Capital and Macroeconomic Instability

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Abstract

We establish a necessary and sufficient condition for local real (in)determinacy of equilibrium in a discrete-time production economy with monopolistic competition and a quadratic nominal price adjustment cost under forward-looking interest rate policy rules, for the case that capital is in an exogenously fixed stock, as well as for the case with endogenous capital accumulation. Using these conditions, we find that (i) indeterminacy is more likely to occur with a greater share of payment to capital in value-added production cost; (ii) indeterminacy can be either more likely or less likely to occur with exogenous capital than with endogenous capital, depending on the cost share of capital; (iii) indeterminacy is more likely to occur when prices are modeled as jump variables than as predetermined variables; and (iv) indeterminacy is more likely to occur with a smaller steady-state inflation rate, or with a smaller steady-state monopolistic markup of price over marginal cost.

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1 Introduction

As central banks around the world have become more independent and transparent in the past fifteen years or so, more systematic conduct of monetary policy has become increasingly popular. Most practices of systematic monetary policy have taken the form of interest rate feedback rules that set the short-term nominal interest rate as an increasing function of the expected future inflation rate. This trend, which began in the industrial and middle-income countries in the late 1980s and spread to the transition and emerging-market economies in the late 1990s, has become a practical phenomenon worldwide.

The past years have also witnessed a surge of academic studies on whether interest rate feedback rules may lead to real indeterminacy of equilibrium, which would open the door to welfare-reducing sunspot fluctuations. This literature, starting with the early contributions by Leeper (1991) and Taylor (1993), has accumulated a number of insightful results that have greatly enhanced our understanding. While most of the studies emphasize the implication for determinacy of how the interest rate affects household’s consumption-savings decision, the channel by which the interest rate affects firm’s investment decision has also begun to draw attention recently.

Dupor (2001) analyzes the issue of local real determinacy in a continuous-time model with a quadratic nominal price adjustment cost and endogenous capital accumulation, where the monetary authority sets the nominal interest rate as a function of the instantaneous rate of inflation. As is known, the instantaneous rate of inflation in a continuous-time setting is given by the right-hand derivative of the log of the price level. Hence, the discrete-time counterpart of a continuous-time interest rate rule that responds to the instantaneous rate of inflation is given by a forward-looking interest rate rule that responds to the expected inflation rate. Carlstrom and Fuerst (2004) obtain a necessary condition for local real determinacy under such a forward-looking interest rate policy rule in a discrete-time model with partial nominal price adjustment and endogenous capital accumulation. While these authors emphasize the endogenous nature of capital in their models, in particular, an implied no-arbitrage condition
that links the (expected) real return on bonds to the (expected) real return on capital, their results indicate that the implications for determinacy can be sensitive to the choice of discrete time versus continuous time in the modeling.

In the present paper, we establish a necessary and sufficient condition for local real (in)determinacy of equilibrium in a discrete-time production economy with monopolistic competition and a quadratic cost of nominal price adjustment under forward-looking interest rate policy rules, for the case that capital is in an exogenously fixed stock, as well as for the case with endogenous capital accumulation. To our knowledge, this paper is the first in the discrete-time sticky-price literature that obtains a necessary and sufficient condition for real (in)determinacy with endogenous capital accumulation under forward-looking interest rate rules. In addition to their own interests, these necessary and sufficient conditions allow us to gain some insights into the determinacy issue that have not been explored before.

Using these necessary and sufficient conditions, we find that:

(i) indeterminacy is more likely to occur with a greater share of payment to capital in value-added production cost, regardless of whether capital itself is modeled as exogenous or as endogenous;

(ii) indeterminacy can be either more likely or less likely to occur with exogenous capital than with endogenous capital, depending on the cost share of capital;

(iii) indeterminacy is more likely to occur when prices are modeled as jump variables than as predetermined variables;

(iv) indeterminacy is more likely to occur with a smaller steady-state inflation rate, or with a smaller steady-state monopolistic markup of price over marginal cost.

Before proceeding to the main text, it is useful to have a birdview over the roadmap of the paper. Section 2 below sets up the model, which incorporates the case with jump prices and the case with predetermined prices in a unified framework, and which allows for both the case with exogenous capital and the case with endogenous capital.

Section 3 characterizes equilibrium conditions for the case with capital in an exogenously
fixed stock. It shows that local determinacy analysis in this case is an analysis of a system of two linear difference equations, a New Phillips curve and a consumption Euler equation, in two jump variables, so that whether there is determinacy depends on whether the two eigenvalues of this linear system both have larger than unit modulus. Capital here matters for determinacy through how its share in value-added production cost affects the New Phillips curve.

Section 4 then characterizes equilibrium conditions for the case with endogenous capital accumulation. It illustrates that, while the endogenous nature of capital brings with it two additional equations, a no-arbitrage condition and a capital accumulation equation, along with two additional variables, a jump variable and a predetermined variable, it introduces at the same time a zero eigenvalue, corresponding to the no-arbitrage condition, and a greater than unit eigenvalue, corresponding to the capital accumulation equation. As a result, local determinacy analysis in this case is essentially an analysis of the log-linear consumption Euler equation and New Phillips curve, in which the no-arbitrage condition might be embedded, and whether there is determinacy here hinges upon whether the two eigenvalues of this two equation system both have greater than unit modulus. Just as in the case with exogenous capital, endogenous capital here matters for determinacy also through how its share in value-added production cost affects the New Phillips curve. Although the detail of this effect is different across the two cases, the qualitative implication for determinacy is similar.

Section 5 further illuminates this point by showing with much detail how determinacy (indeterminacy) of the four equation system is equivalent to determinacy (indeterminacy) of the reduced two equation system in the case with endogenous capital accumulation, and the structural similarity between this reduced system and the two equation system for the case with exogenous capital. It demonstrates that the basic mechanisms within constant-capital and variable-capital models that ensure determinacy are similar, while the specific details are different.

Section 6 presents our main results: the necessary and sufficient condition for local real
(in) determinacy of equilibrium under forward-looking interest rate rules, for the case with capital in a fixed stock, and for the case with endogenous capital accumulation. It then applies these necessary and sufficient conditions to establish several findings that have not been observed before. It provides some intuitions as to why indeterminacy is more likely to occur with a greater share of payment to capital in value-added production cost, and why indeterminacy can be either more likely or less likely to occur with exogenous capital than with endogenous capital, depending on the cost share of capital. It also illustrates why the implications for determinacy of the two conventional timings of price adjustment are different and why the difference is more dramatic in lower frequency models.

Section 7 concludes the paper with some final remarks.

