On the Equivalence of Monetary Policy Rules: Balanced Growth and Transitional Dynamics*

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Abstract

In a simple Ak endogenous growth model with flexible prices where a cash-in-advance constraint applies to both consumption and investment goods, we study the equivalent relation between money growth and interest rate rules. We restrict these monetary policy rules to yield the same balanced growth path equilibria, to exhibit the same equilibrium dynamics, and to have qualitatively equivalent comparative statics results. Our main finding is that an active (passive) interest rate is equivalent to an active (passive) money growth rule, where the central bank raises the rate of money growth by more (less) than one percentage point in respond to a one-percentage point increase in inflation. If the intertemporal elasticity of substitution in consumption is not greater than unity, we find that constant money growth rules cannot mimic interest rate rules.

Key Words: money growth rules, interest rate rules, equivalence.

JEL Classification: E52, O42.

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1 Introduction

The study of monetary policy has long been an important topic in macroeconomics. In his now-famous Carnegie-Rochester paper, Taylor (1993) summarizes the operation of monetary policy by a rule that adjusts the nominal interest rate in response to output and inflation, now widely known as the Taylor rule. Since then the recent literature on monetary policy has almost focused on interest rate rules exclusively [e.g., Benhabib et al]. However, most of the textbook cases in macroeconomics and monetary economics still describe monetary policy in terms of constant money growth rules advocated by Milton Friedman (1959) nearly half a century ago. In practice, monetary policies are usually stated in terms of the short-term interest rate targets (such as the Federal Fund rate in the U.S.) which are achieved by the central banks through their control over different monetary aggregates (e.g., the money supply or more accurately the supply of reserves). However, as pointed out by Eichenbaum (1992), the qualitative and quantitative effects of monetary policy are sensitive to the choice of measures of disturbances to monetary policy (i.e., innovations to short-term interest rates versus innovations to central bank reserves) used in the vector autoregressive analysis. On the other hand, based on the quantity equation, Taylor (1999) argued that a feedback interest rate rule "provides a good description of monetary policy in a fixed money growth regime." Empirically, Fève and Auray (2002) and Minford et. al. (2002) have both shown that the estimated interest rate relation with output and inflation of a Taylor rule is observationally equivalent to the equilibrium relation of the same variables in a monetary economy with exogenous money growth rule. Thus, it is worth investigating the actual relation between these two types of monetary policy rules.

Recently, several papers have attempted to study how money growth and interest rates are related and whether equivalence can be established among simple rules for both. Carlstrom and Fuerst (1995) compares the welfare effects of these two types of monetary policy rules in a flexible price economy where there are cash-in-advance constraints on households' consumption purchases and firms' wage bill. As a result, the competitive equilibrium is subject to a capital accumulation distortion and a portfolio choice distortion. Since these distortions are related to the nominal interest rate, the interest rate peg policy can elimi-
nate them while the money growth peg policy cannot. The former rule therefore is "the benevolent central banker’s preferred policy." Monnet and Weber (2001) explains the correlation between money growth and interest rate can be either negative (the liquidity effect) or positive (the Fisher equation effect) and all it "depends on when the change in money occurs and how long the public expects it to last." This explanation applies to monetary policies that are either stated in terms of money supply growth or in terms of interest rates. In an economy with costly trading, Végh (2001) formally establishes conditions for equivalence between three types of monetary policy rules: a "k-percent" money growth rule, a nominal interest rate rule combined with an inflation target, and a real interest rate rule combined with an inflation target. The criterion for equivalence requires that monetary policy rules "yield exactly the same dynamics in response to, say, a long-term reduction in the inflation rate." Schabert (2003) derives equivalent conditions for interest rate and money growth rules by focusing on the restriction that both rules "implement the same fundamental solution, also known as the bubble free solution, for the perfect foresight equilibrium." The analytical framework adopted is a cash-in-advance model with staggered price setting. Under flexible prices, it is found that a constant money growth rule is equivalent to a passive interest rate rule, while an active interest rate rule behaves like an accommodating money growth rule where money growth responds to the levels of inflation. When prices are sticky, then history dependent interest rate policies are required for equivalence. In addition, constant money growth rules can no longer mimic the Taylor-type interest rate rules. Nevertheless, focusing only on the bubble-free solutions to establish equivalence between interest rate and money growth rules may be too restrictive. As emphasized by Auray and Fève (2002), the equivalent relation can depend "on the relative size of the sunspot variables associated to nominal and real variables" of the dynamic model.

