optimality is not agents’ welfare.
We assume that all agents know the distributions of the real or monetary shocks and form expectations about the return on money conditional on these distributions, and in a rational expectations equilibrium, these expectations are correct. We focus solely on stationary equilibria.

Our goal is identical to that of Poole (1970): can we use the model to tell us if inflation targeting is superior in a stationary welfare sense to money growth targeting, and when? As a benchmark, we start by studying the deterministic case. Here, as noted by Poole, “it obviously makes no difference whatsoever whether the policy prescription is in terms of setting the interest rate or in terms of setting the money stock...”. The best policy, as discussed in Bhattacharya, Haslag, and Russell (2005) is to hold the money stock fixed (zero inflation). When shocks are added to the environment, we find that Poole’s results are validated by our analysis. When the economy is hit with i.i.d shocks to the endowment, we can show, for a general class of CRRA utility functions, that an optimally chosen fixed money growth rate is stationary welfare superior to an optimally chosen fixed inflation rate. Indeed, the percentage gain in stationary welfare in switching from inflation targeting to monetary targeting can be quite high.

The situation is exactly reversed when the shocks are monetary in nature, that is they affect the fraction of agents relocating; here inflation targeting does a better job than monetary targeting. We find that the optimal net inflation rate maybe positive or negative. Interestingly, under inflation rate targeting it may be optimal to pursue an expansionary policy, it is never optimal to do so under money growth targeting. Numerical exercises also reveal that the percentage gain in stationary welfare in switching from monetary targeting to inflation targeting is considerably smaller than in the case with endowment shocks.

In a fairly narrow sense, this paper has few antecedents. Almost all the work done in this area employs models with sticky or staggered prices and very few, as Collard and Della (2005) point out, use welfare criteria to answer Poole’s original question. Prominent examples of work in the rigid prices tradition surveyed in Walsh (1998) and Woodford (2000) include Carlstrom and Fuerst (1996), Ireland (2000), Khan, King, and Wolman (2003), among others. In a recent paper, Collard and Della (2005) answer Poole’s question “in an economy that represents a faithful, general equilibrium rendition of Poole.”: the model is Neo-Keynesian with capital accumulation, staggered prices (monopolistic competition), money-in-the-utility-function, and supply, fiscal, and money demand shocks. Their main findings are: a) contrary to Poole (1970), monetary targeting generates higher welfare for money demand shocks irrespective of the degree of risk aversion, b) for real shocks, interest rate targeting produces higher welfare only when risk aversion is high. Finally, using a deterministic OG model with legal restrictions, Smith (1991, 1994) compares the two targeting procedures in terms of their efficiency properties and goes on to isolate a “tension between efficiency and determinacy” of monetary equilibria reminiscent of the nineteenth century quantity theory versus real bills doctrine controversy.

The plan for the rest of the paper is as follows. In the next section, we outline the baseline model without uncertainty and compute optimal monetary policies. In Section 3, we study the role of endowment uncertainty in shaping the optimal choice of monetary instruments. In Section 4, we do the same with money demand shocks.
2. The environment

2.1. Primitives. We consider an economy consisting of an infinite sequence of two period lived overlapping generations. Time $t$ is discrete and runs from $\{t\}_{-\infty}^{\infty}$. At each date $t$, young agents are symmetrically assigned to one of two locations. Each location contains a continuum of young agents with unit mass, and our assumptions will imply that locations are always symmetric. There is a single good that may be consumed or stored. Each two-period-lived agent is endowed with $w_t > 0$ units of this good at date $t$ when young and nothing when old. Let $c_{2t+1}$ denote the consumption of the final good by a representative old agent born at $t$. All such agents have preferences representable by the utility function $U(c_{2t+1})$,

$$U(c_{2t+1}) = E_t u(c_{2t+1})$$

where $u$ is twice-continuously differentiable, strictly increasing, and strictly concave in its arguments. At points below, we will specialize to $u(c) = \left[ c^{1-\phi} - 1 \right] / (1 - \phi)$, with $\phi > 0$, and $u(c) = \ln c$ when $\phi = 1$.

The assets available to the agents are goods, which they may store, and fiat currency (money). If $\kappa > 0$ units of the good are placed in storage at any date $t \geq 1$, then $x \kappa$ units are recovered from storage at date $t+1$, where $x > 1$. The quantity of money in circulation at the end of period $t \geq 1$, per young agent, is denoted $M_t$. Let $p_t < \infty$ denote the price level at date $t$. Let $\pi_t < 1$ denote the inflation rate between period $t$ and $t+1$. Then the gross real rate of return on money acquired at date $t$ denoted by $R_{m,t} \equiv p_t / p_{t+1} = 1 / \pi_{t,t+1}$.

Also, let $m_t \equiv M_t / p_t$ denote real money balances at date $t$. In addition to the store of value function of money spatial separation and limited communication generate a transactions role of money as in Townsend (1987). As such, money can be valued even if it is dominated in return by storage. The details are outlined below and follow standard conventions setup in Schreft and Smith (1997) or Smith (2002).

2.2. Random relocation. Each period, a fraction $\alpha_t$ of the young agents is relocated to the other location. An agent that is relocated cannot collect the return on any goods she has stored, or that have been stored on her behalf, since goods cannot be transported across locations. However, if an agent is carrying fiat currency when she is relocated, then the currency is relocated with it.

Under the circumstances, there are two strategies a agent can use to transfer income over time. First, it can save on its own, storing some quantity of goods and acquiring some quantity of fiat currency. The drawback is that if she is relocated then she must abandon her stored goods, and if not, then it is stuck holding fiat currency, a “bad” asset (more below on this). Alternatively, she can deposit her entire endowment in a perfectly competitive bank. The bank pools the goods deposited by all the young agents and uses them to acquire a portfolio of stored goods and fiat currency. Banks can transport fiat currency across locations. It issues claims to the agents whose nature, timing and size are contingent on their relocation status. If an agent does not get relocated, then she gets a return on her deposit that is funded by the goods the bank has stored. If she gets relocated, then she gets a return on her deposit that takes the form of a fiat currency payment funded
by the bank’s holdings of fiat currency. Since banks can pool individual risks, it can be checked that the latter strategy always dominates the former one and we will analyze the economy on this basis.

2.3. **Conduct of monetary policy.** We allow the government to conduct monetary policy in one of two possible ways. The first, called “monetary targeting”, is one where the government changes the nominal stock of fiat currency at a fixed non-stochastic gross rate $\mu > 0$ per period, so that $M_t = \mu M_{t-1}$ for all $t$. The second, called “inflation rate targeting”, is one where the government changes the nominal stock of fiat currency in such a way as to keep the long-run gross real rate of return on money fixed at $1/\pi$. If the net money growth rate is positive then the government uses the additional currency it issues to purchase goods, which it gives to current young agents in the form of lump-sum transfers. If the net money growth rate is negative, then the government collects lump-sum taxes from the current young agents, which it uses to retire some of the currency. The tax (+) or transfer (−) is denoted $\tau_t$. The budget constraint of the government is

$$\tau_t = \frac{M_t - M_{t-1}}{p_t} = m_t - m_{t-1}R_{m,t-1}$$  \hspace{1cm} (2.1)

for all $t \geq 1$.