2 A Model with Price Adjustment Cost and Capital

Time is discrete and indexed by \( t = 0, 1, \ldots \). The economy is populated by a large number of household-firm units, each producing a differentiated good and having a lifetime utility,

\[
\sum_{t=0}^{\infty} \rho^t \left[ \log c_t + \log \frac{M_{t+1}}{P_t} - n_t - \frac{\gamma}{2} \left( \frac{P_{t+J}}{P_{t-1+J}} - \pi^* \right)^2 \right], \quad \text{for } \gamma > 0, \tag{1}
\]

where \( P_t \) denotes the nominal price that a unit’s firm charges at \( t \) for the good it produces in the period, and \( J \) is either 0 or 1, corresponding to two price-adjustment timings often used in discrete-time models of monopolistic competition: The specification is such so that, at date \( t \), the firm chooses \( P_{t+J} \) taking \( P_{t-1+J} \) as given; thus, prices effective in the current period are set in the current period if \( J = 0 \), but were set in the previous period if \( J = 1 \). As such, individual prices and, by symmetry, the price level are jump variables if \( J = 0 \), but predetermined variables if \( J = 1 \). The other notations in (1) are standard: \( \rho \in (0, 1) \) is a discount factor, \( c_t \) is the unit’s household consumption in period \( t \), which is a composite of goods produced by all firms, \( n_t \) is the household’s labor supply in period \( t \), \( \bar{P}_t \) is the economy-wide price level, \( \pi^* \) is the steady state value of the gross rate of change in the price
level, and $M_{t+1}$ is the household’s nominal money balances at the end of period $t$.\footnote{1}

At each date $t$, the firm inputs labor and capital services, $\tilde{n}_t$ and $\tilde{k}_t$, to produce its differentiated good $y_t$ according to

$$y_t = \tilde{k}_t^\alpha \tilde{n}_t^\beta, \quad \text{where } \alpha \in [0, 1), \ \alpha + \beta = 1. \quad (2)$$

The firm is an input-price taker, but a monopolistic competitor in its product market. With markup pricing, $\alpha$ and $\beta$ determine respectively the share of payments to capital and to labor in value-added production cost rather than in gross output (i.e., the cost share as oppose to the revenue share), as will be made more transparently below. It is assumed that, given the price $P_t$ that the firm charges for its product, it must produce enough to meet the demand for its good given by the right-hand side of the following equation,

$$y_t = Y_t^d \left( \frac{P_t}{\bar{P}_t} \right), \quad (3)$$

where $Y_t^d$ denotes aggregate output which, as the consumption good, is a composite of individually differentiated goods produced via a Dixit-Stiglitz technology. The function $d(\cdot)$ is assumed to be twice continuously differentiable, positive, decreasing, and satisfy $d(1) = 1$. The equilibrium price elasticity of this demand function faced by the individual firm, denoted here as $\phi$, is assumed to be strictly less than $-1$, that is,

$$\phi \equiv \left. \frac{d'(\frac{P_t}{\bar{P}_t}) \times \frac{P_t}{\bar{P}_t}}{d(\frac{P_t}{\bar{P}_t})} \right|_{P=\bar{P}} < -1,$$

which is a necessary assumption for a well-defined model of monopolistic competition. As we will show below, the ratio $\phi/(\phi + 1)$ determines the steady-state monopolistic distortion measured by the markup of price over marginal cost.

The objective of the household-firm unit is to maximize (1), subject to (2), (3), and a
flow budget constraint,

\[
\frac{M_{t+1} + B_t}{P_t} = \frac{M_t + B_{t-1}R_{t-1}}{P_t} + \frac{P_t}{P_t}y_t - r_t\tilde{k_t} - w_t\bar{n}_t + r_tk_t + w_t\bar{n}_t - \tilde{y}_t - \tau_t,
\]

(4)

where \(\tilde{y}_t\) denotes the household’s demand for the composite good in period \(t\), \(k_t\) is its capital supply at the beginning of period \(t\), which, as the consumption good and aggregate output, is measured in units of the composite of the individually differentiated goods, \(B_{t-1}\) is its bond-holdings acquired in period \(t-1\) and \(R_{t-1}\) is the gross nominal rate of return on holding the bond from \(t-1\) to \(t\), \(w_t\) and \(r_t\) are real wage and real capital rental rate, respectively, and \(\tau_t\) is a real lump-sum tax (or subsidy).^6

This setup allows for both the case where capital is in an exogenously fixed supply and the case with endogenous capital accumulation. In the case with exogenous capital supply (at either aggregate or individual household level), the fixed amount of capital available for firms to hire never depreciates, while new capital good never being produced, though the demand for capital by firms can still be endogenously determined.^7 In this case, we have

\[
k_t = k_0, \quad \text{and} \quad \tilde{y}_t \equiv c_t, \quad \text{for all} \quad t \geq 0,
\]

(5)

where \(k_0\) denotes the household’s initial holding of capital, which is treated as given. In the case with endogenous capital accumulation, both the supply of and the demand for capital are determined endogenously. In this case, (5) is replaced with a capital accumulation equation,

\[
k_{t+1} = i_t + (1 - \delta)k_t, \quad \text{and} \quad \tilde{y}_t \equiv c_t + i_t,
\]

(6)

where \(\delta\) is the capital depreciation rate, and \(i_t\) denotes the household’s investment in units of the composite good during period \(t\).
3 Equilibria with Exogenous Capital Supply

At each date \( t \), a household-firm unit chooses \( c_t, M_{t+1}, B_t, n_t, \tilde{n}_t, \tilde{k}_t, \) and \( P_{t+1} \) to maximize (1) subject to (2)–(5), taking as given the initial conditions \( M_0, B_{-1}, k_0, \) and \( P_{-1} \), as well as the time paths for \( \tau_t, R_t, Y^d_t, w_t, r_t, \) and \( \bar{P}_t \). The Lagrangian is given by

\[
\sum_{t=0}^{\infty} \rho^t \{ \log c_t + \log \frac{M_{t+1}}{P_t} - n_t - \frac{\gamma}{2} \left( \frac{P_{t+1}}{P_{t-1+1}} - \pi^* \right)^2 \} + \mu_t [Y^d_t d(P_t) - \tilde{k}_t^{\alpha} \tilde{n}_t^{\beta}] \\
+ \lambda_t \left[ \frac{M_t + B_{t-1} R_{t-1}}{P_t^2} + \frac{P_t}{P_t} \tilde{k}_t^{\alpha} \tilde{n}_t^{\beta} - r_t \tilde{k}_t - w_t \tilde{n}_t + r_t k_0 + w_t n_t - c_t - \tau_t - \frac{M_{t+1} + B_t}{P_t} \right].
\]

