Banking and Liquidity

(very preliminary)

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Abstract

If agents cannot commit to repay loans, then they will be credit constrained. For instance, young agents may not be able to repay a loan for life-cycle consumption smoothing once their income is high. However, if a banking system provides loans for short-run needs, then a threat of exclusion from future short-run borrowing can be used as a threat to enforce repayment of loans for life-cycle purposes. Additionally, if the rate of inflation is too low, then agents can self-insure for short-term needs using money. Therefore the cost of exclusion from future short-term borrowing is low when inflation is low. Thus, low inflation can shut down the banking system. As a result a money growth rate above the Friedman rule can be welfare superior. This result depends upon there being a set of agents who are constrained for credit reasons other than short term borrowing so that they cannot fully self-insure for short-term needs even at the Friedman rule. Thus a banking system is strictly welfare enhancing even at the Friedman rule.

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1. Introduction

The idea: agents have short term liquidity needs. That is, they have the wealth or income to make a purchase, but they cannot necessarily use those assets or income to make the purchase. This environment allows for agents to use another, more liquid asset, such as money, to make the purchase. However, the money may be costly to accumulate in advance. Instead, an agent can go to a bank and take out a loan, possibly against the other assets, to get some more liquid assets to pay for the temporary need. If these loans are not fully backed by collateral, then the borrower may have an incentive to not repay. The mechanism to make agents honor their debts is exclusion from future borrowing. Thus, *the more easy it is for agents to self-insure using an asset such as money, then the less future value they place on the ability to use the short-term loan market*. Therefore, the lower is inflation, the easier it is for agents to self-insure and the tighter will be the constraint on borrowing in the short-term market.

Long-term borrowing: agents also have needs for long-term borrowing. Here, agents do not have the current assets or income to pay for a purchase, but have human capital that implies higher future labor income. It could also be an entrepreneur who has a project with high future returns, but low current returns and high current costs. These individuals would like to borrow. However, when they are required to pay back the funds, they have no incentive to pay back since their now current income is sufficient and they have no future needs to borrow. Therefore, *
there is necessarily a constraint on long-term borrowing.*

Combining these two ideas, *the banking network that provides short-term borrowing will be more efficient at providing long-term loans since it can use the threat of exclusion from the short-term network as an enforcement mechanism to raise the credit constraint in the long-term market.* Thus, since lower inflation hinders the value to the short-term banking network, it hinders the value to the long-term network.

What can the banking network do? They can identify individuals and record information about them. Thus, they can record whether they gave an agent a loan and whether that agent repaid the loan. They can also identify individuals who lend to the bank, so that these agents can be repaid. Third, it can commit to repaying. The third assumption is probably the most controversial. It is the lack of the ability of individuals to repay that creates problems for the economy in the long-term market (in the short-term market there is the problem of anonymity).
Here the bank can commit. Perhaps, if the bank were to ever not repay, then it would be excluded from being able to operate. Thus, maybe the bank needs some positive profits. For now, ignore this.

2. Model of Banking and Liquidity

2.1. Environment

Time is discrete and the horizon is infinite. There are two types of goods, linear and specific goods. There are two types of agents, buyers and sellers. There is a continuum with measure one of both types of agents. Both types of agents can produce and consume linear goods, with production and utility both being linear. However, only sellers can produce specific goods, at a cost $c(q)$. For simplicity, also assume that $c(q) = q$. Only buyers receive utility from consuming specific goods, with period utility given by $zu(q)$, where $z$ is a preference shock. Let $F(z)$ denote the distribution for $z$. Last, no good is storable and agents discount the future at factor $\beta$.

Each period is split up into two periods: day and night. Linear goods are traded in the day market, while specific goods are traded in the night market. In the night market, buyers and sellers meet pairwise at rate $\alpha$. The buyer then learns $z$. There is a gain to trading specific goods between the buyer and the seller, given by

$$zu(q) - q.$$ 

Let $q^*(z)$, denote the efficient quantity of goods traded, given by

$$zu'(q^*(z)) = 1.$$ 

However, the production of linear goods is limited to the day market and linear goods cannot be stored from day to night. Furthermore, all of these meetings are anonymous, so a buyer cannot write a contract with the seller to guarantee payment in the future for specific goods today. This is the situation that brings about a demand for liquid, tradable assets, in order to compensate an anonymous seller for purchases.

Therefore, let there be a supply of money, $\hat{M}$, that grows at rate $\gamma$, so that

$$\dot{\hat{M}} = (1 + \gamma) \hat{M}.$$ 

Each buyer receives a transfer of money at the start of each period given by

$$\hat{\tau} = \gamma \hat{M}_{-1} = \frac{\gamma}{1 + \gamma} \hat{M}.$$
Sellers do not receive a transfer. Giving the buyers the transfer makes the analysis simpler than giving the seller any transfer.

In the day market, there is a Walrasian market where linear goods and money are traded at $P\left(\hat{M}\right)$ units of money for one linear good.

Introduce banking into the model. An infinite set of banks exists during the night market where agent’s can borrow and lend at an endogenous nominal interest rate of $i$. There is an exogenous upper bound of money that any one individual can borrow, given by $\bar{b}\left(\hat{M}\right)$. Note that it will depend upon $\hat{M}$, which is essential to make the model stationary. The bank is a network of banks that can record the transactions that an individual has with the bank. The banking system can commit to repay lenders, and for now can enforce repayment from borrowers. Thus, the banking system has the following powers:

1. Record-keeping: it can record transactions so that transactions are not anonymous
2. Commitment: it can commit to repay
3. Enforcement: it can enforce contracts so that borrowers repay

The third property will be dropped when we analyze repayment. Therefore, the banking system is a collection of banks that cooperate on enforcement and information sharing, but that compete on the interest rate.

2.2. Timeline

The timeline of the model is as follows:

- Period starts, buyer and sellers holding money, $\hat{m}_b^i$ and $\hat{m}_s^i$, buyers have position at the bank $\hat{b}^i$
- Buyers receive the lump-sum transfer $\hat{\tau}$
- The day market opens:
  - Buyers buy/sell money and linear goods to choose money holdings for night market and settle with the bank
  - Sellers sell money for linear goods, they have no value to money in night market
• The night market opens
  
  – Buyers learn if they meet a seller and the shock \( z \)
  
  – Buyers can then go to a bank to borrow up \( \tilde{b} (\tilde{M}) \) units of money or lend their excess money at rate \( i \)
  
  – Matched buyers make a proposal of \( \hat{d} \) units of money for \( q \) units of specific goods
  
  – Matched sellers either accept or reject

• End of Period

2.3. Decisions

Let the value to a buyer of holding \( \hat{m} \) units of money and having position \( \hat{b} \) at the bank in the day market when the aggregate stock of nominal money is \( \hat{M} \), be \( W_b (\hat{m}, \hat{b}, \hat{M}) \), while that of a seller is \( W_s (\hat{m}, \hat{M}) \). Let the value to a buyer of holding \( \hat{m} \) units of money in the night market when the aggregate stock of nominal money is \( \hat{M} \), be \( V_b (\hat{m}, \hat{M}) \), while that of a seller is \( V_s (\hat{m}, \hat{M}) \). Note that in general the aggregate state would also include the distribution of money across agents, but we can ignore this here since, as will be shown below, all buyers will choose the same amount of money holdings, independent of their current money holdings.