2.4. **The bank’s problem.** The asset holdings of young agents are costlessly intermediated by perfectly competitive banks. These banks hold portfolios of fiat currency and physical assets, which consist of stored goods. Every young agent deposits her after-tax/transfer income in the bank. The banks divide their deposits between stored goods $s_t$ and real balances of fiat currency $m_t$, so that

$$w_t + \tau_t = m_t + s_t.$$  \hspace{1cm} (2.2)

Define $\gamma_t \equiv \frac{m_t}{w_t + \tau_t}$ as the ratio of cash reserves to deposits. Banks announce a return of $d^m_t$ to each mover (one who gets relocated) and $d^n_t$ to each non-mover (one who stays on in the location she was born). These returns satisfy some constraints. First, relocated agents, of whom there are $\alpha_t$, have to be given money and so the bank has to use its holdings of cash reserves to pay them. Consequently,

$$\alpha_t d^m_t \leq \gamma_t R_{m,t}$$  \hspace{1cm} (2.3)

must hold, since money earns a return of $R_{m,t} = \frac{p_t}{p_{t+1}}$ between $t$ and $t + 1$ which the bank takes as given. Similarly, the promised return to the non-movers must satisfy

$$(1 - \alpha_t) d^n_t \leq (1 - \gamma_t) x.$$  \hspace{1cm} (2.4)

Additionally, $\gamma_t \in [0,1]$ must hold. In what follows, we assume that money is a “bad” asset, or that

$$x > R_{m,t} \text{ for all } t > 1.$$  \hspace{1cm} (2.5)

Competition among banks for depositors will, in equilibrium, force banks to choose return schedules and portfolio allocations so as to maximize the expected utility of a representative depositor, subject to the constraints we have described. If $w_t = w$ and $\alpha_t = \alpha$
\[ \forall t \geq 1, \text{ these are known and fixed (as in the standard random relocation model analyzed by Schreft and Smith (1997)), the bank’s problem can be rewritten as} \]
\[
\max_{\gamma_t \in [0, 1]} \left\{ \alpha u \left( \frac{\gamma_t R_{m,t}}{\alpha} \right) + (1 - \alpha) u \left( \frac{(1 - \gamma_t)x}{(1 - \alpha)} \right) \right\}. \tag{2.6}
\]

Define \( I_t \equiv \frac{x}{R_{m,t}} = x \pi_{t,t+1} \) as the gross nominal interest rate between \( t \) and \( t+1 \). Note that \( I_t \) represents the opportunity cost of cash relative to storage. It is then easily checked that the solution to this problem for \( u(c) = c^{1-\phi} - 1 \) is
\[
\gamma_t(I_t) = \frac{\alpha}{\alpha + (1 - \alpha) (I_t)^{1-\phi}} \tag{2.7}
\]

Once the optimal \( \gamma \) is computed using (2.7), the promised returns to movers and non-movers can easily be computed using (2.3)-(2.4). Several points deserve mention here. First, for CRRA utility, notice that the optimal \( \gamma \) does not depend on \( w \). Second, monetary policy influences the optimal \( \gamma \) in the case of CRRA utility only insofar as it determines the relative return on money, \( I_t \). Thirdly, for all \( I > 1, \gamma \gtrless \alpha \text{ iff } \phi \gtrless 1 \). In words, when the consumption of movers and non-movers are complements (substitutes) a lower return on money requires that the share of current income allocated to movers be relatively high (low). Finally, as has been shown by Schreft and Smith (1997), \( \gamma'(I) \gtrless 0, \forall I \text{ iff } \phi \gtrless 1 \). An increase in \( I \) has both income and substitution effects. First, it decreases the combined income available for consumption next period. However, for any fixed share \( \gamma \), it affects movers relatively more. Thus, when the consumptions of movers and non-movers are complements, movers’ share \( \gamma \) must be increased. On the other hand, when the two consumptions are substitutes, it is better to shift consumption from movers to non-movers; hence, \( \gamma \) should be lowered. As an aside, also note that in this setting, nominal interest rate targeting and inflation rate targeting are exactly identical goals.

2.5. **Welfare.** Finally, steady state welfare for CRRA utility can be defined as
\[
W(R_m) = \left\{ \frac{w + \tau (R_m)}{1 - \phi} \right\}^{1-\phi} \left\{ \alpha^\phi [\gamma(R_m) R_m]^{1-\phi} + (1 - \alpha)^\phi [(1 - \gamma(R_m))x]^{1-\phi} \right\} \tag{2.8}
\]

Under monetary targeting, the gross money growth rate is set to \( \mu \). Then in a steady state, \( R_m = 1/\mu \) and so
\[
\tau = \frac{M_t - M_{t-1}}{p_t} = \left( 1 - \frac{1}{\mu} \right) m
\]
holds; since \( \gamma(w + \tau) = m \), we have
\[
\tau(\mu) = \frac{\left( 1 - \frac{1}{\mu} \right) \gamma(\mu)}{1 - \left( 1 - \frac{1}{\mu} \right) \gamma(\mu)} w
\]
where using \( I = x\mu \)
\[
\gamma(\mu) = \frac{\alpha}{\alpha + (1 - \alpha) (x\mu)^{1-\phi}} \tag{2.9}
\]
Then the problem of choosing the optimal ("steady state welfare maximizing") money growth rate under monetary targeting reduces to

$$
\max_{\mu} W(\mu) = \max_{\mu} \frac{\{w + \tau(\mu)\}^{1-\phi}}{1-\phi} \left\{ \alpha^\phi \left[ \frac{\gamma(\mu)}{\mu} \right]^{1-\phi} + (1-\alpha)^\phi \left[ (1-\gamma(\mu))x \right]^{1-\phi} \right\}
$$

Under inflation rate targeting, the return to money is set to $1/\pi$, i.e., $R_{m,t} = 1/\pi \ \forall t$. As such, monetary targeting and inflation targeting are exactly identical goals as is evident from (2.8). We close this section with a fairly well-known result about optimal monetary policy in this environment, the proof of which may be found in Bhattacharya, Haslag, and Russell (2005).

**Proposition 1.** Under inflation rate targeting or equivalently under monetary targeting, the optimal policy is to hold the money stock fixed (zero inflation) if there are no shocks to endowments or liquidity preference.