The resulting first-order conditions, when coupled with the market-clearing conditions for factor inputs and equilibrium symmetry (i.e., \( \tilde{k}_t = k_0, \tilde{n}_t = n_t, P_t = \bar{P}_t \)), imply that,

\[
\frac{1}{c_t} = \lambda_t, \tag{7}
\]

\[
\frac{1}{M_{t+1}} = \lambda_t \frac{\lambda_{t+1}}{P_t}, \tag{8}
\]

\[
\frac{R_t}{\tilde{\pi}_t} = \frac{\lambda_t}{\rho \lambda_{t+1}}, \tag{9}
\]

\[
\frac{1}{w_t} = \lambda_t, \tag{10}
\]

\[
\frac{w_t \tilde{n}_t}{y_t} = \beta \left( 1 - \frac{\mu_t}{\lambda_t} \right), \tag{11}
\]

\[
\frac{r_t \tilde{k}_t}{y_t} = \alpha \left( 1 - \frac{\mu_t}{\lambda_t} \right), \tag{12}
\]
\[
\frac{\gamma \pi_{t+J-1} (\pi_{t+J-1} - \pi^*) - \rho \gamma \pi_{t+J} (\pi_{t+J} - \pi^*)}{\rho^J y_{t+J}} = \phi \mu_{t+J} + \lambda_{t+J},
\]

and the market-clearing condition for the composite good is simply \( c_t = y_t = \tilde{k}_t^\alpha \tilde{n}_t^\beta \). Here we have used \( \pi_t \) to denote the expected inflation rate \( \bar{P}_{t+1}/\bar{P}_t \) so our notion is consistent with the one used in the continuous-time setup where the instantaneous rate of inflation is given by the right-hand derivative of the log of the price level. This also implies that the discrete-time counterpart of a standard continuous-time nominal interest rate rule that responds to the instantaneous rate of inflation is characterized by a forward-looking rule, that is, the monetary authority sets the nominal interest rate as a function of the expected inflation rate, \( R_t = \psi(\pi_t) \). For the purpose of our local analysis, it is without loss of generality to consider a linear rule

\[
R_t = R^* + q(\pi_t - \pi^*),
\]

where \( q = \psi'(\pi^*) \geq 0 \) measures the degree of activeness of policy around the steady state and \( R^* = \psi(\pi^*) \) denotes the steady-state value of the nominal interest rate.

Combining the steady-state versions of (11), (12), and (13), one can show that the ratio \( \phi/(\phi + 1) \), given by the inverse of \( (1 - \mu/\lambda) \), is equal to the steady-state markup of price over unit cost, which is also marginal cost under the assumed constant-return-to-scale production technology. Note also that, with markup pricing, the share of payment to capital (labor) input in value-added production cost (i.e., the cost share) equals the product of the share of capital (labor) input in gross output (i.e., the revenue share) and the steady-state markup of price over marginal cost. It then follows from (12) and (11) that \( \alpha \) and \( \beta \) measure the share of payments to capital and to labor, respectively, in value-added production cost.

Before proceeding further, it is useful to simplify the equilibrium conditions (7)-(13) into a system of equations that are key to our determinacy analysis. One of such equations is
derived by substituting (7), (10), and (11) into (13), which yields

$$\gamma \pi_{t+J-1} (\pi_{t+J-1} - \pi^*) - \rho \gamma \pi_{t+J} (\pi_{t+J} - \pi^*) = \rho^J (1 + \phi) \frac{y_{t+J}}{c_{t+J}} - \rho^J \phi \beta n_{t+J}. \quad (15)$$

Another of such equations is obtained by combining (7) with (9), which gives rise to

$$\frac{R_t}{\pi_t} = \frac{c_{t+1}}{\rho c_t}. \quad (16)$$

Determinacy analysis in this case requires (15) and (16), along with the policy rule (14). Here, capital matters through how its share in value-added production cost ($\alpha = 1 - \beta$) affects the New Phillips curve (15), and policy matters through its interaction with households’ intertemporal consumption decisions, as prescribed by (16).

In deriving (15) and (16), we have used neither (8) nor (12). Equation (8) is not used since under an interest rate policy rule, money supply is endogenously determined, and real money balances $M_{t+1}/\bar{P}_t$ is solved as a residual from (8), once the paths for consumption and the expected inflation rate are pinned down. Yet, (8) does imply that, for real determinacy, not only real quantities but the expected inflation rate have to be determinate in equilibrium. Equation (12) is not used since in the present case with exogenous capital supply, this capital demand equation only serves a residual role to pin down equilibrium capital rental rate.

To proceed, denote by $x^*$ the steady-state value of a variable $x_t$, and $\hat{x}_t \equiv \log x_t - \log x^*$ the percentage deviation of the variable from its steady-state value. It follows that

$$x_t = x^* e^{\hat{x}_t} = x^* (1 + \hat{x}_t) + \mathcal{O}(\|\hat{x}_t\|^2),$$

where $\mathcal{O}(\|\hat{x}_t\|^2)$ summarizes terms of the second or higher orders. Using this expression to rewrite (14), (15), and (16), and dropping all higher than the first-order terms, we have

$$\rho \gamma \pi^{x^2} \hat{\pi}_{t+J} = \gamma \pi^{x^2} \hat{\pi}_{t+J-1} + \rho^J \phi \beta n^* \hat{c}_{t+J}, \quad (17)$$
Local determinacy analysis in this case with exogenous capital supply involves an analysis of the log-linearized New Phillips curve (17) and consumption Euler equation (18) as linear difference equations in two jump variables, \( \pi \) and \( c \), where the log-linearized policy rule is embedded. Thus, whether there is determinacy depends on whether the two eigenvalues of this linear system both have larger than unit modulus. Capital here matters for determinacy through how its share in value-added production cost \( (\alpha = 1 - \beta) \) affects the New Phillips curve.

4 Equilibria with Endogenous Capital Accumulation

At each date \( t \), in addition to those choice variables specified in the previous section, a household also chooses capital to be supplied in the next date, \( k_{t+1} \), to maximize (1) subject to (2), (3), (4), and (6), taking as given the initial conditions and the time paths for aggregate variables, as before. The Lagrangian is given by

\[
\sum_{t=0}^{\infty} \rho^t \left\{ \log c_t + \log \frac{M_{t+1}}{P_t} - n_t - \gamma \left( \frac{P_{t+1}}{P_{t-1}} - \pi^* \right)^2 \right\} + \mu_t \left[ Y_t^d d_t \left( \frac{P_t}{P_{t-1}} \right) - \tilde{k}_t \tilde{n}_t^\beta \right] + \lambda_t \left[ M_t + B_{t-1} R_{t-1} \right] \\
+ \frac{P_t}{P_t} \tilde{k}_t \tilde{n}_t^\beta - r_t \tilde{k}_t - w_t \tilde{n}_t + r_t n_t - k_{t+1} + (1 - \delta) k_t - c_t - \tau_t - \frac{M_{t+1} + B_t}{P_t} \right\}.
\]

The resulting equilibrium conditions contain (7)-(13), as in their exact forms, plus the first-order condition for the household’s endogenous capital supply,

\[
r_{t+1} + 1 - \delta = \frac{\lambda_t}{\rho \lambda_{t+1}}.
\]  

This capital supply equation, when coupled with the bond-holding condition (9), gives rise to a no-arbitrage condition,

\[
\frac{R_t}{\pi_t} = r_{t+1} + 1 - \delta,
\]
which links the expected real return on bonds to the expected real return on capital when there is no arbitrage opportunity between investing in these two types of assets. In addition, there is now the capital accumulation equation which, when combined with the market-clearing condition for the composite good, gives rise to a difference equation,

\[ k_{t+1} = k_t^\alpha n_t^\beta + (1 - \delta)k_t - c_t. \] (21)

Here, capital matters through how its share in value-added production cost affects (21), in addition to (15), and policy matters through interacting with households’ investment decisions, as prescribed by (20), in addition to with their consumption decisions prescribed by (16).