A buyer’s problem is

\[
W_b (\hat{m}_b, \hat{b}, \hat{M}) = \max_{x, \hat{m}'} P (\hat{M}) (x) + V_b (\hat{m}', \hat{M})
\]

such that

\[
x + \frac{1}{P (\hat{M})} \hat{m}' = \frac{1}{P (\hat{M})} [\hat{m}_b + \hat{\tau} - \hat{b} (1 + i)] .
\]

The problem can thus be rewritten as

\[
W_b (\hat{m}, \hat{M}) = \max_{\hat{m}'} \frac{1}{P (\hat{M})} [\hat{m} + \hat{\tau} - \hat{b} (1 + i) - \hat{m}'] + V_b (\hat{m}', \hat{M})
\]

Note that this can be rewritten as

\[
W_b (\hat{m}_b, \hat{M}) = \frac{1}{P (\hat{M})} [\hat{m}_b - \hat{b} (1 + i)] + \max_{\hat{m}'} \left\{ \frac{1}{P (\hat{M})} [\hat{\tau} - \hat{m}'] + V_b (\hat{m}', \hat{M}) \right\}
\]

\[
= \frac{1}{P (\hat{M})} [\hat{m}_b - \hat{b} (1 + i)] + W_b (0, \hat{M})
\]
Similarly, a seller's problem is given by

\[ W_s(m, \hat{M}) = \frac{1}{P(\hat{M})} \hat{m} + W_s(0, \hat{M}) \]

### 2.3.1. Optimal Proposing

Let the buyer's money holdings be denoted \( \hat{m}_b \), and that of the seller, \( \hat{m}_s \). The buyer chooses how much money to borrow/lend, given by \( \hat{b} \). When \( \hat{b} > 0 \) the buyer is borrowing. In addition, the buyer make a take-it-or-leave-it offer of \( \hat{d} \) units of money for \( q \) units of specific good:

\[
\max_{d,b,q} z(u(q) + \beta \frac{1}{P(\hat{M}')} [\hat{m}_b + \hat{b} - \hat{d} - \hat{b}(1 + i)] + W_b(0, \hat{M}'))
\]

subject to

\[
q \leq \beta \frac{1}{P(\hat{M}')} \hat{d}
\]

\[
\hat{d} \leq \hat{m}_b + \hat{b}
\]

and

\[
\hat{b} \leq \bar{b}(\hat{M}).
\]

The first constraint comes from the leaving the seller indifferent about accepting, the second constraint is the cash constraint for the transaction, and the third constraint is the borrowing constraint.

Redefine the problem by setting

\[
m = \beta \frac{\hat{m}}{P(\hat{M}')}\]

then bargaining solves

\[
\max_{b,q} z(u(q) + m_b + b - d - b(1 + i) + W_b(0, \hat{M}'))
\]

subject to

\[
q \leq m_b + b
\]

and

\[
b \leq \bar{b}.
\]
where the acceptance constraint and the cash constraint have been combined into one constraint. Let \( \lambda \) be the multiplier on the acceptance constraint from the seller, and \( \theta \) the borrowing constraint. Then the FOC satisfy, for \( q \)

\[
zu'(q) - 1 = \lambda
\]

and for \( b \)

\[
\lambda - \theta = i.
\]

Define \( q(z, i) \) as the solution to

\[
zu'(q(z, i)) = 1 + i
\]

Then optimal proposing satisfies

\[
q(z, m_b) = \begin{cases} 
  m_b + \bar{b} & m_b + \bar{b} < q(z, i) \\
  q(z, i) & m_b + \bar{b} \geq q(z, i)
\end{cases}
\]

(2.1)

and optimal borrowing satisfies

\[
b(z, m_b) = \begin{cases} 
  \bar{b} & m_b + \bar{b} < q(z, i) \\
  q(z, i) - m_b & m_b + \bar{b} \geq q(z, i)
\end{cases}
\]

(2.2)

If a buyer does not meet a seller, then the buyer solves

\[
\max_{b \geq -m_b} m_b + b - b(1 + i) + \beta W_b(0, \hat{M}')
\]

This results in an optimal decision of

\[
b = -m_b
\]

if \( i > 0 \), otherwise, if \( i = 0 \), the buyer is indifferent about lending.

Thus, the value to the buyer of bringing \( m_b \) units of real money to the night market is given by

\[
V(m_b, \hat{M}) = \alpha \int \left[ \max_{q \leq m_b + \bar{b}} zu(q) - q(1 + i) \right] dF(z) + (1 + i) m_b + \beta W_b(0, \hat{M}')
\]

2.3.2. Day Market Decisions

It is easy to show that a seller will always choose to bring no money to the night market.

**Lemma 2.1.** The optimal choice of money holdings by the seller for the night market is

\[
\hat{m}_s = 0.
\]

(2.3)
Proof See the appendix.

Turning to the buyers, the day market problem for the buyer is given by

\[ W_b(0, \hat{M}) = \max_{\hat{m}_b} \frac{1}{P(\hat{M})} [\hat{\tau} - \hat{m}_b'] + \alpha \int \left[ \max_{q \leq m_b + \bar{b}} zu(q) - q(1 + i) \right] dF(z) + (1 + i) m_b + \beta W_b(0, \hat{M}') \]

where

\[ m_b = \beta \frac{\hat{m}_b'}{P(\hat{M}')}. \]

The FOC results in

\[ \frac{1}{P(\hat{M})} = a \int \left\{ zu'(q) - (1 + i) \right\} \frac{\partial q(z, m_b)}{\partial m_b} \beta \frac{1}{P(\hat{M}')} dF(z) + \beta \frac{1}{P(\hat{M}')} (1 + i). \]

Lemma 2.2. The optimal choice of money holdings for the buyer satisfies define

\[ m_b = \bar{q}(i) - \bar{b}. \]  

where \( \bar{q}(i) \) and \( \bar{z}(i) \) are the solution to

\[ \frac{P(\hat{M}')}{P(\hat{M})} \frac{1}{\beta} - (1 + i) = \alpha \int_{\bar{z}(i)}^{\bar{z}(i)} \left\{ zu'(\bar{q}(i)) - (1 + i) \right\} dF(z) \]

and

\[ \bar{z}u'(\bar{q}(i)) = 1 + i. \]

Essentially, inflation and the interest rate pin down what the agent is willing to purchase in different matches, and the borrowing constraint then lets the agents know how much money they need to bring into the night market.

2.3.3. Banks

It is assumed that there is an infinite amount of banks. Thus, the banks compete on the interest rate. This implies that the interest rate for lending and borrowing is the same, which has been assumed. Therefore, the interest rate just has to clear the loan market. Market clearing requires

\[ \alpha \int b(z, i, m_b) dF(z) = (1 - \alpha) m_b \]

or

\[ \alpha \int_{z(m_b, i, \bar{b})}^{z(m_b, i, \bar{b})} \bar{b}dF(z) + \alpha \int_{z_i}^{z(m_b, i, \bar{b})} [q(z, i) - m_b(\bar{b}, i)] dF(z) = (1 - \alpha) m_b(\bar{b}, i). \]
\[ m_b(\bar{b}, i) = \alpha \left[ 1 - F(\bar{z}) \right] \bar{q} + \int_{z_l}^{\bar{z}} q(z, i) \, dF(z) \]

or

\[ \bar{b} = \bar{q}(i) - \alpha \left[ 1 - F(\bar{z}(i)) \right] \bar{q} + \int_{z_l}^{\bar{z}} q(z, i) \, dF(z) \]  \hspace{1cm} (2.7)

Define the net demand for loans as

\[ B(i, \bar{b}) = \bar{b} - \bar{q}(i) + \alpha \left[ 1 - F(\bar{z}(i)) \right] \bar{q} + \int_{z_l}^{\bar{z}} q(z, i) \, dF(z) \]

2.4. Equilibrium

**Definition 2.3.** An equilibrium of the liquidity model is a list of functions: \( b(\hat{m}_b, \hat{m}_s, \hat{M}) \), \( q(\hat{m}_b, \hat{m}_s, \hat{M}) \), \( d(\hat{m}_b, \hat{m}_s, \hat{M}) \), \( m_s(\hat{M}) \), \( m_b(\hat{M}) \), \( P(\hat{M}) \) and \( i(\hat{M}) \) such that

1. Given \( P(\hat{M}) \) and \( i(\hat{M}) \), \( b(\hat{m}_b, \hat{m}_s, \hat{M}) \), \( q(\hat{m}_b, \hat{m}_s, \hat{M}) \) and \( d(\hat{m}_b, \hat{m}_s, \hat{M}) \) satisfy optimal bargaining. They solve equation (2.1) and (2.2).

2. Given \( P(\hat{M}) \), \( q(\hat{m}_b, \hat{m}_s, \hat{M}) \) and \( d(\hat{m}_b, \hat{m}_s, \hat{M}) \), then \( m_s(\hat{M}) \) and \( m_b(\hat{M}) \) solve equations (2.3) and (2.4)

3. Markets Clear:

\[ \hat{M} = \hat{m}_b + \hat{m}_s. \]

and equation (2.7) holds.