In this paper, we follow the literature to investigate the relation between money growth and interest rate rules. We conduct our analysis in a simple Ak endogenous growth model with flexible prices where a cash-in-advance constraint applies to both consumption and investment goods. By focusing on the equivalence of these policy rules, we require them to yield the same balanced growth path (BGP) equilibria [like in Schabert (2003)] and exhibit qualitatively same equilibrium dynamics [as in Végh (2001)]. Moreover, we add one more criterion, that is, we also restrict the comparative statics results to be qualitatively equivalent [e.g., Wang and Yip (1992)]. Following Schabert (2003), we allow
for an accommodating money growth rule where the money growth rate depends "on the current realization of the inflation rate." Our main finding is that an active interest rate is equivalent to an active money growth rule, where the central bank raises the rate of money growth by more than one percentage point in respond to a one-percentage point increase in inflation. Similarly, a passive money growth rule, where a one-percentage point increase in inflation is associated with a less than one percentage point increase in the nominal money growth rate, mimics a passive interest rate rule. In contrast to Végh (2001) and Schabert (2003), if the intertemporal elasticity of substitution in consumption is not greater than unity, we find that constant money growth rules [Friedman(1959)] cannot be replicated by accommodating interest rate rules proposed by Taylor (1993).

The organization of the paper is as follows. The next section provides the basic model for the analysis. Section 3 establishes conditions for the equivalence between money growth and interest rate rules. Concluding remarks are given in section 4.

2 The Model

In this section, we develop the basic monetary endogenous growth model to study the dynamics of equilibrium under money growth rule and interest rate rule. Money is required in advance to purchase consumption and investment goods, or the so-called Stockman (1981) type cash in advance (CIA) constraint.

2.1 The economic environment

Representative Agents. The economy consists of a continuum of identical representative agents with unit mass, each of whom maximizes his lifetime utility according to

\[ U = \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma} - 1}{1-\sigma} dt \]  

(1)

where \( c \) is consumption, \( \sigma > 0 \) is the inverse of the elasticity of intertemporal substitution and \( \rho > 0 \) is the subjective rate of time preferences.\(^1\) In addition to money the agent can also hold nominal bonds and physical capital. The nominal bonds pay the nominal interest rate \( R > 0 \). The momentary budget constraint of a typical agent is

\(^1\)Time index is omitted to ease the burden of notations.
\[ c + \dot{m} + \dot{b} + \dot{k} = (R - \pi) b + y - \pi m - \tau \]  

(2)

where \( \dot{x} \equiv \frac{dx}{dt} \) is the time derivative of the variable \( x \), \( k \) is the capital, \( b \) is the real bonds holdings, \( m \equiv M/P \) is real money balances defined by deflating the nominal money stock \( (M) \) by the price level \( (P) \), \( \pi \equiv \dot{P}/P \) denotes the inflation rate, \( \tau \) is the lump-sum transfers and \( y \) is the output. The single consumption good prevailing in this economy is produced by a simple \( Ak \) technology.Depreciation rate of physical capital is set to zero, a simplification that affects none of our major results.

Following Stockman (1981), each agent faces an additional liquidity constraint given by

\[ c + \dot{k} \leq m \]  

(3)

By defining the agent’s non-capital wealth as \( a = m + b \), the agent budget constraint can be written as

\[ c + \dot{a} + \dot{k} = (R - \pi) a + Ak - Rm - \tau \]  

(4)

The representative agent’s optimization problem is given by maximizing \( (1) \) subject to \( (3), (4) \), nonnegativity constraints of \( c, k, M \), and the initial asset holdings: \( k(0) = k_0 > 0 \), \( a(0) = a_0 > 0 \).

Let \( \psi \) be the Lagrangian multiplier associated with the general CIA constraint, \( \lambda_k \) and \( \lambda \) be the costate variables of capital and non-capital wealth respectively. Interior solutions of the above problem are characterized by the first-order conditions:

\[ e^{-\sigma} = \lambda + \psi \]  

(5)

\[ \psi = R\lambda \]  

(6)

\[ \lambda + \psi = \lambda_k \]  

(7)

\[ \dot{\lambda} = \lambda (\rho + \pi - R) \]  

(8)

\[ \dot{\lambda}_k = \rho \lambda_k - \lambda A \]  

(9)

and the transversality conditions

\[ \lim_{t \to \infty} e^{-\rho t} \lambda_k k = \lim_{t \to \infty} e^{-\rho t} \lambda a = 0 \]

In addition, the goods market equilibrium condition yields
\[ \dot{k} = Ak - c \]  