The first-order approximation to these four equations and the policy rule gives rise to the following log-linearized system of equilibrium conditions:

\[ \rho \gamma \pi^{*2} \hat{\pi}_{t+1} = \gamma \pi^{*2} \hat{\pi}_{t+1} - 1 + \rho \frac{\phi m^*}{\beta} [\beta \hat{c}_{t+1} + \alpha \hat{r}_{t+1}], \] (22)

\[ \hat{c}_{t+1} = (\rho q - 1) \hat{\pi}_t + \hat{c}_t, \] (23)

\[ \hat{r}_{t+1} = \frac{\rho q - 1}{\rho r^*} \hat{\pi}_t, \] (24)

\[ \hat{k}_{t+1} = -[\beta \delta + (\beta + 1) \frac{c^*}{k^*}] \hat{c}_t + \beta (\delta + \frac{c^*}{k^*}) \hat{r}_t + (1 + \frac{c^*}{k^*}) \hat{k}_t. \] (25)

We argue that local determinacy analysis in this case with endogenous capital accumulation is essentially an analysis of the log-linearized New Phillips curve (22) and consumption Euler equation (23), just as in the case where capital is in fixed supply. What the endogenous nature of capital supply essentially does is to introduce two additional equations, (24) and
an additional jump variable, \( r \), along with a predetermined variable, \( k \), into the system for determinacy analysis. But, as we will show below, by doing so it introduces at the same time a zero eigenvalue, corresponding to (24), and a greater than unit eigenvalue, corresponding to (25), into the system of the four linear difference equations (22)-(25). As a result, whether there is determinacy hinges entirely on whether the two eigenvalues of the system of the consumption Euler equation and the New Phillips curve, in which the no-arbitrage condition might be embedded, both have greater than unit modulus, and capital here matters for determinacy through how its share in value-added production cost affects the New Phillips curve, just as in the case with exogenous capital supply. Although the detail of this effect is different, the qualitative implication for determinacy turns out to be similar.

5 Comparison: A Further Illustration

To further illustrate the comparison laid out in the previous sections, consider \( J = 1 \). The comparison under \( J = 0 \), though a little more complicated, is similar.

By embedding how policy affects households’ intertemporal consumption decisions [(18)] into the New Phillips curve [(17)], the system of the two equations (17) and (18) for the case with exogenous capital supply can be written as

\[
\begin{pmatrix}
\hat{\pi}_{t+1} \\
\hat{c}_{t+1}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{\rho} + \frac{(1+\phi)(pq-1)}{\gamma \pi^{*2} \beta} & \frac{(1+\phi)}{\gamma \pi^{*2} \beta} \\
(pq - 1) & 1
\end{pmatrix}
\begin{pmatrix}
\hat{\pi}_t \\
\hat{c}_t
\end{pmatrix}.
\]

Since both \( \pi_t \) and \( c_t \) are jump variables, this two equation system is determinate if and only if its two eigenvalues both have larger than unit modulus.

By embedding how policy affects households’ consumption and investment decisions [(23) and (24)] into the New Phillips curve [(22)], the four equation system (22)-(25) for the case with endogenous capital accumulation can be written as
This linear system always has a zero eigenvalue, corresponding to the third line, and a greater than unit eigenvalue, $1 + \frac{c^*}{k^*}$, corresponding to the fourth line. It is easy to show that this four equation system is determinate if and only if the two eigenvalues of the following two equation system,

\[
\begin{bmatrix}
\dot{\pi}_{t+1} \\
\dot{c}_{t+1} \\
\dot{r}_{t+1} \\
\dot{k}_{t+1}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\rho} + \frac{\phi(\rho + 1)(\rho - 1)[1 - \rho(1 - \delta)]}{\gamma [\phi(1 - \rho)(1 - \delta)] - (1 + \phi)\rho\delta(1 - \beta)} & \frac{\phi(\rho + 1)(1 - \rho)(1 - \delta)}{\gamma [\phi(1 - \rho)(1 - \delta)] - (1 + \phi)\rho\delta(1 - \beta)} & 0 & 0 \\
\frac{\phi(\rho + 1)(\rho - 1)[1 - \rho(1 - \delta)]}{\gamma [\phi(1 - \rho)(1 - \delta)] - (1 + \phi)\rho\delta(1 - \beta)} & \frac{\phi(\rho + 1)(1 - \rho)(1 - \delta)}{\gamma [\phi(1 - \rho)(1 - \delta)] - (1 + \phi)\rho\delta(1 - \beta)} & 0 & 0 \\
\frac{\rho - 1}{1 - \rho + \rho\delta} & 0 & -[\beta (\delta + (\beta + 1)\frac{c^*}{k^*}) - \beta(\delta + \frac{c^*}{k^*}) 1 + \frac{c^*}{k^*}] & \beta(\delta + \frac{c^*}{k^*}) 1 + \frac{c^*}{k^*}
\end{bmatrix} \begin{bmatrix}
\dot{\pi}_t \\
\dot{c}_t \\
\dot{r}_t \\
\dot{k}_t
\end{bmatrix}.
\]

both have larger than unit modulus, just as in the case of (26). Notice that the two equation system (27) results from removing the last two lines in the above four equation system. This essentially removes a jump variable, $r_t$, and a predetermined variable, $k_t$, along with the two eigenvalues, 0 and $1 + \frac{c^*}{k^*}$, from the original system. Therefore, determinacy (indeterminacy) of the original four equation system is equivalent to determinacy (indeterminacy) of the reduced two equation system (27). Note the similarity between (26) and (27): the second lines (the consumption Euler equation) are identical, and $\beta$ affects only the first lines (the New Phillips curve). The difference between the two first lines is that the first line in (26) embeds how policy affects households’ consumption decisions, while the first line in (27) embeds how policy affects households’ consumption as well as investment decisions. Thus the basic mechanisms within exogenous-capital and endogenous-capital models that ensure determinacy are similar, while the specific details are different.
As we will show in the following section, because of the similarity in the basic mechanisms, a greater cost share of capital (a larger $\alpha$ and thus a smaller $\beta$), regardless of whether capital supply itself is modeled as exogenous or as endogenous, always leads to a smaller determinacy region.

As we will illustrate, also in the next section, because of the difference in the specific details, the determinacy region under exogenous capital supply can be relatively smaller or larger than the determinacy region with endogenous capital accumulation, depending on the cost share of capital.

6 Main Results

6.1 The Necessary and Sufficient Condition for Determinacy

To drive home these points, we derive in this section a necessary and sufficient condition for local real determinacy for the exogenous-capital model, as well as for the endogenous-capital model. To our knowledge, this paper is the first in the discrete-time sticky-price literature to derive a necessary and sufficient condition for real determinacy of equilibrium with endogenous capital accumulation under forward-looking interest rate policy rules.