Notice that this result holds irrespective of the degree of risk-aversion. The intuition for this result is as follows. A planner unconstrained by limited communication would face a rate of return of $x$ since a unit of the good invested in the storage technology this period yields $x$ next period. Such a planner who is deciding to allocate $w$ between the movers and the non-movers would choose an allocation $(c_m, c_n)$ so as to set

$$
\frac{u'(c_m)}{u'(c_n)} = x
$$

The government’s objective, of course, is to choose a $\mu$ that maximizes stationary welfare in a decentralized equilibrium. In such a equilibrium involving money, As $R_m = \frac{1}{\mu}$, using (2.3), (2.4), and the assumed CRRA form of $u$, it is easily checked that

$$
\frac{u'(c_m)}{u'(c_n)} = \left[ \frac{\gamma(\mu)}{1-\gamma(\mu)} \frac{1-\alpha}{\alpha} \frac{1}{\mu x} \right]^{-\phi}
$$

which using (2.9) reduces to

$$
\frac{u'(c_m)}{u'(c_n)} = \mu x.
$$

Thus the government can select an efficient allocation only by setting $\mu = 1$.

As Bhattacharya, Haslag, and Russell (2005) argue, in a OG model, in steady states, every unit of goods devoted to holding money is an unit that is not devoted to acquiring storage; as such, the social opportunity cost of money is the return on storage. Optimality requires that the private opportunity cost of holding money be the same as the social opportunity cost of money. Of course, the private opportunity cost of money is the nominal interest rate, $I = x\mu$. Hence, $\mu = 1$ is the best choice.  

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4 As Bhattacharya, Haslag, and Russell (2005) argue, in a infinitely-lived agent model, the social opportunity cost of providing money is zero (not $x$) so it is optimal for the private opportunity cost of holding money to be zero – the “Friedman rule”. This explains why the Friedman rule $\mu = 1/x$ is not the best choice for the government.
In the next section, we show that the presence of real shocks can alter the very nature of Proposition 1. To foreshadow, inflation rate targeting will deviate from monetary targeting even for logarithmic utility and the prescription for optimal policy will become sensitive to agents’ risk aversion.

3. ENDOWMENT UNCERTAINTY

We now analyze an economy that is identical to the one studied above, except that the endowment $w$ is now assumed to be stochastic. In particular, we assume that $w$ is drawn each period from a i.i.d distribution $f(w)$ with support $[w, \bar{w}]$. Since $w$ is an endowment, we are clearly restricted to choosing distributions whose support lies on $\mathbb{R}_+$. Also, for competitive equilibria to exist, $w$ has to stay bounded.

Shocks to the endowment represent real shocks. Our goal is to investigate how monetary policy should respond to such intrinsic real uncertainty. Recall that at the point at which the bank solves its problem, the current endowment is known. But the realization of next period’s endowment has not happened yet. The bank cares about next period’s endowment because the latter will potentially influence next period’s money demand, hence the next period’s price level and thus the return on money between this period and the next. In this sense, the bank cannot promise a fixed real return to the movers anymore. The bank knows the distribution for $w$ and forms expectations on the return on money conditional on $f(w)$; in a rational expectations equilibrium, these expectations are correct. We will focus on stationary versions of such equilibria below.

3.1. Logarithmic utility. We start by focusing on the case of logarithmic utility. We assume a condition equivalent to (2.5), which is given by

$$x > \frac{w^e}{\mu w},$$

where $w^e$ is the expected value of $w$. This condition ensures that it is never optimal to reserve currency for non-movers. The bank’s problem is now described by

$$\max_{\gamma_t} \int_{w}^{\bar{w}} \left\{ \alpha \ln \left[ \frac{\gamma_t R_{m,t}}{\alpha} \right] + (1 - \alpha) \ln \left[ \frac{(1 - \gamma_t) x}{1 - \alpha} \right] + \ln (w_t + \tau_t) \right\} f(w_{t+1}) \, dw_{t+1}$$

which simplifies to

$$\max_{\gamma_t} \int_{w}^{\bar{w}} \alpha \ln (R_{m,t}) f(w_{t+1}) \, dw_{t+1} + \alpha \ln \left[ \frac{\gamma_t}{\alpha} \right] + (1 - \alpha) \ln \left[ \frac{(1 - \gamma_t) x}{1 - \alpha} \right] + \ln (w_t + \tau_t)$$

Observe that since it takes the return on money and the size of the transfer as given, the bank’s choice of $\gamma$ will only consider the second and the third terms of the previous expression. Then, the choice of $\gamma_t$ is given by

$$\gamma_t = \alpha \text{ for all } t,$$

making the decision rule of the bank identical to that in the non-stochastic endowment case. Even though endowment uncertainty has the potential to affect the real return on
money, under log utility, the choice of \( \gamma \) is separate and is not influenced by this return. An increase in money’s rate of return uncertainty effectively increases its opportunity cost and makes it less desirable. However, whatever the opportunity cost of money be, since the elasticity of substitution between the consumption of movers and non-movers remains unity, banks allocate \( \alpha \) share to be spent on movers’ consumption. This will not be the case in the more general CRRA formulation discussed below.

3.1.1. Monetary targeting. Under monetary targeting, the government fixes the money growth rate at \( \mu \). It follows that the real return to money is given by

\[
R_{m,t} = \frac{m_{t+1}}{\mu m_t} = \frac{w_{t+1} + \tau_{t+1}}{\mu (w_t + \tau_t)} \tag{3.1}
\]

Also, since \( \tau_t = \frac{M_t - M_{t-1}}{p_t} = m_t \left( 1 - \frac{1}{\mu} \right) \), \( m_t = \gamma_t (w_t + \tau_t) \), and \( \gamma_t = \alpha \), we have

\[
\tau_t = \frac{\alpha w_t \left( 1 - \frac{1}{\mu} \right)}{1 - \alpha \left( 1 - \frac{1}{\mu} \right)}
\]

implying \( w_t + \tau_t = w_t / [1 - \alpha \left( 1 - \frac{1}{\mu} \right)] \), and hence

\[
\frac{w_{t+1} + \tau_{t+1}}{\mu (w_t + \tau_t)} = \frac{w_{t+1}}{\mu w_t} \tag{3.2}
\]

Welfare at \( t \) is given by

\[
W_t = \int \alpha \ln(R_{m,t}) f (w_{t+1}) \, dw_{t+1} + \alpha \ln \left[ \frac{\gamma_t}{\alpha} \right] + (1 - \alpha) \ln \left[ \frac{(1 - \gamma_t) x}{1 - \alpha} \right] + \ln (w_t + \tau_t) \tag{3.3}
\]