We now perform a balanced growth analysis to solve for an optimal endogenous monetary growth equilibrium. From (5), (7) and (9), we can solve for

\[ \theta = \frac{\dot{c}}{c} = \frac{1}{\sigma} \left( \frac{\lambda}{\lambda_k} A - \rho \right) \]  

where \( \theta \) is the constant growth rate of per capita consumption. By definition, the rate of growth of each endogenous variable (which may not be necessarily equal) is constant along a balanced growth path (BGP). From the goods market equilibrium, we obtain the consumption to capital ratio

\[ \frac{c}{k} = A - \frac{\dot{k}}{k} \]

which is constant along a BGP. Thus (per capita) consumption and capital have to grow at same rate, \( \theta \). The constancy of the marginal product of capital implies output and capital also have to grow at the same rate. Assuming that the nominal interest rate is positive so that the cash-in-advance constraint always holds with equality, we know that consumption and real money demand grow at the same rate along a BGP. In summary

\[ \frac{\dot{k}}{k} = \frac{\dot{c}}{c} = \frac{\dot{m}}{m} = \frac{\dot{y}}{y} = \theta \]  

For the costate variable, we can combine (8), (9) and (11) to show that

\[ \frac{\dot{\lambda}}{\lambda} = \frac{\dot{\lambda}_k}{\lambda_k} = -\sigma \theta \]  

**Government.** Following Ireland (2003), the Central Bank conducts monetary policy by adjusting a linear combination of the short term nominal interest rate \( R \) and the money growth rate \( \mu \) in response to deviations of inflation from their steady rate values (or targets), according to the policy rule

\[ \alpha (R - R^*) + \beta (\mu - \mu^*) = \gamma_i (\pi - \pi^*) \text{ and } i = \mu \text{ and } R \]  

where \( \alpha, \beta \) and \( \gamma_i \) are the response coefficients chosen by the central bank and \( R^*, \mu^* \) and \( \pi^* \) are the steady rate (or target) values for \( R, \mu, \) and \( \pi \).\(^2\) When \( \alpha = 1, \beta = 0 \) and \( \gamma_R > 0 \), it becomes an interest rate rule. When \( \alpha = 0, \beta = 1 \) and \( \gamma_\mu \neq 0 \), it becomes

\(^2\)The subscript \( i \) is used to distinguish different inflation elasticities under interest rate rule and money growth rule.
the accommodating money growth rule studied by McCallum (1999) and Schabert (2003). Finally, the case of \( \alpha = \gamma_\mu = 0 \) and \( \beta > 0 \) corresponds to the case of constant money growth rule.

2.2 Equilibrium Analysis

2.2.1 Accommodating Money Growth Rules

When the central bank chooses an accommodating money growth rule (i.e. \( \mu = \mu^* + \gamma_\mu (\pi - \pi^*) \)), \( R \) becomes endogenous and equilibrium in money market implies

\[ \dot{m} = (\mu - \pi) m \]  \hspace{1cm} (15)

We assume the CIA constraint is binding in equilibrium, and when the goods market clear, the binding CIA constraint then implies the quantity of real money holdings exactly equals the quantity of aggregate output. Hence, real money balances and physical capital must be growing at the same rate. Therefore, by considering (10) and (15), we can rewrite the inflation as

\[ \pi = \frac{1}{1 - \gamma_\mu} (\mu^* - \gamma_\mu \pi^* - A + z) \]  \hspace{1cm} (16)

where \( z \) is the consumption to capital ratio (i.e. \( z \equiv c/k \)). Then, simple algebra yields

\[ \dot{z} = \left( \frac{A}{\sigma p} + z - \frac{\rho + \sigma A}{\sigma} \right) \]  \hspace{1cm} (17)

where \( p \) is ratio of the shadow price of capital to that of non-capital. \( (i.e. \ p \equiv \lambda_k/\lambda) \). Using (6), (7) and the definition of \( p \), one can derive the following relationship

\[ R = p - 1 \]  \hspace{1cm} (18)

Then, we can also derive the following differential equation for \( p \).

\[ \dot{p} = \left[ p - \frac{A}{p} - 1 - \frac{1}{1 - \gamma_\mu} (\mu^* - \gamma_\mu \pi^* - A + z) \right] \]  \hspace{1cm} (19)

where we have used (16) and (18) to substitute away the \( \pi \) and \( R \) respectively. Then, (17) and (19) constitute the dynamic system that completely characterizes the model’s equilibrium under money growth rules.
Comparative Statics  Solving (17) and (19), a BGP equilibrium consists of a pair of positive real numbers \((\bar{p}, \bar{z})\) characterized by

\[(\bar{p})^2 - \beta_1 \bar{p} + \beta_2 = 0\]  \hfill (20)

and

\[\bar{z} = \frac{\rho}{\sigma} + A \left(1 - \frac{1}{\sigma \bar{p}}\right)\]  \hfill (21)

where \(\beta_1 \equiv \frac{\rho}{\sigma} + 1 + \frac{\mu^* - \gamma_\mu \pi^*}{1 - \gamma_\mu}\), \(\beta_2 \equiv (\frac{1 - \tilde{\sigma}}{\tilde{\sigma}}) A\) and \(\tilde{\sigma} \equiv \sigma (1 - \gamma_\mu)\). Totally differentiate (20) with respect to \(\bar{p}\) and \(\mu^*\), we have

\[\frac{d\bar{p}}{d\mu^*} = \frac{\bar{p}}{2(1 - \gamma_\mu)} (\bar{p} - \beta_1/2).\]  \hfill (22)