**Proposition 1.** With exogenous capital supply, there is local real determinacy of equilibrium if and only if

$$0 < \rho q - 1 < \frac{2\gamma \pi^2 \beta (1 + \rho)}{-\rho^j (1 + \phi)} \equiv \bar{B}_1. \quad (28)$$

Otherwise, there is a continuum of equilibria.

**Proposition 2.** With endogenous capital accumulation, there is local real determinacy of equilibrium if and only if

$$0 < \rho q - 1 < \frac{\gamma \pi^2 (1 - \rho) [\rho \delta - \phi (1 - \rho) - (1 + \phi) \beta \rho \delta]}{(1 - \beta) \rho^j \phi (1 + \phi)} \equiv \bar{B}_2, \quad \text{for} \quad \beta < \beta^*, \quad (29)$$
\[ 0 < \rho q - 1 < \frac{2\gamma^{*2}(1 + \rho)[\rho\delta - \phi(1 - \rho) - (1 + \phi)\beta\rho\delta]}{[2 - \beta(1 + \rho - \rho\delta)]\rho\phi(1 + \phi)} \equiv \bar{B}_2, \text{ for } \beta \geq \beta^*, \quad (30) \]

where
\[ \beta^* \equiv \frac{4\rho}{(1 + \rho)^2 + \rho(1 - \rho)\delta} \in (0, 1). \]

Otherwise, there is a continuum of equilibria.

Propositions 1 and 2 say that there is equilibrium real determinacy if and only if policy is active, in the spirit of the Taylor principle, but not more active than what is specified by the upper bound in (28) for the constant-capital model or in (29)-(30) for the variable-capital model.\(^{10}\) Clearly, these upper bounds are proportional to \(\gamma\), and thus the greater (smaller) this price adjustment cost coefficient is, the larger (smaller) is the determinacy region, with either exogenous capital or endogenous capital, and under both timing conventions for price-setting.

Besides their own interests, these necessary and sufficient conditions make it feasible to compare the determinacy region under exogenous capital supply and the determinacy region with endogenous capital accumulation, and allow us to gain some insights into the determinacy issue that have not been explored before.

### 6.2 Implication of the Cost Share of Capital

The necessary and sufficient conditions established in Propositions 1 and 2 for the case where capital is in an exogenously fixed supply and for the case with endogenous capital accumulation reveal an essential implication of capital for indeterminacy that is first observed here.

It can be verified that the upper bound in (28) for the fixed-capital model and the upper bound in (29)-(30) for the variable-capital model are each increasing in \(\beta\) and thus decreasing in \(\alpha\) (note that \(\bar{B}_2\) and \(\bar{B}_2\) meet only at \(\beta = \beta^*\)). Therefore, regardless of whether capital itself is modeled as exogenous or as endogenous, a larger share of capital in value-added
production cost (a larger \( \alpha \) and thus a smaller \( \beta \)) always results in a smaller determinacy region. This is true under both conventions about the timing of price-setting.

### 6.3 Implication of Exogenous versus Endogenous Capital

Examining the upper bound in (28) for the constant-capital model and the upper bound in (29)-(30) for the variable-capital model also reveals a theoretical possibility not explored before that the reduction in the determinacy region due to the use of capital in production can be either greater or smaller when capital supply is modeled as exogenous than as endogenous. In other words, from the theoretical point of view, indeterminacy can be either less likely or more likely to occur in the constant-capital model than in the variable-capital model.

For instance, it can be shown that, if the cost share of capital is very large, then the determinacy region with exogenous capital is smaller than that with endogenous capital. This can be illustrated by considering a close-to-one \( \alpha \), and thus a close-to-zero \( \beta \). Then, \( \bar{B}_1 \) is close to zero and the determinacy region defined by (28) is close to an empty set, while \( \bar{B}_2 \) and \( \bar{\bar{B}}_2 \) are each bounded from below by strictly positive numbers (\( \bar{B}_2 \) is what really matters here) and the determinacy region defined by (29)-(30) stays as a non-empty interval [what really matters here is (29)]. To understand this contrast, compare the first line in (26) with the first line in (27). As \( \alpha \) gets larger and thus \( \beta \) gets smaller, the response of inflation expectation to even small variations in current inflation and the output gap becomes unboundedly sensitive in the case with exogenous capital supply [the first line in (26)], but stays bounded in the case with endogenous capital supply [the first line in (27)]. Since production uses mostly capital, the effects of changes in aggregate economic conditions are drastic changes in inflation expectation if capital is in fixed supply, but are more evenly split between changes in inflation expectation and changes in expected quantity variables if capital is in varying supply.

In the case that the share of (variable) labor in value-added production inputs is moderate or large, the effects of changes in aggregate demand conditions are always split somewhat
evenly between changes in inflation expectation and changes in expected quantity variables, regardless of whether capital is in varying or fixed supply, and thus the sensitivity of inflation expectation to variations in current inflation and output gap is bounded in both the case with endogenous capital supply and the case with exogenous capital supply. In fact, the determinacy region with exogenous capital is larger than the determinacy region with endogenous capital if the share of labor (capital) in value-added production inputs is very large (small), or, more specifically, if

$$\max\{\beta^*, \beta^{**}\} < \beta < 1,$$  \hfill (31)

where

$$\beta^{**} \equiv \frac{1 - \rho - \rho \delta}{1 + \rho - \rho \delta} \in (0, 1),$$

and $\beta^*$ is as defined in Proposition 2.\textsuperscript{11}

Again, the analysis conducted in this section holds regardless of which convention about the timing of price-setting is used.

### 6.4 Implication of the Timing of Price Adjustment

Recall that our theoretical approach allows us to model the two conventions about the timing of price adjustment often used in models of monopolistic competition in a unified framework. This unified approach leads to the following point first made here.

The necessary and sufficient conditions (28) and (29)-(30) presented in Proportions 1 and 2 reveal that the choice of one timing convention versus the other in models with quadratic nominal price adjustment costs has an implication for local real determinacy of equilibrium. In particular, these conditions show that determinacy is more likely to occur when prices are modeled as predetermined variables (i.e., when $J = 1$) than as jump variables (i.e., when $J = 0$). With a smaller discount factor (say, in a lower-frequency model), the increase in the likelihood of determinacy due to price being preset can be more dramatic.\textsuperscript{12} This conclusion
holds regardless of whether capital is modeled as exogenous or as endogenous.

6.5 Implication of Steady-State Inflation Rate

It is clear that both the upper bound in (28) for the constant-capital model and the upper bound in (29)-(30) for the variable-capital model are proportional to $\pi^2$, and thus a greater steady-state inflation rate implies a larger region for local real determinacy.

This is true regardless of whether capital is modeled as exogenous or as endogenous, and regardless of which of the two conventions about the timing of price adjustment is used. This implication of our determinacy result has not been observed before. Local determinacy analysis in discrete-time models with partial nominal price adjustment is typically based on log-linearization around a zero steady-state inflation rate and thus, by design, is not able to explore the implication of steady-state inflation rate for determinacy.