In any stationary equilibrium where \( m_b \) and \( i \) are constant, then

\[ \frac{P(\hat{M})}{P(\hat{M}')} = 1 + \gamma. \]

Then given \( i \), the equilibrium is defined by

\[ \frac{1 + \gamma}{\beta} - (1 + i) = \alpha \int_{\bar{z}(i)}^{\bar{z}_h} \{ zu'(\bar{q}(i)) - (1 + i) \} \, dF(z) \]  \hspace{1cm} (2.8)

\[ \bar{z}(i) = \frac{1 + i}{u'(\bar{q}(i))} \]

\[ m_b = \bar{q}(i) + \bar{b} \]
and
\[ \bar{b} = \bar{q}(i) - \alpha \left[ 1 - F(\bar{z}(i)) \right] \bar{q} + \int_{z_1}^{z_2} q(z, i) dF(z) \]
\[ zu'(q(z, i)) = 1 + \gamma. \]

Before getting into the results the following properties of \( B \) will be useful.

**Lemma 2.4.** \( B'(i) < 0. \)

**Proof**
\[ B'(i) = -q'(i) + \alpha \left[ 1 - F(\bar{z}(i)) \right] \bar{q}'(i) + \int_{z_1}^{z_2} q_i(z, i) dF(z) \]

Doing some algebra,
\[ q_i(z, i) = \frac{1}{zu''(q(z, i))} < 0 \]
and
\[ \bar{q}'(i) = \frac{[\alpha [1 - F(\bar{z})] - 1]}{\alpha u''(\bar{q}) \int_{z_1}^{z_2} zdF(z)} > 0. \]

Thus
\[ B'(i) = -q'(i) + \alpha [1 - F(\bar{z})] \bar{q}'(i) + \alpha \int_{z_1}^{z_2} q_i(z, i) dF(z) \]
\[ = -q'(i) [1 - \alpha [1 - F(\bar{z})]] + \alpha \int_{z_1}^{z_2} q_i(z, i) dF(z) \]
\[ = \frac{[\alpha [1 - F(\bar{z})] - 1]^2}{\alpha u''(\bar{q}) \int_{z_1}^{z_2} zdF(z)} + \alpha \int_{z_1}^{z_2} \frac{1}{zu''(q(z, i))} dF(z) < 0 \quad (2.9) \]

\[ \blacksquare \]

**Proposition 2.5.** Define
\[ \bar{b}_h = \bar{q}(i_{\text{max}}) - \alpha \int_{z_1}^{z_2} q(z, i_{\text{max}}) dF(z) \]
where
\[ i_{\text{max}} = \frac{1 + \gamma}{\beta} - 1. \]

Define
\[ \bar{b}_l = \bar{q}(0) - \alpha \left[ \int_{z_1}^{z_2} q(z, 0) dF(z) + [1 - F(\bar{z}(0))] \bar{q}(0) \right] \]

An equilibrium exists, is unique, and \( i > 0 \) if \( \bar{b} \in (\bar{b}_l, \bar{b}_h) \). Furthermore, if \( \bar{b} \leq \bar{b}_l \) then an equilibrium exists and \( i = 0. \)
Proof First, assume that $\bar{b} \leq \bar{b}_h$. Then net demand for loans is given by

$$B(i) = \bar{b} - \bar{q}(i) + \alpha \left[ 1 - F(\bar{z}(i)) \right] \bar{q}(i) + \int_{z_l}^{\bar{z}(i)} q(z,i) dF(z)$$

First, when $i = 0$, then if $\bar{b} > \bar{b}_l$, then by assumption

$$B(0) = \bar{b} - \bar{q}(0) + \alpha \left[ 1 - F(\bar{z}(0)) \right] \bar{q}(0) + \int_{z_l}^{\bar{z}(i)} q(z,0) dF(z)$$

which is positive by assumption. Furthermore, if $b < \bar{b}_h$, then

$$B(i_{max}) = \bar{b} - \bar{q}(i_{max}) + \int_{z_l}^{z_h} q(z,i_{max}) dF(z)$$

which is negative by assumption. It has already been shown that $B'(i) < 0$. Thus, there exists a unique $i > 0$, such that $B(i) = 0$.

To finish off the proof, if $\bar{b} \leq \bar{b}_l$, then

$$B(0) = 0.$$

and the only equilibrium is $i = 0$.

Corollary 2.6. If $b \in (\bar{b}_l, \bar{b}_h)$, then $d\bar{q}/d\bar{b} > 0$, $d\bar{z}/d\bar{b} > 0$, and $d\bar{z}/d\bar{b} > 0$.

Proof If $b \in (\bar{b}_l, \bar{b}_h)$, then there exists a unique $i > 0$, such that $B(i) = 0$. Since

$$B(i) = \bar{b} - \bar{q}(i) + \alpha \left[ 1 - F(\bar{z}(i)) \right] \bar{q}(i) + \int_{z_l}^{\bar{z}(i)} q(z,i) dF(z)$$

and $B'(i) < 0$, and $B$ is increasing in $\bar{b}$, then $di/d\bar{b} > 0$. Furthermore, as shown earlier, $\bar{q}'(i) > 0$, so that $d\bar{q}/d\bar{b} > 0$, which also implies $d\bar{z}/d\bar{b} > 0$.

Corollary 2.7. If $b \in (\bar{b}_l, \bar{b}_h)$, then $d\bar{q}/d\bar{\gamma} > 0$ and $d\bar{z}/d\bar{\gamma} < 0$.

Proof If $b \in (\bar{b}_l, \bar{b}_h)$, then there exists a unique $i > 0$, such that $B(i) = 0$. From equation (2.8) then $d\bar{q}/d\bar{\gamma} < 0$. From equation (2.9) then $dB/d\bar{q} < 0$. Therefore $dB/di > 0$.

Thus, we can think about the borrowing constraint affecting the market clearing nominal interest rate. For low $\bar{b}$, $i$ is zero and raising the borrowing constraint has no affect. Eventually, the borrowing constraint loosens enough that the demand for loans is larger than the supply.
so that $i$ must increase to equate supply and demand. At the same time, raising the borrowing constraint will result in agents being able to purchase more in constrained matches, but lowers the amount traded in unconstrained matches. However, agents value this trade off as evidenced by equation (2.8).

An increase in the inflation rate lowers the demand for purchases, which lowers the demand for loans. Thus, an increase in $\gamma$ will result in an increase in $i$ whenever $i > 0$, or $\bar{b} \geq \bar{b}_l$. However, the net affect is that $\bar{q}$ falls.

### 2.5. Welfare

Welfare is given by

$$W = \frac{\alpha}{1-\beta} \left[ \int_{z_l}^{z_h} \{zu(q(z,i)) - q(z,i)\} dF(z) + \int_{z_l}^{z_h} \{zu(\bar{q}(i,\gamma)) - \bar{q}(i,\gamma)\} dF(z) \right]$$

**Proposition 2.8.** Welfare is decreasing in $\gamma$. The optimal rate of money growth is $1 + \gamma = \beta$. If $1 + \gamma > 0$, then welfare is increasing in $\bar{b}$ for $\bar{b} \in (\bar{b}_l, \bar{b}_h)$ and the optimum is $\bar{b} = \bar{b}_h$. When $1 + \gamma = \beta$, then welfare is independent of $\bar{b}$.

**Proof** See the appendix.

Welfare is decreasing in $\gamma$, which is standard. The interesting point is that (1) at the Friedman rule, then the banking system has no value, and (2) away from the Friedman rule, the banking system has value, by redistributing money is such a way that even the agents in the best match are not constrained. However, the efficient banking system requires a strictly positive nominal interest rate. This results in distorting the unconstrained matches. The reason for this is that when the Friedman rule is not being run, then the economy as a whole does not hold enough real money to make efficient purchases in all matches. However, the banking system allows the value to money to be equalized across all matches. For this to happen, the interest rate has to be positive to draw money from the agents in bad matches to the agents in good matches.

It would seem to be interesting to examine the effect of an increase in the spread of the type of matches, $z_h - z_l$. This would seem to imply that more re-distribution would have to take place, raising the nominal interest rate.
3. Model with young agents

The result that the banking system has no value to redistribute liquid assets at the Friedman rule seems unsatisfactory. The reason is that it seems sensitive to the Lagos-Wright environment. This environment has two negative aspects:

1. The model is built up to remove any distribution of asset holdings across agents, in particular, liquid assets. Therefore, this environment seems ill suited to address redistribution.