Using (3.1) and (3.2) and \( \gamma_t = \alpha \), \( W_t \) in (3.3) reduces to

\[
W_t = \int \alpha \ln \left[ \frac{w_{t+1}}{\mu w_t} \right] f (w_{t+1}) \, dw_{t+1} + (1 - \alpha) \ln x + \ln w_t - \ln \left[ 1 - \alpha \left( 1 - \frac{1}{\mu} \right) \right] \tag{3.4}
\]

and finally to

\[
W_t = \alpha \int \{ \ln (w_{t+1}) \} f (w_{t+1}) \, dw_{t+1} + (1 - \alpha) \ln (xw_t) - \ln \left[ 1 - \alpha \left( 1 - \frac{1}{\mu} \right) \right] - \alpha \ln \mu.
\]

Since \( w \) is drawn from a time-invariant i.i.d. distribution, stationary welfare is given by

\[
W_f (w) = \alpha \int \{ \ln w \} f (w) \, dw + (1 - \alpha) \ln x + (1 - \alpha) \int \{ \ln w \} f (w) \, dw
\]

\[
- \ln \left[ 1 - \alpha \left( 1 - \frac{1}{\mu} \right) \right] - \alpha \ln \mu
\]

\[
= (1 - \alpha) \ln x + \int \{ \ln w \} f (w) \, dw - \ln \left[ 1 - \alpha \left( 1 - \frac{1}{\mu} \right) \right] - \alpha \ln \mu \tag{3.5}
\]
What \( \mu \) is the best from the standpoint of stationary welfare?\footnote{By stationary welfare, we mean the indirect utility accruing to two-period-lived agents in a steady state, ignoring the initial old, if any.} The one that maximizes the last two terms in (3.5). The exercise yields

\[
\tilde{\mu} = 1.
\]

The optimal monetary policy is to keep the money supply fixed. It is interesting to note that the optimal prescription for the money growth rate coincides with that in the economy with no real shocks studied in Section 2.5. As (3.4) makes clear, this has to do with the special “additive and separable” nature of log utility. In addition to equating the social opportunity cost of money with its private opportunity cost, the government’s choice of \( \mu \) should also attempt to reduce endowment and rate-of-return of uncertainty over time. However, given the private sector behavior, a reduction in endowment uncertainty \((w + \tau = w/(1 - \alpha(1 - \frac{1}{\mu}))\) by decreasing \( \mu \) below unity increases rate of return \((\frac{w_{t+1}}{\mu w_t})\) uncertainty. With a unit elasticity of consumption substitution (here it is going to be intertemporal) at \( \mu = 1 \), the marginal cost of decreasing \( \mu \) in order to reduce endowment uncertainty equals its marginal cost in terms of increased rate of return uncertainty. Hence \( \mu = 1 \) turns out to be optimal even on this margin.

As we demonstrate below, unlike in the deterministic case, the prescription for optimal monetary policy will be different from zero inflation under inflation rate targeting.

3.1.2. Inflation rate targeting. Under inflation rate targeting, the government fixes the inflation rate at \( \pi \). It follows that the real return to money is given by

\[
R_{m,t} = \frac{1}{\pi}
\] (3.6)

Then using \( \gamma_t = \alpha \) and (3.6), in (3.4), we get

\[
W_t = -\int_w^\infty \alpha \ln \pi f(w_{t+1})dw_{t+1} + (1 - \alpha) \ln x + \ln (w_t + \tau_t)
\]

\[
= -\alpha \ln \pi + (1 - \alpha) \ln x + \ln (w_t + \tau_t) = -\alpha \ln \pi + (1 - \alpha) \ln x + \ln \frac{m_t}{\alpha} \tag{3.7}
\]

Since \( \tau_t = \frac{M_t - M_{t-1}}{\pi} = m_t - \frac{m_{t-1}}{\pi} \), we get

\[
m_t = \alpha (w_t + \tau_t) = \alpha \left( w_t + m_t - \frac{m_{t-1}}{\pi} \right)
\]

implying

\[
m_t = -\frac{\alpha}{(1 - \alpha) \pi} m_{t-1} + \frac{\alpha}{1 - \alpha} w_t \tag{3.8}
\]

Notice that (3.8) represents an AR(1) process for real balances where \( w_t \) is a stochastic “forcing function”. The invariant (long run) distribution will depend on \( w \) and \( \pi \). Denote
the stationary distribution by $\Omega(m; f(w), \pi)$. A necessary condition for its existence is
\[
\left| \frac{\alpha}{(1 - \alpha) \pi} \right| \leq 1;
\]
then $m_\infty$ will not depend on $m_0$. For future reference note that the above stochastic process for $m$ implies that
\[
\bar{m} = \frac{\alpha}{1 - \frac{\alpha}{(1 - \alpha) \pi}} \bar{w}
\]
\[
\sigma_m^2 = \frac{\alpha \sigma_w^2}{1 - \left[ \frac{\alpha}{(1 - \alpha) \pi} \right]^2}
\quad (3.9)
\]
Using (3.7), stationary welfare is written as
\[
\int W f(w) \, dw = -\alpha \ln \pi - \ln \alpha + (1 - \alpha) \ln x + \int m(f(w), \pi) \{\ln m\} \Omega(m; f(w), \pi) \, dm
\]
where the support of $\Omega(m; f(w), \pi)$ given by $[\bar{m}(f(w), \pi), m(f(w), \pi)]$ corresponds to the support of $w$ given by $[w, \bar{w}]$.

What is the best $\pi$? The one that solves $\frac{d}{d\pi} \int W f(w) \, dw = 0$. In general, it is not possible to characterize $\Omega(.)$ analytically for all distributions $f(w)$; as such, we cannot get a generalized closed form expression for $\pi$. We will report results from several numerical experiments below. It can be argued that the optimal value of inflation rate, $\pi > 1$. First, note that inflation rate targeting completely insures against the rate of return uncertainty. So, if endowments were constant, equating social opportunity cost of money with its private opportunity cost will call for $\pi = 1$. However, note that the income of a young generation depends not only on the current endowment but also on the transfers that they get as a result of monetary injections. So the volatility of income depends on the volatility of endowment as well as transfers. The combined volatility can be decreased by increasing the rate of inflation above unity as can be seen from (3.9) above. Hence, $\pi > 1$.