From (11) and the definition of \(p\), common growth rate is given by

\[\theta = \frac{1}{\sigma} \left[\frac{A}{\bar{p}} - \rho\right].\]  \hfill (23)

Hence,

\[\frac{d\theta}{d\mu^*} = -\frac{A}{\sigma (\bar{p})^2} \frac{d\bar{p}}{d\mu^*} = -\frac{A}{2\sigma \mu^* \bar{p} (1 - \gamma_\mu)} (\bar{p} - \beta_1/2).\]  \hfill (24)

Solving the quadratic equation (20), we get the roots:

\[\bar{p} = \frac{\beta_1 \pm \sqrt{\Delta}}{2},\]  \hfill (25)

where \(\Delta \equiv (\beta_1)^2 - 4 (\frac{1 - \tilde{\sigma}}{\tilde{\sigma}}) A\). When \(\tilde{\sigma} < 0\) or \(\tilde{\sigma} > 1\), one of the roots is negative and has to be rejected due to the nonnegativity restriction of \(\bar{p}\). In this case, a unique BGP equilibrium exists with \(\bar{p} > \beta_1/2\). Hence, according to (24), an increase in the growth rate of nominal money supply will reduce (increase) the long-run common growth rate of other aggregates when \(\tilde{\sigma} > 1(<0)\). Figures 1a and 1b provide the graphical representation of our comparative statics result for the case where \(\tilde{\sigma} > 1\) and \(\tilde{\sigma} < 0\) respectively.

[INSERT FIGURE 1a AND 1b HERE]

From (17) and (19), it is straightforward to show that the equilibrium locus \(\dot{z} = 0\) is upward sloping. On the other hand, the \(\dot{p} = 0\) locus is upward (downward) sloping when \(\tilde{\sigma} > 1(<0)\). Specifically, when it is upward sloping, the \(\dot{p} = 0\) locus should have a steeper slope than the \(\dot{z} = 0\) locus. An increase in nominal money growth shifts the \(\dot{p} = 0\) locus
to the left so that \( dp/d\mu > 0(\leq) \) for \( \tilde{\sigma} > 1(\leq) \). On the other hand, when \( 0 < \tilde{\sigma} < 1 \), two possible values of \( \bar{p} \) emerge as shown in Figure 2.

\[ \text{[INSERT FIGURE 2 HERE]} \]

This is due to the fact that the relative magnitudes of the slopes of the two equilibrium loci depend on the values of \( \bar{p} \). Let \( \bar{p}_1 \) and \( \bar{p}_2 \) denote the two roots with \( \bar{p}_1 > \bar{p}_2 \).\(^5\) It is then straightforward to derive the following results

\[ \bar{p}_1 > \frac{\beta_1}{2} > \bar{p}_2, \quad \theta (\bar{p}_2) > \theta (p_1) \quad \text{and} \quad \frac{d\theta (\bar{p}_2)}{d\mu^*} > 0 > \frac{d\theta (\bar{p}_1)}{d\mu^*}. \]

**Local Dynamics**  Linearizing the system of (17) and (19) in the neighborhood of a BGP:

\[
\begin{bmatrix}
\dot{p} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
\bar{p} \left[ 1 + \frac{A}{(\bar{p})^2} \right] & -\frac{\bar{p}}{1-\gamma}\mu \\
-\frac{A}{\sigma (\bar{p})^2} & -\frac{\bar{p}}{\bar{z}}
\end{bmatrix} \begin{bmatrix}
p - \bar{p} \\
z - \bar{z}
\end{bmatrix}. \tag{26}
\]

The determinant and trace of the Jacobian matrix \( J \) are

\[
\det J = \bar{p} \bar{z} \left[ 1 - \frac{A}{(\bar{p})^2} \left( \frac{1 - \tilde{\sigma}}{\tilde{\sigma}} \right) \right], \tag{27}
\]

\[ \text{Trace } J = \bar{p} \left[ 1 + \frac{A}{(\bar{p})^2} \right] + \bar{z} > 0. \tag{28} \]

To determine the sign of \( \det J \), we consider \( \dot{p} = 0 \) at equilibrium. From (20), we have

\[
\det J = 2\bar{z} (\bar{p} - \beta_1/2). \]

It is clear that \( \det J \) must be positive when \( \tilde{\sigma} > 1 \), since the fraction inside the bracket must be smaller than one by the fact that \( \bar{p} \geq 0 \). By inspecting (27), \( \det J \) is also positive when \( \tilde{\sigma} < 0 \). The BGP equilibrium in these two cases is therefore a source so that local indeterminacy cannot occur. But when \( 0 < \tilde{\sigma} < 1 \), \( \det J > 0 (< 0) \) for the BGP with \( \bar{p}_1 \) (\( \bar{p}_2 \)) (where \( \bar{p}_1 > \bar{p}_2 \)). Thus the BGP equilibrium can be either a source or a saddle in this subcase. As a result, local indeterminacy cannot be ruled out.