2. The model assumes that agents can purchase an unlimited amount of real money. Thus, at the Friedman rule agents can insure against any possible shock, so there is no need to redistribute. However, if agents are constrained in how much money they can purchase, then there will be some value to redistributing, even at the Friedman rule.

To address these issues, assume that each period, a new set of young agents are born. For the first period of life, young agents are constrained in the amount of linear goods that they can produce,

\[ x \leq \bar{x}. \]

Assume that \( \eta \) young agents of each type are born each period. This implies that the measure of young agents, \( \mu \), is given by

\[ \mu = \frac{\eta}{1 + \eta}. \]

Since the population is growing, let \( \gamma \) denote the growth rate of the per capita nominal money stock,

\[ \frac{\dot{M}'}{N'} = (1 + \gamma) \frac{\dot{M}}{N}. \]

3.1. Young Agent decisions

Since a young agent is exactly like an old agent once he has made his money holding decision, then the optimal proposing for a young agent is unchanged. Turning to the young agent’s decision of money holdings, his problem is given by

\[
W_y \left( 0, \frac{\dot{M}}{N} \right) = \max \left\{ \hat{m}_b, \hat{m}_0 \right\} \frac{1}{P \left( \frac{\dot{M}}{N} \right)} \left[ \hat{\tau} - \hat{m}_b \right] + \alpha \int \left[ \max_{q \leq m_b + \hat{b}} zu(q) - q \left( 1 + i \right) \right] dF(z) + \left( 1 + i \right) m_b + \beta W_b \left( 0, \frac{\dot{M}'}{N'} \right)
\]
such that

\[ m_b \leq \beta \frac{P \left( \bar{M}/N \right)}{P \left( \bar{M}'/N' \right)} \bar{x} = \bar{m} (\bar{x}, \gamma) \]

where

\[ m_b = \beta \frac{\bar{m}_b'}{P \left( \bar{M}'/N' \right)} \]

The assumption has been made that the young do not receive any money transfer. The reason is to get away from inflation being optimal since it redistributes to the young. The only difference of this problem with an old agent’s problem is the constraint in \( m_b \). Thus, if

\[ \bar{m} (\bar{x}, \gamma) < \bar{q} (i, \gamma) - \bar{b} \]

the young agent will be constrained, otherwise, he is not, and the introduction of the young does not affect the equilibrium. Define \( \bar{q} (\bar{m}) = \bar{m} + \bar{b} \). Likewise, define \( \bar{z} (\bar{m}) \).

### 3.2. Old agent decisions

Old agent decisions have changed in an interesting way. If

\[ i + 1 < \frac{1 + \gamma}{\beta} \]

then the old agent problem is the same. However, now it will be a possibility in equilibrium that

\[ 1 + i = \frac{1 + \gamma}{\beta} \]

in which case the old agent is indifferent about money holdings above a certain threshold.

### 3.3. Loan Market Clearing

The primary impact of the young is to change the loan market clearing condition. The supply of funds is now

\[ S (i, \bar{m}, \bar{b}) = (1 - \mu) \left[ \bar{q} (i, \gamma) - \bar{b} \right] + \mu \left[ \bar{q} (\bar{m}) - \bar{b} \right] \]

\[ = (1 - \mu) \bar{q} (i, \gamma) + \mu \bar{q} (\bar{m}) - \bar{b} \]

while the demand is

\[ D (i, \bar{m}) = (1 - \mu) \alpha \left[ [1 - F (\bar{z} (\bar{m}))] \bar{q} (i, \gamma) + \int_{z_l}^{\bar{z}} q (z, i) dF (z) \right] + \]

\[ \mu \alpha \left[ [1 - F (\bar{z} (\bar{m}))] \bar{q} (\bar{m}) + \int_{z_l}^{\bar{z}(\bar{m})} q (z, i) dF (z) \right] \]
Thus, the market clearing condition is

\[
\bar{b} = (1 - \mu) \bar{q} (i, \gamma) + \mu \bar{q} (\bar{m}) - (1 - \mu) \alpha \left[ 1 - F(\bar{z} (i, \gamma)) \right] \bar{q} (i, \gamma) + \int_{z_i}^{\bar{z}} q (z, i) dF (z)
\]

When \( q (\bar{m}) = \bar{q} (i, \gamma) \), then this has the same result as in the old agent model.

### 3.4. Equilibrium

First, examine the money market clearing condition,

\[
\bar{M} = (1 - \mu) \bar{m}_b + \mu \bar{m}_y
\]

\[
\frac{\bar{M}}{N} = \frac{P \left( \bar{M}' / N' \right)}{\beta} \left[ (1 - \mu) \bar{m}_b + \mu \bar{m} \right]
\]

Therefore

\[
\hat{\tau} = \frac{\gamma}{1 + \gamma} \frac{P \left( \bar{M}' / N' \right)}{\beta} \left[ (1 - \mu) \bar{m}_b + \mu \bar{m} \right]
\]

The equilibrium conditions are

\[
\frac{1 + \gamma}{\beta} - (1 + i (\gamma)) = \alpha \int_{z_i}^{z_h} \{ z u' (\bar{q} (i, \gamma)) - (1 + i (\gamma)) \} dF (z) \quad (3.1)
\]

\[
\bar{z} (i, \gamma) = \frac{1 + i (\gamma)}{w' (\bar{q} (i, \gamma))} \quad (3.2)
\]

\[
m_b = \bar{q} (i) + \bar{b} \quad (3.3)
\]

\[
\bar{m} (\bar{x}, \gamma) = \min \left\{ \beta, 1 + \gamma \bar{x}, \bar{m}_b \right\} \quad (3.4)
\]

\[
\bar{q} (\bar{m}) = \bar{m} + \bar{b} \quad (3.5)
\]

and

\[
\bar{b} = (1 - \mu) \bar{q} (i, \gamma) + \mu \bar{q} (\bar{m}) - (1 - \mu) \alpha \left[ 1 - F(\bar{z} (i, \gamma)) \right] \bar{q} (i, \gamma) + \int_{z_i}^{\bar{z}} q (z, i) dF (z)
\]

\[
\mu \alpha \left[ 1 - F(\bar{z} (\bar{m})) \right] \bar{q} (\bar{m}) + \int_{z_i}^{\bar{z}(\bar{m})} q (z, i) dF (z)
\]

\[
zu' (q (z, i)) = 1 + i. \quad (3.7)
\]
Claim 1. Define

\[ \bar{b}_l^y = (1 - \mu) \left[ \bar{q}(0) - \bar{b} - \alpha \left( \int_{z_l}^\bar{z} q(z,0) dF(z) + [1 - F(\bar{z}(i,\gamma))] \bar{q}(0,\gamma) \right) \right] + \mu \left[ \bar{q}(\bar{m}) - \bar{b} - \alpha \left( \int_{z_l}^\bar{z} q(z,0) dF(z) + [1 - F(\bar{z}(\bar{m}))] \bar{q}(\bar{m}) \right) \right] \]

and

\[ \bar{b}_h^y = (1 - \mu) \left[ \bar{q}(i_{\text{max}}) - \bar{b} - \alpha \left( \int_{z_l}^\bar{z} q(z,i_{\text{max}}) dF(z) + [1 - F(\bar{z}(i,\gamma))] \bar{q}(i_{\text{max}},\gamma) \right) \right] + \mu \left[ \bar{q}(\bar{m}) - \bar{b} - \alpha \left( \int_{z_l}^\bar{z} q(z,i_{\text{max}}) dF(z) + [1 - F(\bar{z}(\bar{m}))] \bar{q}(\bar{m}) \right) \right] \]

An equilibrium exists and it is unique. Furthermore, if \( \bar{b} < \bar{b}_l^y \), then \( i = 0 \), if \( \bar{b} \geq \bar{b}_h^y \), then \( i = i_{\text{max}} \), and if \( \bar{b} \in (\bar{b}_l^y, \bar{b}_h^y) \), then \( i \in (0, i_{\text{max}}) \).

Proof The proof is simply by construction.

Claim 2. In equilibrium, if \( \bar{b} \in (\bar{b}_l^y, \bar{b}_h^y) \), then \( di/d\mu > 0 \), and \( di/d\bar{x} < 0 \).