3.2. CRRA utility. We now generalize the above analysis to CRRA utility
\[
U(c) = \frac{c^{1-\phi}}{1 - \phi}
\]
In this case, the bank’s problem at date $t$ reduces to
\[
\max_{\gamma_t} \frac{\left( w_t + \tau_t \right)^{1-\phi}}{1-\phi} \left[ \int_{\bar{w}} \left\{ \alpha \left( \frac{\gamma_t}{\alpha} R_{m,t} \right)^{1-\phi} + (1 - \alpha) \left( \frac{(1 - \gamma_t) x}{1 - \alpha} \right)^{1-\phi} \right\} f(w_{t+1}) \, dw_{t+1} \right].
\quad (3.10)
\]
It is possible to rewrite (3.10) as
\[
\max_{\gamma_t} \frac{\left( w_t + \tau_t \right)^{1-\phi}}{1-\phi} \left[ \alpha \left( \frac{\gamma_t}{\alpha} \right)^{1-\phi} \int_{\bar{w}} (R_{m,t})^{1-\phi} f(w_{t+1}) \, dw_{t+1} + (1 - \alpha) \left( \frac{(1 - \gamma_t) x}{1 - \alpha} \right)^{1-\phi} \right]
\]
since the bank’s choice of $\gamma_t$ cannot depend on the future realization of $w$ except through the return on money which the bank takes as given. The first order conditions to the bank’s problem yields

$$
\left(\frac{\gamma_t}{\alpha}\right)^{-\phi} \int_{w}^{\bar{w}} (R_{m,t})^{1-\phi} f(w_{t+1}) \, dw_{t+1} = x \left(\frac{1 - \gamma_t}{1 - \alpha}\right)^{-\phi}.
$$

Denote $\Phi_t \equiv \int_{w}^{\bar{w}} (R_{m,t})^{1-\phi} f(w_{t+1}) \, dw_{t+1}$; then it is easily checked that

$$
\gamma_t(w_{t+1}) = \frac{1}{(1 - \gamma_t)^{\frac{1-\phi}{1-\phi}} + 1 + \frac{1}{1 - \alpha} x^{\frac{1-\phi}{1-\phi}} \Phi_t^{\frac{1}{1-\phi}}}
$$

(3.11a)

It is clear from (3.11a) that in general, and unlike in the case of logarithmic utility, the bank’s choice of $\gamma$ is time-dependent. Note that a mean-preserving spread in $R_{m,t}$ effectively increases the opportunity cost of money $I_t$. Then, as discussed before, $\gamma'(I) \geq 0, \forall I$ iff $\phi \leq 1$, can be alternatively rewritten as $\gamma'(\sigma_w^2) \geq 0, \forall I$ iff $\phi \leq 1$, where $\sigma_w^2$ is the variance of endowment process.

3.2.1. Inflation rate targeting. Under inflation rate targeting, $R_{m,t} = \frac{1}{\pi}$ for all $t > 1$. Then

$$
\int_{w}^{\bar{w}} (R_{m,t})^{1-\phi} f(w_{t+1}) \, dw_{t+1} = (\pi)^{\phi-1}
$$

and it follows from (3.11a) that

$$
\gamma(\pi) = \frac{1}{1 + \frac{1-\phi}{\alpha} (\pi x)^{\frac{1-\phi}{1-\phi}}},
$$

(3.12a)

which is time-independent. Since $\tau_t = m_t - \frac{m_{t-1}}{\pi}$, and $m_t = \gamma(\pi)(w_{t+\tau_t})$, we get, as before (see (3.8)),

$$
m_t = -\frac{\gamma(\pi)}{1 - (\gamma(\pi))\pi} m_{t-1} + \frac{\gamma(\pi)}{1 - (\gamma(\pi))\pi} w_t.
$$

(3.13)

As described above, given the stochastic process for $w$, one can then find the unconditional long-run distribution of $m$, $\Omega(m; f(w), \pi)$. To that end, we continue to require that

$$
\left|\frac{\gamma(\pi)}{1 - (\gamma(\pi))\pi}\right| \leq 1.
$$

Then, stationary welfare under an inflation target $\pi$ is given by

$$
W^\pi = \frac{1}{1 - \phi} \left[ \alpha \left(\frac{\gamma(\pi)}{\alpha}\right)^{1-\phi}(\pi)^{\phi-1} + (1 - \alpha) \left(\frac{1 - \gamma(\pi)}{1 - \alpha}\right)^{1-\phi} \frac{1}{m(f(w), \pi)} \left(\frac{m}{\gamma(\pi)}\right)^{1-\phi} \Omega(m; \pi) \, dm \right]
$$

(3.14)

which, using (3.12a) reduces to

$$
W^\pi = \frac{1}{1 - \phi} \alpha^\phi(\pi)^{\phi-1} \left[ 1 + \frac{1 - \alpha}{\alpha} (\pi x)^{\frac{1-\phi}{1-\phi}} \right] \frac{1}{m(f(w), \pi)} m^{1-\phi} \Omega(m; \pi) \, dm.
$$

(3.15)
Thus far, we have not imposed additional structure on the distribution \( f(w) \); however, any analysis of optimal \( \pi \) requires us to make additional functional assumptions. A reasonably simple form is the uniform over \([\bar{w}, \bar{w}]\). In this case, the invariant distribution takes the form of a trapezoidal distribution\(^6\), i.e.,

\[
\Omega (m, \pi) = \begin{cases} 
\frac{m - m}{\bar{m} - m} \bar{h}, & \text{if } m \in [m, m^*] \\
\bar{h}, & \text{if } m \in [m^*, \bar{m}] \\
\frac{m^* - m}{m - m^*} \bar{h}, & \text{if } m \in [m, m^*] 
\end{cases} \tag{3.16}
\]

where

\[
\bar{h} \equiv \frac{2}{(\bar{m} + m) - (m^* + m^*)}; \quad \bar{m} \equiv \chi \frac{\bar{w} - \frac{\chi}{\bar{\pi}} w}{1 - (\frac{\chi}{\bar{\pi}})^2}; \quad m \equiv \chi \frac{w - \frac{\chi}{\bar{\pi}} \bar{w}}{1 - (\frac{\chi}{\bar{\pi}})^2} \tag{3.17}
\]

\[
m^* = \frac{\bar{w}}{1 + \frac{\chi}{\bar{\pi}}}; \quad m = \frac{w}{1 + \frac{\chi}{\bar{\pi}}}; \quad \chi = \frac{\gamma (\pi)}{1 - \gamma (\pi)} \tag{3.18}
\]

Finally, using (3.16)-(3.18), one can reduce (3.15) to

\[
W^\pi = \frac{1}{1 - \Phi} \alpha^\pi (\pi)^{\phi - 1} \left[ 1 + \frac{1 - \alpha}{\alpha} (\pi x)^{\phi - 1} \right] \bar{h} \times \left[ \int_{m}^{m^*} m_{1-\phi} \frac{m - m}{m^* - m} dm + \int_{m^*}^{\bar{m}} m_{1-\phi} dm + \int_{m^*}^{\bar{m}} m_{1-\phi} \frac{\bar{m} - m}{m^* - m^*} dm \right]. \tag{3.19}
\]

No closed form characterizations of the optimal choice of \( \pi \) are possible. We will conduct several numerical experiments below.