\(^5\)In order to ensure that both roots satisfy the transversality conditions, an upper bound on the nominal money growth rate given by

\[
\frac{(1 - \tilde{\sigma}) A}{\rho} > 1 + \mu^*
\]

is needed when \( 0 < \tilde{\sigma} < 1 \).
Proposition 1  In the Ak model with accommodating money growth rule, we have non-superneutrality in the growth-rate sense. When $\tilde{\sigma} > 1$ or $\tilde{\sigma} < 0$ there exists a unique balanced growth path equilibrium that is locally determinate and faster money growth retards (promotes) economic growth in the former (latter) case. When $0 < \tilde{\sigma} < 1$, then dual BGP equilibria may emerge in which one of them is locally determinate where faster money growth lowers economic growth, and the other is indeterminate where faster money growth raises economic growth.

2.2.2  Interest Rate Rules

When the central bank choose an interest rate rule, $m$ becomes endogenous and the policy rule will become

$$R = \gamma_R (\pi - \pi^*) + R^*$$  \hspace{2cm} (29)

We refer the policy rule as active if $\gamma_R > 1$ and passive if $\gamma_R < 1$. We further assume $\gamma_R \neq 1$. As in Meng (2002), that fiscal policy is Ricardian so that the present discounted value of total government liabilities converges to zero both in and off equilibrium. For details, see Benhabib et al. (2001) and the reference cited therein.

Combining (18) and (29), we can now rewrite the inflation as

$$\gamma_R (\pi - \pi^*) + R^* = p - 1$$  \hspace{2cm} (30)

$$\pi = \frac{1}{\gamma_R} (p - 1 - R^*) + \pi^*$$

which is the counterpart to (16). Next, we used (30) and (18) to substitute away the $\pi$ and $R$ again to derive the following differential equation in $p$

$$\frac{\dot{p}}{p} = \left[ p - \frac{A}{p} - 1 - \frac{1}{\gamma_R} (p - 1 - R^*) - \pi^* \right]$$  \hspace{2cm} (31)

$$= \left[ p - \frac{A}{p} - \frac{p}{\gamma_R} + \Omega \right]$$

where $\Omega \equiv \frac{R^* + 1 - \gamma_R (1 + \pi^*)}{\gamma_R}$.

Comparative Statics  Solving (17) and (31), a BGP equilibrium consists of a pair of positive real numbers $(\bar{p}, \bar{z})$ characterized by

$$\beta_3 (\bar{p})^2 + \Omega \bar{p} - A = 0$$  \hspace{2cm} (32)
\[
\ddot{z} = \frac{\rho}{\sigma} + A \left( 1 - \frac{1}{\sigma \bar{p}} \right)
\]  
(33)

where \( \beta_3 = \left( 1 - \frac{1}{\gamma_R} \right) \). Solving the quadratic equation (32), we get the roots

\[
\bar{p} = \frac{-\Omega \pm \sqrt{\Delta}}{2 \beta_3},
\]  
(34)

where \( \Delta \equiv \Omega^2 + 4 \left( 1 - \frac{1}{\gamma_R} \right) A \). When \( \gamma_R > 1 \), one of the roots is negative and has to be rejected due to the nonnegativity restriction of \( \bar{p} \). In this case, a unique BGP equilibrium exists with \( \bar{p} = \left( -\Omega + \sqrt{\Delta} \right) / 2 \beta_3 \) (global indeterminacy does not occur). On the other hand, when \( \gamma_R < 1 \), two possible values of \( \bar{p} \) emerge (global indeterminacy occurs). Let \( \bar{p}_1 \) and \( \bar{p}_2 \) denote the two roots with \( \bar{p}_1 > \bar{p}_2 \).

Totally differentiate (11) with respect to \( R^* \),

\[
\frac{d\theta}{dR^*} = -\frac{A}{\sigma (\bar{p})^2} \frac{d\bar{p}}{dR^*}
\]  
(35)

The following lemma is useful to determinate the sign of \( \frac{d\bar{p}}{dR^*} \) under active and passive interest rate rules.