Proof The net savings of the young is always less than the old. Thus, when \( \mu \) increases, then the net savings for the economy falls, and \( i \) must increase to compensate. Furthermore, when \( \bar{x} \) falls, then the net savings of the economy falls and \( i \) must increase.

This is really one of the most important results of the paper. For a given \( \bar{b} \), then the more young there are, or the more they are constrained, then the utility of the old will be increasing whenever \( i < i_{\text{max}} \). Thus, if \( \bar{b} < \bar{b}_l^y \), then the more constrained are the young, then the better off are the old since the nominal interest rate is higher. On average an old agent is a lender, so he accumulates more money than he needs, and the higher interest rate raises his utility.

3.5. Welfare

Welfare in this model is given by

\[ W = \frac{-\eta}{1 + \eta \beta} [\mu \bar{m} + (1 - \mu) m_0] + \frac{\alpha \mu}{1 - \beta} \left[ \int_{z_l}^{\bar{z}} \{ z u(q(z,i)) - q(z,i) \} dF(z) + \left( \int \{ z u(\bar{q}(\bar{m})) - \bar{q}(\bar{m}) \} dF(z) \right) \right] + \frac{\alpha (1 - \mu)}{1 - \beta} \left[ \int_{z_l}^{\bar{z}} \{ z u(q(z,i)) - q(z,i) \} dF(z) + \left( \int \{ z u(\bar{q}(i,\gamma)) - \bar{q}(i,\gamma) \} dF(z) \right) \right] \]
Proposition 3.1. Welfare is strictly increasing in $\bar{b}$, if

$$\bar{b} < \bar{q}(i, \gamma) - \bar{m}$$

even at the Friedman rule, and even for $\bar{b} < \bar{b}_i^q$

Proof At the Friedman rule, then $i_{\text{max}} = 0$. This maximizes the welfare of the old. But $\bar{m} < \bar{q}(0)$. Then the young gain from $\bar{b} > 0$.

The gain to welfare by setting $\bar{b} = \bar{q}(0) - \bar{m}$ is given by

$$\Delta_w = \frac{\alpha \mu}{1 - \beta} \left\{ \int_{\bar{z}(\bar{m})}^{\bar{z}_h} \{zu(\bar{q}(z,0)) - q(z,0)\} dF(z) - \int_{\bar{z}(\bar{m})}^{\bar{z}_h} \{zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m})\} dF(z) \right\} > 0.$$ 

Thus the gain is greater, the greater is $\mu$, and the lower is $\bar{x}$. In general, the extra gain from the young over the old for a given $\gamma$, is to set $\bar{b} = \bar{b}_i^q$, so that $i = i_{\text{max}}$, and then the extra gain in welfare is from

$$\Delta_w = \frac{\alpha \mu}{1 - \beta} \left\{ \int_{\bar{z}(\bar{m})}^{\bar{z}_h} \{zu(\bar{q}(z,\text{max})) - q(z,\text{max})\} dF(z) - \int_{\bar{z}(\bar{m})}^{\bar{z}_h} \{zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m})\} dF(z) \right\}.$$ 

Of course, as $\gamma$ increases, then $\bar{z}(\bar{m})$ approaches $\bar{z}_h$.

4. Enforcement

Now suppose that the only way that a bank can get a borrower to repay is by being threatened with a more severe borrowing constraint,

$$\tilde{b}_d < \bar{b}.$$ 

We still assume that the agent can use the banking system, just not as much, and the agent will have to self-insure more. However, the $\bar{q}$ will still be the same, the agent simply has to produce for more money. Suppose that an agent has just not repaid his debt, then his welfare is given by.

$$W^d_{\tilde{b}}(\tilde{m}_b, \tilde{M}) = \frac{\gamma}{\beta(1 - \beta)} [\tilde{m}_b - m_b] - \frac{1}{\beta} m_b + \frac{i}{1 - \beta} m_b + \frac{\alpha}{1 - \beta} \left[ \int_{\bar{z}_i}^{\bar{z}(i)} \{zu(q(z, i)) - q(z, i)(1 + i)\} dF(z) + \int \{zu(\bar{q}(i)) - \bar{q}(i)(1 + i)\} dF(z) \right]$$

where

$$\tilde{m}_b = \beta \frac{1}{P(M')} \tilde{M}$$
This agent will now choose

\[ m_b = [\bar{q}(i, \gamma) - \bar{b}_d] \]

while

\[ \bar{m}_b = [\bar{q}(i, \gamma) - \bar{b}] \]

Thus we get

\[ W_d^b(\bar{m}_b, \bar{M}') = \frac{\gamma}{\beta(1-\beta)} [\bar{b}_d - \bar{b}] - \frac{1}{\beta} [\bar{q}(i, \gamma) - \bar{b}_d] + \frac{i}{1-\beta} [\bar{q}(i, \gamma) - \bar{b}_d] + \]

\[ \frac{\alpha}{1-\beta} \left[ \int_{\tilde{z}_1} z u (q(z,i)) - q(z,i) (1+i) \right] dF(z) + \int \{ z u (\bar{q}(i)) - \bar{q}(i) (1+i) \} dF(z) \]

Compare this to the welfare of someone who repaid,

\[ W_b(\bar{m}_b, \bar{M}') = -\frac{1}{\beta} [\bar{q}(i, \gamma) - \bar{b}] + \frac{i}{1-\beta} [\bar{q}(i, \gamma) - \bar{b}] + \]

\[ \frac{\alpha}{1-\beta} \left[ \int_{\tilde{z}_1} z u (q(z,i)) - q(z,i) (1+i) \right] dF(z) + \int \{ z u (\bar{q}(i)) - \bar{q}(i) (1+i) \} dF(z) \]

The difference is given by

\[ \Delta = \frac{[\bar{b} - \bar{b}_d]}{\beta(1-\beta)} [\gamma + 1 - \beta - \beta i] \]

In addition there is the current cost of having to repay the loan, this is given by

\[ -\bar{b}(1+i) \]

thus an agent will only repay if

\[ \Delta_p = \frac{[\bar{b} - \bar{b}_d]}{\beta(1-\beta)} [\gamma + 1 - \beta - \beta i] - \bar{b}(1+i) \geq 0 \]

set \( \bar{b}_d = 0 \), then this becomes

\[ \frac{[\gamma + 1 - \beta - \beta i]}{\beta(1-\beta)} \geq (1+i) \]

Define

\[ \frac{1+\gamma}{\beta} = \rho \]

then the condition becomes

\[ \frac{[\rho - 1 - i]}{(1-\beta)} \geq (1+i) \]

When \( i \) is zero, this comes down to

\[ \rho > (2-\beta) \]
so when agents are really patient, then anything goes. Thus, there is a trade-off between inflation
and patience. For given patience, then some borrowing can take place. The upper bound on \( i \) is
given by

\[
i \leq \bar{i} = \frac{\rho}{2 - \beta} - 1
\]

Next, how does \( i \) vary with \( \rho \)?

\[
\frac{di}{d\rho} = \frac{1}{(2 - \beta)} > 0
\]

When is there a borrowing constraint? When \( \bar{i} < i_{\text{max}} = \rho - 1 \)

\[
\beta < 1
\]

which is always the case, so that the borrowing constraint is always less than \( \bar{b}_h \).

**Claim 3.** If

\[
\rho > (2 - \beta)
\]

then \( i > 0, \) and \( \bar{b} > 0. \) However, \( i < i_{\text{max}}, \) so that \( \bar{b} < \bar{b}_h. \)

Since welfare is increasing in \( i \) for both agents, then the more borrowing that can be sustained,
then the higher will be welfare. However, the borrowing constraint cannot be set at the optimal.
The reason is that at the optimal then

\[
\frac{1 + \gamma}{\beta} = 1 + i
\]

so that there is no cost to holding money. Money that is not used in purchases is lent out, which
compensates for deviating from the Friedman rule. Thus, there is no cost to holding money is
such a case and no agent can commit to repay the optimal amount of borrowing.