3.2.2. Monetary targeting. Under monetary targeting, the government fixes the money growth rate at \( \mu \). It follows that the real return to money is given by

\[
R_{m, t} = \frac{m_{t+1} \gamma^t (w_{t+1} + \tau_{t+1})}{\mu m_t (w_t + \tau_t)} \tag{3.20}
\]

Also, since \( \tau_t = \frac{M_t - M_{t-1}}{\mu_t} = m_t \left( 1 - \frac{1}{\mu} \right), m_t = \gamma_t (w_t + \tau_t) \), we have

\[
\tau_t = \frac{\gamma_t w_t \left( 1 - \frac{1}{\mu} \right)}{1 - \gamma_t \left( 1 - \frac{1}{\mu} \right)} \text{ and } w_t + \tau_t = \frac{w_t}{1 - \gamma_t \left( 1 - \frac{1}{\mu} \right)}.
\]

Then (3.20) reduces to

\[
R_{m, t} = \frac{1}{\mu} \frac{\gamma_{t+1} w_{t+1}}{\gamma_t w_t} \left( 1 - \gamma_t \left( 1 - \frac{1}{\mu} \right) \right) \tag{3.21}
\]

\(^6\)There is no exact known closed form distribution; numerous simulations suggest the trapezoidal form.
Using (3.21), note that
\[
\int_0^w (R_{m,t})^{1-\phi} f(w_{t+1}) dw_{t+1} = \left( \frac{1}{\mu w_t} \right)^{1-\phi} \left[ \frac{1}{\gamma_t} - \left( 1 - \frac{1}{\mu} \right) \right]^{1-\phi} \int_0^w \left( \frac{\gamma(w_{t+1})w_{t+1}}{1-\gamma(w_{t+1})(1-\frac{1}{\mu})} \right)^{1-\phi} f(w_{t+1}) dw_{t+1}
\]

Stationary welfare in this case is then given by
\[
W^\mu = \frac{1}{1-\phi} \int_0^w \left[ \alpha \left( \frac{1}{\mu} \right)^{1-\phi} \left( \frac{1}{\alpha} - \frac{\gamma(w_{t}w_{t+1})}{1-\gamma(w_{t+1})(1-\frac{1}{\mu})} \right)^{1-\phi} + (1-\alpha) \left( \frac{1}{1-\alpha} \frac{(1-\gamma(w_{t+1})w_{t+1})}{1-\gamma(w_{t+1})(1-\frac{1}{\mu})} \right)^{1-\phi} \right] f(w) dw \quad (3.22)
\]

As before, the optimal \( \mu \) is chosen by solving \( \frac{\partial W^\mu}{\partial \mu} = 0 \).

3.3. Numerical results with real shocks. Since analytical results are hard to come by here, we resort to reporting results from several numerical experiments. We adopt the following parametric specification: \( x = 1.04, \alpha = 0.2, \) and \( f(w) \) is i.i.d and uniform with support [0.9, 1.1]. We work with the CRRA form, allowing \( \phi \) to vary in some experiments. Our goal is to compare our results to Poole (1970) who argued that when the shocks are real in nature, fix the money supply; if the shocks are monetary, fix the interest rate. Our results are as follows:

- When the shocks are to endowments, targeting the money stock is welfare-superior to targeting the inflation rate.
- The above result holds for a wide range of risk-aversion, from \( \phi = 0.5 \) to 2.
- The optimal inflation rate (\( \pi \)) stays above 1 for the entire range of \( \phi \) suggesting that positive inflation can be optimal.
- Even for log utility, the optimal inflation rate is not zero inflation.
- The optimal money growth rate calls for inflation for low values (\( \phi < 1 \)) of risk-aversion, zero inflation for \( \phi = 1 \), and deflation for high values of risk-aversion (\( \phi > 1 \)).
- The percentage gain in stationary welfare in switching from inflation targeting to monetary targeting varies from near 5% to about 15% as \( \phi \) varies between 0.5 and 2.1.
- The intuition behind the result that monetary targeting works better than inflation targeting is the following. Consider the logarithmic utility case. Then, under monetary targeting old agents that carry money consume from the endowment of the current young. If the current endowment is relatively high, banks’ real money allocation is also relatively high which occurs through a fall in the price level because money supply is predetermined. Thus, while the old non-movers from last period consume stored goods from the last period, the old movers consume an amount that is proportional to the current endowment. Thus, there is an intertemporal aggregate consumption smoothing under monetary targeting. Under inflation targeting, on the other hand, the amount of money saved for old varies
with the current endowment as the price level is fixed. Thus, each period both movers and non-movers consume out of savings from their own income, and thus there is no intertemporal smoothing of consumption.

• The intuition behind the result that as the risk aversion increases, $\bar{\mu}$ decreases while $\bar{\pi}$ increases is the following. As risk aversion increases, an optimal policy gives more weight to reducing the volatility of current income. A lower $\mu$ decreases income volatility (as can be observed from in the log case: $w + \tau = \frac{w}{1-\alpha(1-\frac{1}{\bar{\mu}})}$), while in the inflation targeting regime a higher $\pi$ decreases volatility as is obvious from equation (3.8).

4. Liquidity shocks

We now pursue another variation on the standard random relocation model by introducing liquidity shocks. Specifically, we assume that $\alpha_t$, the fraction of young agents relocating to the other location, is drawn each period from an i.i.d distribution $g(\alpha)$ with support $[\alpha, \bar{\alpha}]$. As described earlier, shocks to $\alpha$ represent money demand or liquidity shocks. We assume that these shocks are realized each period *before* the bank makes its portfolio decisions. Our goal as before is to investigate how monetary policy should respond to shocks to liquidity preference. Analogous to the setting with endowment uncertainty, the bank cares about next period’s liquidity demand because it will potentially influence next period’s price level and thus the return on money between this period and the next. As before, we assume the bank knows the distribution for $\alpha$ and forms expectations on the return on money conditional on $g(\alpha)$; in a rational expectations equilibrium, these expectations are correct. We will focus on stationary versions of such equilibria below. As we demonstrate below, the impact of such liquidity shocks is entirely different from the endowment shocks studied earlier, even though at first blush it may seem that they ought to have similar effects (after all, both shocks work through liquidity demand and the return on money).

4.1. Logarithmic utility. We hold $w$ fixed for all $t$. The bank’s problem is now described by

$$\max_{\alpha_t} \int_{\alpha} \alpha_t \ln \left[ \frac{\gamma_t p_t}{\alpha_t p_{t+1}} \right] + (1 - \alpha_t) \ln \left[ \frac{(1 - \gamma_t) x}{1 - \alpha_t} \right] + \ln [w + \tau_t] \, g(\alpha_{t+1}) \, d\alpha_{t+1}$$

which reduces to

$$\max_{\alpha_t} \int_{\alpha} \alpha_t \ln \left[ \frac{p_t}{p_{t+1}} \right] g(\alpha_{t+1}) \, d\alpha_{t+1} + \alpha_t \ln \left[ \frac{\gamma_t}{\alpha_t} \right] + (1 - \alpha_t) \ln \left[ \frac{(1 - \gamma_t) x}{1 - \alpha_t} \right] + \ln [w + \tau_t]$$

(4.1)

---

7Smith (2002) and Antinolfi, Huybens, and Keister (2001) consider settings where such shocks are realized *after* the bank has made its portfolio decisions. In such situations, “banking crises” may arise, i.e., if the realized value of the liquidity shock is “too high”, the bank may run out of all its cash reserves and even be forced to prematurely liquidate storage. These issues are the subject matter of a companion paper.
Note that bank’s choice of $\gamma$ will only consider the second and the third term. It is easy to verify that the optimal choice of $\gamma_t$ is given by

$$\gamma_t = \alpha_t \text{ for all } t.$$  \hspace{1cm} (4.2)

Now the choice rule for $\gamma$ is state-contingent irrespective of the monetary policy regime.