**Lemma 2** \(|\Omega| > (\leq) \sqrt{\Delta} \) for passive (active) interest rate rules.

**Proof.** From the definition of \( \Delta \) and notice that \( 1 - \frac{1}{\gamma_R} > (\leq) 0 \) for the case of active (passive) interest rate rules, and the result follows. ■

For active interest rate rule, the only positive root is \( \bar{p} = \left( -\Omega + \sqrt{\Delta} \right) / 2 \beta_3 \) so that simple differentiation yields

\[
\frac{d\bar{p}}{dR^*} = \frac{\beta_3 - 1}{2 \beta_3} \left[ 1 - \frac{\Omega}{\sqrt{\Delta}} \right] < 0 \text{ and } \frac{d\theta}{dR^*} > 0
\]

Therefore, an increase in \( R^* \) will increase the growth rate and the corresponding figure is provided in figure 3.

[INSERT FIGURE 3 HERE]

For passive interest rate rule, we have

\[
\frac{d\bar{p}_1}{dR^*} = \frac{\beta_3 - 1}{2 \beta_3} \left[ 1 + \frac{\Omega}{\sqrt{\Delta}} \right] > 0 > \frac{d\bar{p}_2}{dR^*} = \frac{\beta_3 - 1}{2 \beta_3} \left[ 1 - \frac{\Omega}{\sqrt{\Delta}} \right]
\]

and hence

\[
\frac{d\theta (\bar{p}_2)}{dR^*} > 0 > \frac{d\theta (\bar{p}_1)}{dR^*}
\]
and the corresponding figure is provided in figure 4.

[INSERT FIGURE 4 HERE]

**Local Dynamics**  Because of the block recursive nature, the dynamics under interest rate rule can be completely described by the differential equation in $p$ only. We linearize (31) around the BGP to yield

$$\dot{p} = \eta (p - \bar{p})$$

where $\eta \equiv \bar{p} \left[ (1 - \frac{1}{\gamma_R}) + \frac{A}{(\bar{p})^2} \right]$ is the eigenvalue to the differential equation (31). For active interest rate rule $\gamma_R > 1$, it is easy to show that $\eta > 0$ and hence the dynamics is determinate. To further investigate the case of passive interest rate rule, we use (32) and (34) to simplify $\eta$ and we have the following lemma

**Lemma 3** $\text{sgn} \{\eta\} = \text{sgn} \left\{ \frac{2A}{\Omega} - \bar{p} \right\} = \text{sgn} \left\{ \frac{\sqrt{\Delta} \left[ \sqrt{\Delta} \pm \Omega \right]}{2\Omega \left[ 1 - \frac{1}{\gamma_R} \right]} \right\}$.

**Proof.** We first note that $\eta$ can be written as $\left( \frac{\Omega}{\bar{p}} - \frac{2A}{\Omega} - \bar{p} \right)$ by (32). Thus, we obtain the first equality. To obtain the second equality, we use (34) to substitute away $\bar{p}$ and yield $\frac{\Delta \pm \Omega \sqrt{\Delta}}{2\Omega \left( 1 - \frac{1}{\gamma_R} \right)}$. Factorize $\sqrt{\Delta}$ out gives the result. \[\blacksquare\]

Next, first notice that the non-negativity constraint of $\bar{p}$ implies that $\Omega > 0$ so that we have $\Omega > \sqrt{\Delta}$ under passive interest rate rules ($\gamma_R < 1$). This together with Lemma 2 above then yield $\eta < (>) 0$ for the low- (high-) growth equilibrium $\bar{p}_1 (\bar{p}_2)$. We now summarize our characterization of the dynamics for interest rate rules in the following proposition:

**Proposition 4** *In the Ak model with active interest rate rule ($\gamma_R > 1$), there is a unique BGP equilibrium and it is locally determinate. When interest rate rule becomes passive ($\gamma_R < 1$), dual BGP equilibria emerge. For the low-growth-rate equilibrium ($\bar{p}_1$), the local dynamics is a saddle and hence local indeterminacy occurs. For the high-growth-rate equilibrium ($\bar{p}_2$), the local dynamics is a source and hence locally determinate.*

### 3 Equivalence

To establish equivalence between money growth rules and interest rate rules, we require:
1. Both types of money policy rules yield the same BGP equilibria,
2. the BGP equilibria exhibit qualitatively same equilibrium dynamics,
3. the comparative statics results are qualitatively equivalent.