6.6 Implication of Steady-State Monopolistic Distortion

Another interesting implication of Propositions 1 and 2, as can be observed from examining the upper bounds in (28) and (29)-(30), is that a greater $\phi$ always leads to a larger region for local real determinacy. It follows that indeterminacy is less likely to occur, the greater is the steady-state monopolistic distortion, or, the steady-state markup of price over marginal cost. This holds true for both the case with exogenous capital and the case with endogenous capital, and under both conventions about the timing of price adjustment.

It is of some interest to compare this implication of our local real determinacy result to those obtained in discrete-time monopolistic competition models with partial nominal price adjustment. The necessary condition for local real determinacy with endogenous capital accumulation and indivisibility in labor obtained by Carlstrom and Fuerst (2004) does not depend on steady-state monopolistic distortion. The necessary and sufficient condition for local real determinacy in the constant-capital model of Woodford (2003) [Chapter 4, Proposition 4.5] is independent of steady-state monopolistic distortion when labor supply elasticity
is infinite, but the implied determinacy region is in fact smaller with a greater steady-state monopolistic distortion when the labor supply elasticity is finite.

This comparison is a manifestation that the quadratic price adjustment cost setup and the partial price adjustment setup may have different implications for equilibrium dynamics and determinacy, possibly not as the conventional wisdom alluded to.

7 Conclusion

As more and more central banks have moved into the framework of setting the nominal interest rate in response to changes in the expected inflation rate for the conduct of monetary policy, it is of paramount interest to know whether such policy practice may lead to real indeterminacy of equilibrium in a modern production economy where capital is an important source of production input. The present paper addresses this issue by deriving a necessary and sufficient condition for local real determinacy of equilibrium in a discrete-time economy with monopolistic competition and quadratic nominal price adjustment cost under forward-looking interest rate policy rules, for the case with capital is in an exogenously fixed amount, as well as for the case with endogenous capital accumulation.

Using these necessary and sufficient conditions, we find that indeterminacy is more likely to occur with a greater share of capital in value-added production cost, regardless of whether capital itself is modeled as exogenous or as endogenous, and that indeterminacy can be either more likely or less likely to occur with exogenous capital than with endogenous capital, depending on the cost share of capital. We demonstrate that these results arise because the basic mechanisms within exogenous-capital and endogenous-capital models that ensure determinacy are similar, while the specific details are slightly different.

It is worth noting that such nuances in the specific details could be made even smaller by adding capital adjustment costs in the variable-capital model. Our specification of the two extreme models in the paper is only meant to be pathological. While the constant-capital
model (or, a variable-capital model with an infinite capital adjustment cost) can be viewed as applicable for analysis in the short run during which it is extremely difficult to install new capital stock, the variable-capital model with no capital adjustment cost is simply a theoretical abstraction. A more realistic specification would be to allow for capital accumulation subject to some adjustment cost, so new capital stock can be installed in the medium to long run. Given the similarity in the basic mechanisms within these models with capital that ensure determinacy, our choice to consider the two opposite extremes allows to characterize analytically the essential implication of capital for indeterminacy without involving explicitly capital adjustment cost or other complications that would make analytical results unobtainable.13

These necessary and sufficient conditions also lead us to several other findings that have not been observed before. While different conventions on the timing of price adjustment have been used in the literature, we find that indeterminacy is more likely to occur when prices are modeled as jump variables than as predetermined variables, and this contrast is shaper in lower-frequency models. From a policy perspective, perhaps more interesting are our findings that indeterminacy is more likely to occur with a smaller steady-state inflation rate, or with a smaller steady-state monopolistic distortion. These results, although correct in our model, perhaps should not be taken as to literally suggesting that macroeconomic stability is always more likely to be ensured among inflation forecast targeting countries that have higher inflation rates or greater monopolistic distortions. Caution should be applied when interpreting these theoretical results, since in actuality most central banks that have adopted such policy regime set the nominal interest rate in response to not only movements in inflation but also movements in other endogenous variables, such as the output gap. It is known that a positive response to the output gap by the nominal interest rate helps ensure real determinacy in a constant-capital model with partial nominal price adjustment [e.g., Woodford (2003), Chapter 4]. We have explored this issue in our present setting with a quadratic nominal price adjustment cost, and our preliminary findings suggest that this is

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true for both the case where capital is in an exogenously fixed supply and the case with endogenous capital accumulation. This issue is so important that we feel it deserves the full attention of a separate paper.
8 Appendix

To help exposition, we shall prove Proposition 2 first.

Proof of Proposition 2: We introduce some auxiliary notations to facilitate the proof:

\[\epsilon \equiv \rho q - 1, \ a \equiv \frac{\phi_0}{\gamma^{i+2}}, \ b \equiv -a \left(1 + \frac{\alpha}{\rho/\beta^2}\right).\]

We first prove the theorem for the case with \(J = 1\). The system (22)-(25) can be written as

\[
\begin{bmatrix}
\hat{\pi}_{t+1} \\
\hat{c}_{t+1} \\
\hat{r}_{t+1} \\
\hat{k}_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\rho} - b\epsilon & a & 0 & 0 \\
\epsilon & 1 & 0 & 0 \\
\frac{\epsilon}{\rho^2} & 0 & 0 & 0 \\
0 & -[\beta\delta + (\beta + 1)\frac{\epsilon}{k}] & \beta(\delta + \frac{\epsilon}{k}) & 1 + \frac{\epsilon}{k}
\end{bmatrix}
\begin{bmatrix}
\hat{\pi}_t \\
\hat{c}_t \\
\hat{r}_t \\
\hat{k}_t
\end{bmatrix}.
\]

The above 4 \times 4 matrix has four eigenvalues, two of which are independent of the policy rule: a zero eigenvalue and a larger than unit eigenvalue given by \(1 + \frac{\epsilon}{k}\). The other two are policy dependent and given by the two eigenvalues of the upper left 2 \times 2 sub-matrix, which can be obtained by solving for the two roots of the following quadratic equation in \(\lambda\)

\[D(\lambda) = \lambda^2 - \left(1 + \frac{1}{\rho} - b\epsilon\right)\lambda + \left(\frac{1}{\rho} - b\epsilon - a\epsilon\right) = 0.\] (32)

It follows from \(a < 0, b > 0, \) and \(b + a > 0\) that, for any \(\epsilon < 0,\)

\[D(0) = \frac{1}{\rho} - (b + a)\epsilon > 0, \ D(1) = -a\epsilon < 0, \ D(+\infty) > 0,\]

implying that there are two real roots of (32), one is strictly between 0 and 1 and the other is larger than 1. Hence a necessary condition for determinacy is for \(\epsilon \geq 0.\) This can be verified.
by computing the two quadratic roots explicitly as

\[
\lambda_1(\epsilon) = \frac{(1 + \frac{1}{\rho} - be) + \sqrt{(1 - \frac{1}{\rho} + be)^2 + 4a\epsilon}}{2},
\]

\[
\lambda_2(\epsilon) = \frac{(1 + \frac{1}{\rho} - be) - \sqrt{(1 - \frac{1}{\rho} + be)^2 + 4a\epsilon}}{2}.
\]