4.1.1. Optimal monetary targeting. Under monetary targeting, the government fixes the money growth rate at $\mu$. It follows that the real return to money is given by (3.1). Also, since $\tau_t = \frac{M_t - M_{t-1}}{\mu} = m_t \left(1 - \frac{1}{\mu}\right)$, $m_t = \gamma_t (w + \tau_t)$, and $\gamma_t = \alpha_t$, we have

$$\tau_t = \frac{\alpha_t w \left(1 - \frac{1}{\mu}\right)}{1 - \alpha_t \left(1 - \frac{1}{\mu}\right)}$$

implying $w + \tau_t = w/[1 - \alpha_t \left(1 - \frac{1}{\mu}\right)]$, and hence

$$R_{m,t} = \frac{w + \tau_{t+1}}{\mu (w + \tau_t)} = \frac{\alpha_{t+1}}{\alpha_t} \frac{1 - \alpha_t \left(1 - \frac{1}{\mu}\right)}{\mu \left[1 - \alpha_{t+1} \left(1 - \frac{1}{\mu}\right)\right]}$$  \hspace{1cm} (4.3)

Then using (4.3), we can write indirect utility as

$$W_t = \int_0^{\tilde{\alpha}} \alpha_t \ln \left[ \frac{\alpha_{t+1}}{\alpha_t} \frac{1 - \alpha_t \left(1 - \frac{1}{\mu}\right)}{\mu \left[1 - \alpha_{t+1} \left(1 - \frac{1}{\mu}\right)\right]} \right] g(\alpha_{t+1}) \, d\alpha_{t+1}$$

$$+ (1 - \alpha_t) \ln x + \ln w - \ln \left[1 - \alpha_t \left(1 - \frac{1}{\mu}\right)\right]$$

which reduces to

$$W_t = \alpha_t \int_0^{\tilde{\alpha}} \ln \left[ \frac{\alpha_{t+1}}{1 - \alpha_{t+1} \left(1 - \frac{1}{\mu}\right)} \right] g(\alpha_{t+1}) \, d\alpha_{t+1} + (1 - \alpha_t) \ln x$$

$$+ \ln w - \alpha_t \ln \alpha_t - (1 - \alpha_t) \ln \left[1 - \alpha_t \left(1 - \frac{1}{\mu}\right)\right] - \alpha_t \ln \mu$$

Define $\int_0^{\tilde{\alpha}} \alpha g(\alpha) \, d\alpha \equiv \alpha_E$; then it can be checked that stationary welfare is given by

$$\int W g(\alpha) \, d\alpha = (1 - \alpha_E) \ln x + \ln w - \alpha_E \ln \mu$$

$$- \int_0^{\tilde{\alpha}} \left\{ (\alpha - \alpha_E) \ln \alpha + (1 - (\alpha - \alpha_E)) \ln \left[1 - \alpha \left(1 - \frac{1}{\mu}\right)\right] \right\} g(\alpha) \, d\alpha$$
What $\mu$ maximizes stationary welfare? The one that solves \( \frac{d}{d\mu} \int W f(\alpha) \, d\alpha = 0 \), and is implicitly defined by

\[
\tilde{\mu} = \frac{1}{\alpha_E} \int_a^\alpha \frac{1 - (\alpha - \alpha_E)}{1 - \alpha \left(1 - \frac{1}{\mu}\right)}ag(\alpha) \, d\alpha. \tag{4.4}
\]

**Proposition 2.** Suppose $g(\alpha) = \frac{\alpha - a}{b - a}$, i.e., $g$ is uniform over $[a, b]$; then $\tilde{\mu} = 1$ cannot be a solution to (4.4).

**Proof.** First off, it is easy to check that

\[
\alpha_E = \int_a^b \alpha g(\alpha) \, d\alpha = \frac{1}{6} (b - a) (a + 2b)
\]

Set $\tilde{\mu} = 1$ in the rhs of (4.4). Then

\[
\frac{1}{\alpha_E} \int_a^b \frac{1 - (\alpha - \alpha_E)}{1 - \alpha \left(1 - \frac{1}{\mu}\right)}ag(\alpha) \, d\alpha = \frac{1}{\alpha_E} \int_a^b (1 - (\alpha - \alpha_E))ag(\alpha) \, d\alpha
\]

Tedious algebra verifies that the rhs of the above expression reduces to

\[
\frac{1}{6} (a + 2b)^{-1} (4b^3 - 3a^2 - a^3 - 9b^2 - 6ab - 3a^2b + 36)
\]

which is not equal to 1. \qed

Clearly, constant money supply, unlike in the case of real endowment shocks, is no longer optimal.

4.1.2. *Inflation rate targeting.* Under inflation rate targeting, the government fixes the inflation rate at $\pi$. It follows that the real return to money is given by (3.6). In this case, using (4.1), indirect utility reduces to

\[
W_t = -\alpha_t \ln \pi + (1 - \alpha_t) \ln x + \ln \left(\frac{m_t}{\alpha_t}\right). \tag{4.5}
\]

Since $\tau_t = \frac{M_t - M_{t-1}}{\pi} = m_t - \frac{m_{t-1}}{\pi}$, we get

\[
m_t = \alpha_t (w + \tau_t) = \alpha_t \left( w + m_t - \frac{m_{t-1}}{\pi} \right)
\]

implying, analogous to (3.8),

\[
m_t = -\frac{\alpha_t}{(1 - \alpha_t) \pi} m_{t-1} + \frac{\alpha_t}{1 - \alpha_t} w
\]

or

\[
\left(\frac{m_t}{\alpha_t}\right) = -\frac{\alpha_{t-1}}{(1 - \alpha_t) \pi} \left(\frac{m_{t-1}}{\alpha_{t-1}}\right) + \frac{w}{1 - \alpha_t}
\]
Note that $m$ and $\alpha$ are jointly distributed, and from \eqref{eq:4.5}, it is clear that we seek the invariant distribution of $\frac{m}{\alpha}$ which will depend on the distribution of $\alpha$ and the value of $\pi$. Denote it as $\Omega \left( g(\alpha), \pi \right)$. Then, stationary welfare is given by

$$
\int W g(\alpha) \, d\alpha = -\alpha E \ln \pi - (1 - \alpha E) \ln x + \int \left\{ \ln \left( \frac{m}{\alpha} \right) \right\} \Omega \left( g(\alpha), \pi \right) d\left( \frac{m}{\alpha} \right)
$$

and optimal choice of $\pi$ is given by the solution to $\frac{d}{d\pi} \int W f(\alpha) \, d\alpha = 0$.