Of interest, for \( \beta < 1, \) then can \( \rho > 1 \) result in superior welfare to \( \rho = 1? \) It would seem that
\( \bar{x} \) and \( \mu \) can be chosen so that welfare is superior for \( \rho > 1. \)

The last issue is whether the young would prefer to borrow from the old in the day market,
borrowing some general goods to make purchases, or whether it is optimal for the lending to take
place in the night market. Alternatively, maybe the life-cycle lending could be sustained for other
purposes that raises welfare. This, along with the analysis of some type of collateral or durable
good is left for future research.
A. Detailed Proofs for Model without Young

A.1. Analysis of Welfare and borrowing constraint

Examining the welfare to buyers and sellers. Assume that \( \hat{\tau}_s = 0 \), then

\[
W_s(\hat{m}, \hat{M}) = \frac{1}{P(\hat{M})}\hat{m}.
\]

Turning to the buyers, then

\[
W_b(\hat{m}_b, M') = \frac{1}{P(M')}\hat{m}_b + \frac{\gamma}{\beta(1 - \beta)}[\hat{m}_b - m_b] - \frac{1}{\beta}m_b + \frac{i}{1 - \beta}m_b + \frac{\alpha}{1 - \beta}\int z_i \{zu(q(z,i)) - q(z,i)(1 + i)\}dF(z) + \int \{zu(\bar{q}(i)) - \bar{q}(i)(1 + i)\}dF(z)
\]

where

\[
\hat{m}_b = \beta \frac{1}{P(M')}\hat{M}
\]

Putting the market clearing condition then

\[
W_b(\hat{m}_b, M') = \frac{1}{P(M')}\hat{m}_b - \frac{1}{\beta}(\bar{q}(i) - \bar{b}) + \frac{i}{1 - \beta}(\bar{q}(i) - \bar{b}) + \frac{\alpha}{1 - \beta}\int z_i \{zu(q(z,i)) - q(z,i)(1 + i)\}dF(z) + \int \{zu(\bar{q}(i)) - \bar{q}(i)(1 + i)\}dF(z)
\]

Treating buyers and sellers symmetrically, then

\[
W = \frac{1}{P(M)}\hat{m}_s + \frac{1}{P(M)}\hat{m}_b - \frac{1}{\beta}(\bar{q}(i) - \bar{b}) + \frac{i}{1 - \beta}(\bar{q}(i) - \bar{b}) + \frac{\alpha}{1 - \beta}\int z_i \{zu(q(z,i)) - q(z,i)(1 + i)\}dF(z) + \int \{zu(\bar{q}(i)) - \bar{q}(i)(1 + i)\}dF(z)
\]

or

\[
W = \frac{1}{P(M)}\frac{1}{1 + \gamma}\hat{M} - \frac{1}{\beta}[\bar{q}(i) - \bar{b}] + \frac{i}{1 - \beta}[\bar{q}(i) - \bar{b}] + \frac{\alpha}{1 - \beta}\int z_i \{zu(q(z,i)) - q(z,i)(1 + i)\}dF(z) + \int \{zu(\bar{q}(i)) - \bar{q}(i)(1 + i)\}dF(z)
\]

or

\[
W = \frac{1}{\beta}[\bar{q}(i) - \bar{b}] - \frac{1}{\beta}[\bar{q}(i) - \bar{b}] + \frac{i}{1 - \beta}[\bar{q}(i) - \bar{b}] + \frac{\alpha}{1 - \beta}\int z_i \{zu(q(z,i)) - q(z,i)(1 + i)\}dF(z) + \int \{zu(\bar{q}(i)) - \bar{q}(i)(1 + i)\}dF(z)
\]
or

\[
W = \frac{i(\gamma)}{1-\beta} \left[ \bar{q}(i, \gamma) - \bar{b} \right] + \\
\frac{\alpha}{1-\beta} \left[ \int_{z_1}^{z(i)} \{uz(q(z, i)) - q(z, i)(1+i)\} dF(z) + \int_{\bar{z}(i, \gamma)}^{z_h} \{uz(\bar{q}(i, \gamma)) - \bar{q}(i, \gamma)(1+i(\gamma))\} dF(z) \right]
\]

Putting in the market clearing condition for loans, then

\[
W = \frac{\alpha}{1-\beta} \left[ \int_{z_1}^{z(i)} \{uz(q(z, i)) - q(z, i)\} dF(z) + \int_{\bar{z}(i, \gamma)}^{z_h} \{uz(\bar{q}(i, \gamma)) - \bar{q}(i, \gamma)\} dF(z) \right]
\]

**Proposition A.1.** Welfare is increasing in \( \bar{b} \).

**Proof** The welfare function is given by

\[
W = \frac{\alpha}{1-\beta} \left[ \int_{z_1}^{z(i)} \{uz(q(z, i)) - q(z, i)\} dF(z) + \int_{\bar{z}(i, \gamma)}^{z_h} \{uz(\bar{q}(i, \gamma)) - \bar{q}(i, \gamma)\} dF(z) \right]
\]

The equations that describe the model are:

\[
1 + \frac{\gamma}{\beta} - (1 + i(\gamma)) = \alpha \int_{\bar{z}(i)}^{z_h} \{uz'(\bar{q}(i, \gamma)) - (1 + i(\gamma))\} dF(z) \tag{A.1}
\]

\[
\bar{z}(i, \gamma) = \frac{1 + i(\gamma)}{w'(\bar{q}(i, \gamma))} \tag{A.2}
\]

\[
m_b = \bar{q}(i) + \bar{b} \tag{A.3}
\]

and

\[
\bar{b} = \bar{q}(i, \gamma) - \alpha \left[ [1 - F(\bar{z}(i, \gamma))] \bar{q}(i, \gamma) + \int_{z_1}^{\bar{z}} q(z, i) dF(z) \right] \tag{A.4}
\]

\[
zu'(q(z, i)) = 1 + i \tag{A.5}
\]

The value of the borrowing constraint has no direct value on welfare. The value of the borrowing constraint comes from changing \( \bar{q} \), in equilibrium

\[
\bar{b} = \bar{q}(i, \gamma) - \alpha \left[ [1 - F(\bar{z}(i, \gamma))] \bar{q}(i, \gamma) + \int_{z_1}^{\bar{z}} q(z, i) dF(z) \right]
\]

\[
= [1 - \alpha [1 - F(\bar{z})]] \bar{q}(i, \gamma) - \alpha \int_{z_1}^{\bar{z}} q(z, i) dF(z)
\]

When \( \bar{b} \) increases, then there is less supply of real money to be loaned out for a fixed \( \bar{q} \) and \( i \). To raise the net supply of funds, then \( i \) must increase, which raises \( \bar{q} \) and lowers \( q(z, i) \). To see this, from the FOC for money holdings

\[
-di = \alpha \int_{\bar{z}(i)}^{z_h} \{zu''(\bar{q}(i, \gamma)) d\bar{q} - di\} dF(z)
\]

\[
20
\]
or

\[ [\alpha [1 - F(\bar{z})] - 1] di = d\tilde{q} \alpha''(\bar{q}(i, \gamma)) \int_{\bar{z}(i)}^{z_h} \{z\} dF(z) \]

so that

\[ \frac{d\tilde{q}}{di} = -\frac{[1 - \alpha [1 - F(\bar{z})]]}{\alpha u''(\bar{q}(i, \gamma)) \int_{\bar{z}(i)}^{z_h} zdF(z)} > 0 \]

and

\[ zu''(q(z, i)) dq(z, i) = di. \]

so that

\[ \frac{dq(z, i)}{di} = \frac{1}{zu''(q(z, i))} \]

Thus, the effect of raising \( i \) on the net supply of loans is

\[ = [1 - \alpha [1 - F(\bar{z})]] \bar{q}_i(i, \gamma) - \alpha \int_{z_l}^{\bar{z}} q_i(z, i) dF(z) \]

\[ = [1 - \alpha [1 - F(\bar{z})]] \left[ -\frac{[1 - \alpha [1 - F(\bar{z})]]}{\alpha u''(\bar{q}(i, \gamma)) \int_{\bar{z}(i)}^{z_h} zdF(z)} - \alpha \int_{z_l}^{\bar{z}} \frac{1}{zu''(q(z, i))} dF(z) \right] \]

\[ = -\left[ \frac{[1 - \alpha [1 - F(\bar{z})]]^2}{\alpha u''(\bar{q}(i, \gamma)) \int_{\bar{z}(i)}^{z_h} zdF(z)} \right] - \alpha \int_{z_l}^{\bar{z}} \frac{1}{zu''(q(z, i))} dF(z) > 0 \]