Clearly, \(\lambda_1(0) = \frac{1}{\rho} > 1\) and \(\lambda_2(0) = 1\). It can also be verified that, for any \(\epsilon < 0\),

\[
\lambda_1(\epsilon) > \lambda_1(0) > 1, \quad 0 < \lambda_2(\epsilon) < \lambda_2(0) = 1.
\]

This validates our claim. On the other side, it is straightforward to show that a policy that is marginally more active than the benchmark case with \(\epsilon = 0\) suffices to ensure determinacy. To see this, compute the derivatives of the two quadratic roots with respect to \(\epsilon\) and evaluate them at \(\epsilon = 0\). We have

\[
\left. \frac{\partial \lambda_1(\epsilon)}{\partial \epsilon} \right|_{\epsilon = 0} = -\frac{b(\frac{1}{\rho} - 1) - a}{\frac{1}{\rho} - 1} < 0,
\]

\[
\left. \frac{\partial \lambda_2(\epsilon)}{\partial \epsilon} \right|_{\epsilon = 0} = -\frac{a}{\frac{1}{\rho} - 1} > 0.
\]

Of course, only the signs of the right-derivatives obtained above are of value, given the global results for the case with \(\epsilon < 0\). The signs of the right-derivatives imply that when policy becomes a little bit more active than the benchmark case with \(\epsilon = 0\), the larger-than-unit root decreases from its value of \(1/\rho\) (at \(\epsilon = 0\)), but remains larger than 1 as long as policy is not too much more active, while the unit root (at \(\epsilon = 0\)) rises above 1, as the roots are continuous functions of \(\epsilon\). Thus such a policy would lead to determinacy.

To generalize this local result to the global one characterized by the theorem, we first note that \(D(1) = -a\epsilon \geq 0\) for any \(\epsilon \geq 0\). This combined with the above observation implies that (32) can never have two real roots with one larger than 1 and the other smaller than
Therefore, determinacy obtains if and only if (32) has two real roots with both larger than 1, or two real roots with both smaller than −1, or a pair of complex roots the module of which is larger than 1. We proceed next to characterize the range for ε under which one of the three mutually exclusive possibilities is realized.

We first note that, for either of the first two possibilities, it is necessary that $D(−1) > 0$, which is true if and only if

$$\epsilon < \frac{1}{\rho} + 1 \frac{a}{b + \frac{a}{2}}. \quad (33)$$

We next note that $\lambda^* = \frac{1 + \frac{1}{\rho} - b \epsilon}{2}$ is the value of $\lambda$ that minimizes $D(\lambda)$. Clearly, $\lambda^* > 1$ if and only if

$$\epsilon < \frac{1}{\rho} - 1 \frac{b}{b}, \quad (34)$$

which is a necessary condition for both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ to be real and larger than 1, while $\lambda^* < −1$ if and only if

$$\epsilon > \frac{1}{\rho} + 3 \frac{b}{b}, \quad (35)$$

which is a necessary condition for both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ to be real and smaller than −1.

Third, we note that the term under the square-root operator in the expressions for $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ can be viewed as a quadratic function of $\epsilon$:

$$\Delta(\epsilon) = b^2 \epsilon^2 - \left[ 2 \left( \frac{1}{\rho} - 1 \right) b - 4a \right] \epsilon + \left( \frac{1}{\rho} - 1 \right)^2, \quad (36)$$

and that $\Delta(\epsilon) = 0$ always has two distinguished and strictly positive real roots,

$$\epsilon_1 = \left[ \sqrt{\left( \frac{1}{\rho} - 1 \right) b - a - \sqrt{-a}} \right]^2 \frac{b}{b},$$

$$\epsilon_2 = \left[ \sqrt{\left( \frac{1}{\rho} - 1 \right) b - a + \sqrt{-a}} \right]^2 \frac{b}{b}. $$
Thus, in order for \( \lambda_1(\epsilon) \) and \( \lambda_2(\epsilon) \) to be two real numbers with absolute values large than 1, it is necessary that
\[
0 < \epsilon \leq \epsilon_1 \quad \text{or} \quad \epsilon \geq \epsilon_2, \tag{37}
\]
while under (37), both \( \lambda_1(\epsilon) \) and \( \lambda_2(\epsilon) \) must be real. It is easy to verify that
\[
\epsilon_1 < \frac{\frac{1}{\rho} - 1}{b} < \frac{\frac{1}{\rho} + 1}{b + \frac{a}{2}}. \tag{38}
\]
This implies that, if \( 0 < \epsilon \leq \epsilon_1 \), then \( \Delta(\epsilon) \geq 0 \), \( \lambda^* > 1 \), and \( D(1) > 0 \), and thus both \( \lambda_1(\epsilon) \) and \( \lambda_2(\epsilon) \) must be real and larger than 1. On the other hand, it is easy to show that
\[
\epsilon_2 > \frac{\frac{1}{\rho} - 1}{b}. \tag{39}
\]
This implies that, if \( \epsilon \geq \epsilon_2 \), then \( \Delta(\epsilon) \geq 0 \), \( \lambda^* < 1 \), and \( D(1) > 0 \), and thus both \( \lambda_1(\epsilon) \) and \( \lambda_2(\epsilon) \) must be real but smaller than 1. We have therefore established that both \( \lambda_1(\epsilon) \) and \( \lambda_2(\epsilon) \) are real and larger than 1 if and only if \( 0 < \epsilon \leq \epsilon_1 \).

This also implies that, in order for \( \lambda_1(\epsilon) \) and \( \lambda_2(\epsilon) \) to be both real and smaller than \(-1\), it is necessary that \( \epsilon \geq \epsilon_2 \), besides that (33) and (35) have to hold. On the other side, if \( \epsilon \geq \epsilon_2 \), and (33) and (35) also hold, then \( \Delta(\epsilon) \geq 0 \), \( \lambda^* < -1 \), and \( D(-1) > 0 \), and thus both \( \lambda_1(\epsilon) \) and \( \lambda_2(\epsilon) \) must be real and smaller than \(-1\). We have therefore established that both \( \lambda_1(\epsilon) \) and \( \lambda_2(\epsilon) \) are real and smaller than \(-1\) if and only if \( \epsilon \geq \epsilon_2 \), and (33) and (35) hold.