4.2. **CRRA utility.** We briefly sketch the main elements of the analysis when preferences are generalized to the CRRA form. The bank’s problem is now described by

$$
\max_{\gamma_t} \left\{ \frac{w + \tau t}{1 - \phi} - \alpha_t \left( \frac{\gamma_t}{\alpha_t} \right)^{1-\phi} \int_{\alpha}^{\gamma_t} (R_{m, t})^{1-\phi} g(\alpha_{t+1}) \, d\alpha_{t+1} + (1 - \alpha_t) \left( \frac{(1 - \gamma_t) x}{1 - \alpha_t} \right)^{1-\phi} \right\}
$$

The first order conditions to this problem yields

$$
\gamma_t = \frac{\alpha_t}{(1 - \alpha_t) \left( \int_{\alpha}^{\gamma_t} (R_{m, t})^{1-\phi} g(\alpha_{t+1}) \, d\alpha_{t+1} \right)^{-\frac{1}{\phi}}} \left( \frac{1 - \gamma_t}{\alpha_t} \right)^{1-\phi} + \alpha_t \tag{4.6}
$$

Here, it is possible that $\gamma_t \geq \alpha_t$.

4.2.1. **Inflation rate targeting.** Under inflation rate targeting,

$$
\int_{\alpha}^{\gamma_t} (R_{m, t})^{1-\phi} g(\alpha_{t+1}) \, d\alpha_{t+1} = (\pi)^{\phi-1}
$$

implying [from \eqref{eq:4.7}]

$$
\gamma(\alpha_t; \pi) = \frac{\alpha_t}{\alpha_t + (1 - \alpha_t) (\pi x)^{\frac{1-\phi}{\phi}}} \tag{4.7}
$$

and

$$
m_t = -\frac{\gamma(\alpha_t; \pi)}{(1 - \gamma(\alpha_t; \pi)) \pi} m_{t-1} + \frac{\gamma(\alpha_t; \pi)}{(1 - \gamma(\alpha_t; \pi))} w
$$

$$
= -\frac{\alpha_t}{(1 - \alpha_t) (\pi x)^{\frac{1-\phi}{\phi}}} m_{t-1} + \frac{\alpha_t}{(1 - \alpha_t) (\pi x)^{\frac{1-\phi}{\phi}}} w
$$

As before, we can find the unconditional joint distribution of $\{m, \alpha\} : \Omega (m, \alpha; \pi)$. Then, stationary welfare, $\int W g(\alpha) \, d\alpha$, is given by

$$
W^\pi = \frac{1}{1 - \phi} \int \left[ \alpha_t \left( \frac{\gamma(\alpha_t; \pi)}{\alpha_t} \right)^{1-\phi} (\pi)^{\phi-1} + (1 - \alpha_t) \left( \frac{(1 - \gamma(\alpha_t; \pi)) x}{1 - \alpha_t} \right)^{1-\phi} \right] \left( \frac{m}{\gamma(\alpha_t; \pi)} \right)^{1-\phi} \Omega (m_t, \alpha_t; \pi) \, dm \, d\alpha \tag{4.9}
$$
which reduces to
\[ W^\pi = \frac{1}{1 - \phi} \int \int \alpha^\phi (\pi)^{\phi - 1} \left[ 1 + \frac{1 - \alpha}{\alpha} (\pi x)^{\frac{1 - \phi}{\phi}} \right] m^{1 - \phi} \Omega (m, \alpha; \pi) \, dm \, d\alpha \]

As before, the optimal \( \pi \) is the one that solves \( \partial W^\pi / \partial \pi = 0 \).

4.2.2. Monetary targeting. Under monetary targeting, the return on money evolves as
\[ R_{m,t} = \frac{1}{\mu} \gamma_{t+1} \left( 1 - \frac{1}{\mu} \right) \frac{1 - \gamma_t \left( 1 - \frac{1}{\mu} \right)}{1 - \gamma_{t+1} \left( 1 - \frac{1}{\mu} \right)} \]

and indirect utility is given by
\[ W_\mu^\mu = \int_{\alpha} \left\{ \left( w + \tau_t \right)^{1 - \phi} \left[ \alpha_t \left( \gamma(\alpha; \mu) \right)^{1 - \phi} \int_{\alpha} \left( R_{m,t} \right)^{1 - \phi} g(\alpha_{t+1}) \, d\alpha_{t+1} \right] + (1 - \alpha_t) \left( \frac{1 - \gamma(\alpha; \mu) x}{1 - \alpha_t} \right)^{1 - \phi} \right\} g(\alpha_t) \, d\alpha_t. \]

Finally, stationary welfare is given by
\[ W^\mu = \frac{w^{1 - \phi}}{1 - \phi} \int_{\alpha} \left[ \left( \frac{1}{\mu} \right)^{1 - \phi} \left( \int_{\alpha} \alpha^\phi g(\alpha) \, d\alpha \right) \left( \frac{\gamma(\alpha; \mu)}{1 - \gamma(\alpha; \mu) \left( 1 - \frac{1}{\mu} \right)} \right)^{1 - \phi} \right. \\
\left. + (1 - \alpha) \phi \left( \frac{1 - \gamma(\alpha; \mu) x}{1 - \gamma(\alpha; \mu) \left( 1 - \frac{1}{\mu} \right)} \right)^{1 - \phi} \right] g(\alpha) \, d\alpha. \]

As before, the optimal \( \mu \) is the one that solves \( \partial W^\mu / \partial \mu = 0 \).

4.3. Numerical results with money demand shocks. Below we report results from several numerical experiments. We adopt the following parametric specification: \( x = 1.04, w = 1 \), and \( g(\alpha) \) is i.i.d and uniform with support \([0.18, 0.22]\). We work with the CRRA form, allowing \( \phi \) to vary in some experiments. Our results are as follows:

- in the case of liquidity shocks, for the entire range of risk-aversion \( \phi \) going from 0.5 to 2.1, a policy of inflation-rate targeting produces the higher welfare.

- The optimal net inflation rate maybe positive or negative.
- The optimal net money growth rate is always negative; that is while under inflation rate targeting, it may be optimal to pursue an expansionary policy, it is never optimal to do so under money growth targeting.
- The percentage gain in stationary welfare in switching from monetary targeting to inflation targeting varies from near 0.5% to about 7% as \( \phi \) varies between 0.5 and 2.1. The gain is smaller than in the case with endowment shocks.
References