Returning to welfare

\[ W = \frac{\alpha}{1 - \beta} \left[ \int_{z_l}^{\bar{z}(i, \gamma)} \{zu(q(z, i)) - q(z, i)\} dF(z) + \int_{\bar{z}(i, \gamma)}^{z_h} \{zu(\bar{q}(i, \gamma)) - \bar{q}(i, \gamma)\} dF(z) \right] \]

thus,

\[ \frac{dW}{db} = \frac{\partial W}{\partial \tilde{q}} \frac{\partial \tilde{q}}{\partial i} \frac{\partial i}{\partial b} \]

First the last term, from above

\[ \bar{b} = [1 - \alpha [1 - F(\bar{z})]] \bar{q}(i, \gamma) - \alpha \int_{z_l}^{\bar{z}} q(z, i) dF(z) \]

so that

\[ \bar{b} = \left\{ [1 - \alpha [1 - F(\bar{z})]] \bar{q}_i(i, \gamma) - \alpha \int_{z_l}^{\bar{z}} q_i(z, i) dF(z) \right\} di \]

\[ \bar{b} = \left\{ -\left[ \frac{[1 - \alpha [1 - F(\bar{z})]]^2}{\alpha u''(\bar{q}(i, \gamma)) \int_{\bar{z}(i)}^{z_h} zdF(z)} \right] - \alpha \int_{z_l}^{\bar{z}} \frac{1}{zu''(q(z, i))} dF(z) \right\} di \]

so that

\[ \frac{di}{db} = \left\{ \frac{[1 - \alpha [1 - F(\bar{z})]]^2}{\alpha u''(\bar{q}(i, \gamma)) \int_{\bar{z}(i)}^{z_h} zdF(z)} + \alpha \int_{z_l}^{\bar{z}} \frac{1}{zu''(q(z, i))} dF(z) \right\} > 0 \]
Putting in the second term

\[
\frac{\partial W}{\partial \bar{q}i} = \frac{[1 - \alpha [1 - F(\bar{z})]]}{\alpha u'(\bar{q}(i,\gamma)) z_0}(\bar{q}(i,\gamma)) + \int_{\bar{z}}^{z_0} \frac{1}{z u'(q(z,i))} dF(z)
\]

Putting in the first term:

\[
\frac{\partial W}{\partial \bar{q}i} = \frac{\alpha}{1 - \beta} \left[ \int_{\bar{z}(i,\gamma)}^{z_0} \{z u'(\bar{q}(i,\gamma)) - 1\} dF(z) \right]
\]

Compare this to the FOC for money holdings:

\[
\frac{1 + \gamma}{\beta} - (1 + i(\gamma)) + i(\gamma) \alpha [1 - F(\bar{z})] = \alpha \int_{\bar{z}(i,\gamma)}^{z_0} \{z u'(\bar{q}(i,\gamma)) - 1\} dF(z)
\]

or

\[
\frac{1 + \gamma - \beta [1 + i(\gamma) [1 - \alpha [1 - F(\bar{z})]]]}{\beta (1 - \beta)} = \alpha \int_{\bar{z}(i,\gamma)}^{z_0} \{z u'(\bar{q}(i,\gamma)) - 1\} dF(z)
\]

Thus,

\[
\frac{dW}{db} = \frac{\partial W}{\partial \bar{q}} \frac{\partial i}{\partial \bar{q}} \frac{\partial i}{\partial i} \frac{\partial i}{\partial \bar{b}} = \frac{1 + \gamma - \beta [1 + i(\gamma) [1 - \alpha [1 - F(\bar{z})]]]}{\beta (1 - \beta)} \Omega > 0
\]

\[\blacksquare\]

A.2. The effect of money growth on welfare

Once again, welfare is given by:

\[
W = \frac{\alpha}{1 - \beta} \left[ \int_{\bar{z}(i,\gamma)}^{z_0} \{z u(z, i) - q(z, i)\} dF(z) + \int_{\bar{z}(i,\gamma)}^{z_0} \{z u'(\bar{q}(i,\gamma)) - \bar{q}(i,\gamma)\} dF(z) \right]
\]

and the equilibrium conditions are:

\[
\frac{1 + \gamma}{\beta} - (1 + i(\gamma)) = \alpha \int_{\bar{z}(i,\gamma)}^{z_0} \{z u'(\bar{q}(i,\gamma)) - 1 + i(\gamma)\} dF(z) \quad (A.6)
\]

\[
\bar{z}(i,\gamma) = \frac{1 + i(\gamma)}{u'(\bar{q}(i,\gamma))} \quad (A.7)
\]

\[
m_b = \bar{q}(i) + \bar{b} \quad (A.8)
\]

and

\[
\bar{b} = \bar{q}(i,\gamma) - \alpha \left[1 - F(\bar{z}(i,\gamma))\right] \bar{q}(i,\gamma) + \int_{\bar{z}(i,\gamma)}^{z_0} q(z, i) dF(z) \quad (A.9)
\]
\[ zu' (q(z,i)) = 1 + i. \] (A.10)

Once again, money growth has no direct effect on welfare, but it does have an affect via \( \bar{q} \), thus,

\[ \frac{dW}{d\gamma} = \frac{\partial W}{\partial \bar{q}} \frac{\partial \bar{q}}{\partial \gamma} \]

When \( \gamma \) changes, then from the FOC for money holdings,

\[ \frac{1}{\beta} d\gamma - [1 - \alpha [1 - F(\bar{z})]] di = \left[ \alpha \int_{\bar{z}(i)}^{z_{th}} \left\{ zu''(\bar{q}(i,\gamma)) \right\} dF(z) \right] d\bar{q} \]

From the loan market clearing, then

\[ [1 - \alpha [1 - F(\bar{z})]] d\bar{q} - \left[ \alpha \int_{\bar{z}(i)}^{z_{th}} \frac{1}{zu''(q(z,i))} dF(z) \right] di = 0 \]

thus,

\[ di = \frac{[1 - \alpha [1 - F(\bar{z})]]^2}{\alpha \int_{\bar{z}(i)}^{z_{th}} \frac{1}{zu''(q(z,i))} dF(z)} d\bar{q}. \]

Putting this in the FOC for money holdings, then

\[ \frac{1}{\beta} d\gamma - \frac{[1 - \alpha [1 - F(\bar{z})]]}{\alpha \int_{\bar{z}(i)}^{z_{th}} \frac{1}{zu''(q(z,i))} dF(z)} d\bar{q} = \left[ \alpha \int_{\bar{z}(i)}^{z_{th}} \left\{ zu''(\bar{q}(i,\gamma)) \right\} dF(z) \right] d\bar{q} \]

or

\[ \frac{d\bar{q}}{d\gamma} = \frac{1}{\beta} \left[ \alpha \int_{\bar{z}(i)}^{z_{th}} \left\{ zu''(\bar{q}(i,\gamma)) \right\} dF(z) \right] + \frac{[1-\alpha[1-F(\bar{z})]]}{\alpha \int_{\bar{z}(i)}^{z_{th}} \frac{1}{zu''(q(z,i))} dF(z)} < 0 \]

Therefore,

\[ \frac{dW}{d\gamma} = \frac{\partial W}{\partial \bar{q}} \frac{\partial \bar{q}}{\partial \gamma} = \frac{1}{\beta^2 (1 - \beta)} \left[ \alpha \int_{\bar{z}(i)}^{z_{th}} \left\{ zu''(\bar{q}(i,\gamma)) \right\} dF(z) \right] + \frac{[1-\alpha[1-F(\bar{z})]]}{\alpha \int_{\bar{z}(i)}^{z_{th}} \frac{1}{zu''(q(z,i))} dF(z)} < 0 \]

\[ \square \]