Recall first the quadratic equations (20) and (32) which are now reproduced here for convenience. For the case of accommodating money growth rules, we have

\[(\bar{p})^2 - \left(1 + \frac{\rho}{\sigma} + \frac{\mu^* - \gamma \mu \pi^*}{1 - \gamma \mu}\right) \bar{p} + \left(\frac{1 - \hat{\sigma}}{\sigma}\right) A = 0.\]

For interest rate rules, we have

\[(\bar{p})^2 - \left(1 + \frac{R^*}{1 - \gamma R} - \frac{\gamma R \pi^*}{1 - \gamma R}\right) \bar{p} + \left(\frac{\gamma R}{1 - \gamma R}\right) A = 0.\]

For the two quadratic equations to yield identical BGP equilibrium solutions, we can impose the following conditions

\[1 - \hat{\sigma} = \frac{\gamma R}{1 - \gamma R}, \quad \text{(36)}\]

\[\frac{\rho}{\sigma} + \frac{\mu^* - \gamma \mu \pi^*}{1 - \gamma \mu} = \frac{R^* - \gamma R \pi^*}{1 - \gamma R}. \quad \text{(37)}\]

Rearranging (36), we have

\[\gamma_R = 1 - \sigma (1 - \gamma_{\mu}) = 1 - \hat{\sigma}. \quad \text{(38)}\]

Then substituting (38) into (37) and assuming the inflation target remains unchanged regardless of the types of monetary policy rules being practiced, we obtain

\[R^* = \sigma \mu^* + [\rho + (1 - \sigma) \pi^*]. \quad \text{(39)}\]

We first note that (38) implies that the interest-rate and money-growth targets are positively related in policy design, revealing the long-run "Fisher equation view" emphasized by Monnet and Weber (2001). We assume the exogenous target can be manipulated so that (37) always hold, immediate results can then be inferred from (38), a passive interest rule \(\gamma_R < 1\) and a money growth rule with \(\gamma_{\mu} < 1\) can lead to identical BGP equilibrium solution. Notice that constant money rule \((\gamma_{\mu} = 0)\) is a subcase in this situation. Similarly, active interest rule \(\gamma_R > 1\) and an accommodating money growth rule with \(\gamma_{\mu} > 1\) can lead to identical BGP equilibrium solutions. Moreover, the restriction that \(\gamma_R > 0\) requires that \(\hat{\sigma} < 1\). We also like to point out that (39) implies that the comparative statics of \(R^*\) and \(\mu^*\) are positively correlated. This is important for establishing qualitative equivalent comparative statics results in later analysis.
We next establish equivalence for the equilibrium dynamics between the monetary policy rules. For active interest rate rules, the BGP equilibrium is unique and it is locally determinate. Under money growth rules, when \( \tilde{\sigma} < 0 \), the BGP equilibrium exhibits the same dynamic properties. For the comparative statics, we have shown that

\[
\text{sgn}(\frac{d\bar{p}}{d\mu^*}) = \text{sgn}(\frac{d\bar{p}}{dR^*}).
\]

Thus we can claim that active interest rate rules is equivalent to an accommodating money growth rule with \( \tilde{\sigma} < 0 \).

For both passive interest rate rules and accommodating money growth rules with \( 0 < \tilde{\sigma} < 1 \), dual BGP equilibria emerge and in each case one of the BGP equilibria is locally indeterminate while the other determinate.\(^6\) In addition, an increase in the target money growth rate (\( \mu^* \)) and an increase in the target nominal interest rate (\( R^* \)) both produce the same comparative statics results:

\[
\frac{d\bar{p}_2}{dt^*} < 0 < \frac{d\bar{p}_1}{dt^*},
\]

where \( \bar{p}_1 > \bar{p}_2 \) and \( i = \mu, R \). Table 1 provides a summary for the comparative statics results.

[INSERT TABLE 1 HERE]

We now summarize our findings on equivalence in the following proposition:

**Proposition 5** Consider the following two types of monetary policy rules:

1. accommodating money growth rules: \( \mu = \mu^* + \gamma_\mu (\pi - \pi^*) \),
2. interest rate rules: \( R = \gamma_R (\pi - \pi^*) + R^* \), \( \gamma_R > 0 \).

Then an active interest rate rule (\( \gamma_R > 1 \)) is equivalent to an accommodating money growth rule with \( \tilde{\sigma} < 0 \) where \( \tilde{\sigma} \equiv \sigma (1 - \gamma_\mu) \), while a passive interest rate rule (\( \gamma_R < 1 \)) is equivalent to an accommodating money growth rule with \( 1 > \tilde{\sigma} > 0 \).

A word of caution should be added to the above proposition on equivalence of passive-type monetary policy rules. For passive interest rate rules, the high-growth equilibrium is determinate while the low-growth equilibrium exhibits indeterminacy. On the other hand, for passive-type accommodating money growth rules, the high-growth equilibrium

\(^6\)It is noted that the low-growth equilibrium under accommodating money growth rules is locally indeterminate while it is determinate under passive interest rate rules.
becomes indeterminate while the low-growth equilibrium is determinate. Consequently, if we focus only on the bubble-free or determinate equilibrium as in Schabert (2003), then the policy implications (based on the comparative statics results) are completely different from the two types of monetary policy rules.