**B. Welfare in Model with young agents and enforcement**

Once again,

\[ W_s (\hat{m}, \hat{M}/N) = \frac{1}{P(\hat{M}/N)} \hat{m} \]
For an old buyer

\[
W_b^O \left( \hat{m}, \frac{\hat{M}}{N} \right) = \frac{1}{P} [\hat{\tau} + \hat{m}] - \frac{P'}{P \beta} m_b + (1 + i) m_b + \\
\alpha \left[ \int z^{(z)} \{ z u (q (z, i)) - q (z, i) (1 + i) \} dF (z) + \int \{ z u (\bar{q} (i, \gamma)) - \bar{q} (i, \gamma) (1 + i (\gamma)) \} dF (z) \right] + W_b^O \left( 0, \frac{\hat{M}'}{N'} \right)
\]

\[
W_b^O \left( \hat{m}, \frac{\hat{M}}{N} \right) = \frac{1}{P} [\hat{\tau} + \hat{m}] - \frac{1 + \gamma}{\beta} m_b + (1 + i) m_b + \\
\alpha \left[ \int z^{(z)} \{ z u (q (z, i)) - q (z, i) (1 + i) \} dF (z) + \int \{ z u (\bar{q} (i, \gamma)) - \bar{q} (i, \gamma) (1 + i (\gamma)) \} dF (z) \right] + W_b^O \left( 0, \frac{\hat{M}'}{N'} \right)
\]

A subtlety is the money transfer. From above

\[
\hat{\tau} = \frac{\gamma}{1 + \gamma} \frac{P \left( \hat{M}' / N' \right)}{\beta} \left[ (1 - \mu) m_b + \mu \hat{m} \right]
\]

But since this is divided just across the old, then the transfer that the old agent gets is

\[
\hat{\tau}^O = \frac{1}{1 - \mu} \frac{\gamma}{1 + \gamma} \frac{P \left( \hat{M}' / N' \right)}{\beta} \left[ (1 - \mu) m_b + \mu \hat{m} \right] = \frac{\gamma}{1 + \gamma} \frac{P \left( \hat{M}' / N' \right)}{\beta} \left[ m_b + \frac{\mu}{1 - \mu} \hat{m} \right]
\]

Thus,

\[
W_b^O \left( \hat{m}, \frac{\hat{M}}{N} \right) = \frac{1}{P} \hat{m} + \frac{\gamma}{\beta} \left[ m_b + \frac{\mu}{1 - \mu} \hat{m} \right] - \frac{1 + \gamma}{\beta} m_b + (1 + i) m_b + \\
\alpha \left[ \int z^{(z)} \{ z u (q (z, i)) - q (z, i) (1 + i) \} dF (z) + \int \{ z u (\bar{q} (i, \gamma)) - \bar{q} (i, \gamma) (1 + i (\gamma)) \} dF (z) \right] + W_b^O \left( 0, \frac{\hat{M}'}{N'} \right)
\]

\[
W_b^O \left( \hat{m}, \frac{\hat{M}}{N} \right) = \frac{1}{P} \hat{m} + \frac{\gamma}{\beta} \frac{\mu}{1 - \mu} \hat{m} - \frac{1 - \beta}{\beta} m_b + i m_b + \\
\alpha \left[ \int z^{(z)} \{ z u (q (z, i)) - q (z, i) (1 + i) \} dF (z) + \int \{ z u (\bar{q} (i, \gamma)) - \bar{q} (i, \gamma) (1 + i (\gamma)) \} dF (z) \right] + \beta W_b^O \left( 0, \frac{\hat{M}'}{N'} \right)
\]

and

\[
W_b^O \left( 0, \frac{\hat{M}'}{N'} \right) = \frac{\gamma}{(1 - \beta)} \frac{\mu}{\beta} \frac{1 - \mu}{1 - \beta} \hat{m} - \frac{1}{\beta} m_b + \frac{i}{1 - \beta} m_b + \\
\frac{\alpha}{1 - \beta} \left[ \int z^{(z)} \{ z u (q (z, i)) - q (z, i) (1 + i) \} dF (z) + \int \{ z u (\bar{q} (i, \gamma)) - \bar{q} (i, \gamma) (1 + i (\gamma)) \} dF (z) \right]
\]
For a young agent:

\[
W_b^y \left( 0, \frac{\dot{M}}{N} \right) = -\frac{1 + \gamma}{\beta} \bar{m} + (1 + i) \bar{m} + \alpha \left[ \int_{z_l}^{z_i} \left\{ zu(q(z,i)) - q(z,i) (1 + i) \right\} dF(z) + \int \left\{ zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m}) (1 + i (\gamma)) \right\} dF(z) \right] + \beta W_b^O \left( 0, \frac{\dot{M}'}{N'} \right)
\]

Examine the per-period utility:

\[
W = \frac{1}{1 + \eta \beta} [(1 - \mu) m_b + \mu \bar{m}] + \mu \left[ -\frac{1 + \gamma}{\beta} \bar{m} + (1 + i) \bar{m} + \alpha \left[ \int_{z_l}^{z_i} \left\{ zu(q(z,i)) - q(z,i) (1 + i) \right\} dF(z) + \int \left\{ zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m}) (1 + i (\gamma)) \right\} dF(z) \right] \right] + (1 - \mu) \left[ \frac{\gamma}{\beta} \mu \bar{m} - \frac{1 - \beta}{\beta} m_b + im_b + \alpha \left[ \int_{z_l}^{z_i} \left\{ zu(q(z,i)) - q(z,i) (1 + i) \right\} dF(z) + \int \left\{ zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m}) (1 + i (\gamma)) \right\} dF(z) \right] \right] + W_b(0)
\]

Grouping the \( \bar{m} \) terms then

\[
W = \frac{1}{1 + \eta \beta} [(1 - \mu) m_b + \mu \bar{m}] + \mu \left[ -\frac{1 - \beta}{\beta} \bar{m} + im_b + \alpha \left[ \int_{z_l}^{z_i} \left\{ zu(q(z,i)) - q(z,i) (1 + i) \right\} dF(z) + \int \left\{ zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m}) (1 + i (\gamma)) \right\} dF(z) \right] \right] + (1 - \mu) \left[ -\frac{1 - \beta}{\beta} m_b + im_b + \alpha \left[ \int_{z_l}^{z_i} \left\{ zu(q(z,i)) - q(z,i) (1 + i) \right\} dF(z) + \int \left\{ zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m}) (1 + i (\gamma)) \right\} dF(z) \right] \right] + W_b(0)
\]

Using the market clearing condition in the loan market, then the \( m_i \), can be taken out.

\[
W = \frac{1}{1 + \eta \beta} [(1 - \mu) m_b + \mu \bar{m}] + \mu \left[ -\frac{1 - \beta}{\beta} \bar{m} + im_b + \alpha \left[ \int_{z_l}^{z_i} \left\{ zu(q(z,i)) - q(z,i) (1 + i) \right\} dF(z) + \int \left\{ zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m}) (1 + i (\gamma)) \right\} dF(z) \right] \right] + \beta W_b(0)
\]

Now solving for \( W(0) \), then divide the last terms by \( 1 - \beta \),

\[
W = \frac{1}{1 + \eta \beta} [(1 - \mu) m_b + \mu \bar{m}]
\]
\[ \mu \left[ -\frac{1}{\beta \bar{m}} + \frac{\alpha}{1 - \beta} \left[ \frac{\int_{z_i} \{zu(q(z,i)) - q(z,i)\} dF(z)}{\int \{zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m})\} dF(z)} \right] \right] + \\
(1 - \mu) \left[ -\frac{1}{\beta m_b} + \frac{\alpha}{1 - \beta} \left[ \frac{\int_{z_i} \{zu(q(z,i)) - q(z,i)\} dF(z)}{\int \{zu(\bar{q}(i,\gamma)) - \bar{q}(i,\gamma)\} dF(z)} \right] \right] + \\
\beta W_b(0) \]

and a final grouping

\[ W = \frac{-\eta}{1 + \eta \beta} \left[ \mu \bar{m} + (1 - \mu) m_b \right] + \frac{\alpha \mu}{1 - \beta} \left[ \frac{\int_{z_i} \{zu(q(z,i)) - q(z,i)\} dF(z)}{\int \{zu(\bar{q}(\bar{m})) - \bar{q}(\bar{m})\} dF(z)} \right] + \\
\frac{\alpha (1 - \mu)}{1 - \beta} \left[ \frac{\int_{z_i} \{zu(q(z,i)) - q(z,i)\} dF(z)}{\int \{zu(\bar{q}(i,\gamma)) - \bar{q}(i,\gamma)\} dF(z)} \right] \]