**Corollary 6** An active interest rate rule \((\gamma_R > 1)\) is equivalent to an accommodating money growth rule with \(\gamma_\mu > 1\), while a passive interest rate rule \((\gamma_R < 1)\) is equivalent to an accommodating money growth rule with \(1 > \gamma_\mu > 1 - 1/\sigma\).

According to the above corollary, we may infer that a constant money growth rule \((\gamma_\mu = 0)\) and an inflation-stabilizing money growth rule \((\gamma_\mu < 0)\) are equivalent to a passive interest rate rule. However, it can be shown that when \(\gamma_\mu < 0\), the comparative statics results are not qualitatively equivalent. In addition, since most of the empirical evidence shows that the intertemporal elasticity of substitution in consumption is not likely to be greater than unity (i.e., \(\sigma \geq 1\)), we believe that it is likely to have \(\gamma_\mu > 1 - 1/\sigma \geq 0\). This may reveal that the Friedman constant money growth rule cannot mimic the Taylor rule in practice.

**Proposition 7** If the intertemporal elasticity of substitution in consumption is not greater than unity (i.e., \(\sigma \geq 1\)), then constant money growth rules cannot mimic any interest rate rules.

Finally, we would like to examine a special case of passive interest rate rules: the interest rate pegging, i.e., \(\beta = \gamma_R = 0\), so that \(R = R^*\) according to (14). Using (6), (7) and the definition of \(p\), we can show that \(p\) is always constant:

\[
p = 1 + R^*. \tag{40}
\]

As a result, we have \(\dot{p}/p = 0\) for all time so that \(\pi\) is always constant:

\[
\pi^* = R^* - \frac{A}{1 + R^*}. \tag{41}
\]

To establish equivalence with money growth rules, we follow (38) and (39) to derive

\[
\gamma_\mu = 1 - 1/\sigma, \tag{42}
\]

\[
\sigma R^* + \frac{(1 - \sigma)A}{1 + R^*} = \rho + \sigma \mu^*. \tag{43}
\]
To sharpen our focus, we consider the special case where preferences are logarithmic (i.e., \( \sigma = 1 \)). Then (42) implies that \( \gamma_\mu = 0 \) and we have constant money growth rules. The BGP equilibrium solution of \( p \) is

\[
\bar{p} = 1 + R^* = 1 + \rho + \mu^*,
\]

so that the comparative statics results for both monetary policy rules are identical:

\[
d\bar{p}/dR^* = d\bar{p}/d\mu^* = 1.
\]

For equilibrium dynamics, it can be easily shown that the BGP equilibrium is a unique source locally under constant money growth rules. Thus equivalence can be established between these two types of pegging rules with logarithmic preferences.

**Proposition 8** When the felicity function is logarithmic \( (\sigma = 1) \), then nominal interest rate pegging policies \( (\gamma_R = 0) \) are equivalent to constant money growth rules \( (\gamma_\mu = 0) \).

### 4 Concluding Remarks

In a simple Ak endogenous growth model with flexible prices where a cash-in-advance constraint applies to both consumption and investment goods, we have investigated the equivalent relation between money growth and interest rate rules. We have considered general money growth rules that allow the money growth rate to respond to the rate of inflation. In the analysis, we have restricted these monetary policy rules to yield the same balanced growth path (BGP) equilibria, to exhibit the same equilibrium dynamics, and to have qualitatively equivalent comparative statics results. Our main finding is that an active (passive) interest rate is equivalent to an active (passive) money growth rule, where the central bank raises the rate of money growth by more (less) than one percentage point in respond to a one-percentage point increase in inflation. In contrast to Végh (2001) and Schabert (2003), if the intertemporal elasticity of substitution in consumption is not greater than unity, we find that constant money growth rules [Friedman(1959)] cannot mimic interest rate rules.

To close the paper, we would like to suggest an interesting extension of our analysis by turning to the case where prices are no longer flexible. As pointed out in Schabert (2003), when prices are sticky, the conditions for equivalence between money growth and interest rate rules are very different from the case of flexible prices. Specifically, real
money balances are now sluggish so that the dimension of the state variable system is augmented. As a result, the fundamental solution of the model is history dependent for a money growth regime, but not under an interest rate rule. It is then found that "the Taylor (1993) rule cannot mimic Friedman’s k-percent money growth rule." We plan to study this extension in an accompanying paper.
References


[12] Schabert, A. "On the Equivalence of Money Growth and Interest Rate Policy", University of Cologne (2003), mimeo