Finally, it is clear that \( \lambda_1(\epsilon) \) and \( \lambda_2(\epsilon) \) are a pair of complex conjugates the module of which is larger than 1 if and only if
\[
\epsilon_1 < \epsilon < \epsilon_2, \tag{40}
\]
and
\[
(1 + \frac{1}{\rho} - be)^2 - (1 - \frac{1}{\rho} + b\epsilon)^2 - 4a\epsilon > 4. \tag{41}
\]
It can be shown that (41) holds if and only if
\[ \epsilon < \frac{1}{\rho} - 1. \]  

(42)

Taking the above three cases together, we have established the necessary and sufficient condition for determinacy as
\[ \epsilon \in \left( 0, \min \left\{ \epsilon_2, \frac{1}{b + a}, \frac{1}{b + \frac{a}{2}} \right\} \right) \cup \left( \max \left\{ \epsilon_2, \frac{1}{b}, \frac{1}{b + \frac{a}{2}} \right\}, \frac{1}{\rho} - 1 \right), \]

(43)

with the understanding that if the first element under the “max” operator is greater than the second one, then the second interval is left-closed. Using the steady-state real rate of return on capital \( r^* = 1/\rho - 1 + \delta \), and going through some algebra, we show that

\[ \beta < \frac{4\rho}{(1 + \rho)^2 + \rho(1 - \rho)\delta} \implies \frac{1}{b + a} < \epsilon_2 < \frac{1}{b + \frac{a}{2}} < \frac{1}{b}, \]

(44)

\[ \beta = \frac{4\rho}{(1 + \rho)^2 + \rho(1 - \rho)\delta} \implies \frac{1}{b + a} = \epsilon_2 = \frac{1}{b + \frac{a}{2}} = \frac{1}{b}, \]

(45)

\[ \beta > \frac{4\rho}{(1 + \rho)^2 + \rho(1 - \rho)\delta} \implies \frac{1}{b} < \epsilon_2 < \frac{1}{b + \frac{a}{2}} < \frac{1}{b + a}. \]

(46)

Under (44), the second interval in (43) is an empty set while the first interval reduces to
\[ \left( 0, \frac{1}{b + \frac{a}{2}} \right). \]

Under (45) or (46), the union of the two intervals in (43) collapses into a single interval
\[ \left( 0, \frac{1}{b + \frac{a}{2}} \right). \]
These, together with the steady-state employment,

$$n^* = \frac{\beta(\frac{1}{\rho} - 1 + \delta)}{\frac{\rho}{1+\phi} \left(\frac{1}{\rho} - 1 + \delta\right) - \alpha \delta},$$

completes the proof of the theorem for the case with $J = 1$.

We turn now to proving the theorem for the case with $J = 0$. The system (22)-(25) can be written as

$$\begin{bmatrix}
\hat{\pi}_{t+1} \\
\hat{c}_{t+1} \\
\hat{r}_{t+1} \\
\hat{k}_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\rho} & \frac{a}{\rho} & \frac{aa}{\beta \rho} & 0 \\
\frac{\epsilon}{\rho} & \frac{ae}{\rho} + 1 & \frac{aae}{\beta \rho} & 0 \\
\frac{e}{\rho^2 \epsilon^2} & \frac{ae}{\rho^2 \epsilon^2} & \frac{aae}{\beta \rho^2 \epsilon^2} & 0 \\
0 & -\left[\beta \delta + (\beta + 1) \frac{c^*}{k^*}\right] & \beta (\delta + \frac{c^*}{k^*}) & 1 + \frac{c^*}{k^*}
\end{bmatrix}
\begin{bmatrix}
\hat{\pi}_{t-1} \\
\hat{c}_{t} \\
\hat{r}_{t} \\
\hat{k}_{t}
\end{bmatrix}.$$

Through some algebra, it can be shown that the four eigenvalues of the above $4 \times 4$ matrix can be obtained by solving for the four roots of the following fourth-order polynomial equation in $\lambda$

$$\lambda \left(\lambda - 1 - \frac{c^*}{k^*}\right) F(\lambda) = 0,$$

where $F(\lambda)$ is a quadratic equation in $\lambda$ given by

$$F(\lambda) = \lambda^2 - \left( 1 + \frac{1}{\rho} - \frac{b}{\rho} \epsilon \right) \lambda + \left( \frac{1}{\rho} - \frac{b}{\rho} \epsilon - \frac{a}{\rho} \epsilon \right).$$

(47)

Thus, there are two policy-independent eigenvalues, a zero eigenvalue and a larger than unit eigenvalue given by $1 + \frac{c^*}{k^*}$, which are the same as in the case with $J = 1$. The rest two eigenvalues are policy dependent and can be obtained by solving for the two roots of the quadratic equation $F(\lambda) = 0$. Notice the similarity between the two functions $F(\cdot)$ and $D(\cdot)$: if we correspond $b/\rho$ and $a/\rho$ in $F(\cdot)$ to $b$ and $a$ in $D(\cdot)$, respectively, then the two functions become identical. The rest of the proof is similar. This completes the proof of the theorem. Q.E.D.
Proof of Proposition 1: We again introduce some auxiliary notations to facilitate the proof:

\[ \epsilon \equiv \rho q - 1, \ b \equiv \frac{-\phi n^*}{\gamma \pi^2 J^2}, \ a \equiv -b, \]

where the steady-state employment level in this case is given by

\[ n^* = \frac{\beta (1 + \phi)}{\phi}. \]

For the case with \( J = 1 \), equations (17) and (18) can be written as

\[
\begin{bmatrix}
\hat{\pi}_{t+1} \\
\hat{c}_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\rho} - b \epsilon & a \\
\epsilon & 1
\end{bmatrix}
\begin{bmatrix}
\hat{\pi}_t \\
\hat{c}_t
\end{bmatrix},
\]

while for the case with \( J = 0 \), they can be written as

\[
\begin{bmatrix}
\hat{\pi}_t \\
\hat{c}_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\rho} & a \\
\epsilon & \frac{a \phi}{\rho} + 1
\end{bmatrix}
\begin{bmatrix}
\hat{\pi}_{t-1} \\
\hat{c}_t
\end{bmatrix}.
\]

The corresponding characteristic functions of the two systems are identical in form to the functions \( D(\cdot) \) and \( F(\cdot) \) presented in the proof of Theorem 1, with an identical definition for \( \epsilon \), modified definitions for \( b \) and \( a \), and an additional relationship \( b + a = 0 \) holding here. Following the same procedure as set out above, it is straightforward to show that determinacy obtains if and only if

\[ \rho q - 1 \in \left( 0, \frac{\rho + 1}{b + \frac{a}{2}} \right). \]

Substituting in \( b, a, \) and the steady-state employment \( n^* \) at the beginning of this proof gives rise to the result in the theorem. Q.E.D.
References


Notes

In a recent analysis, King and Wolman (2004) demonstrate how discretionary monetary policy can lead to multiple equilibria and sunspot fluctuations by generating dynamic complementarity between forward-looking private agents and a discretionary monetary authority.


This approach of modeling nominal price adjustment cost has been used in both the continuous-time and the discrete-time literature. For continuous-time examples, see, besides Dupor (2001), Benhabib, Schmitt-Grohé, and Uribe (2001a&b), among others. For discrete-time examples, see, for example, Rotemberg (1982), Hairault and Porter (1993), Ireland (2000), and Kim (2000). Our choice of using this approach here allows for dealing with two price-adjustment timings often used in the literature in a unified framework in a transparent way. See Section 2 below for detail. In their recent work, John and Wolman (1999, 2004) investigate the issue of equilibrium existence and determinacy in a menu cost framework, which generalizes Dotsey, King, and Wolman’s (1999) setting by allowing for a non-degenerate distribution of fixed costs of price adjustment, which is in turn a general equilibrium version of the menu cost models of Caplin and Spulber (1987), Caplin and Leahy (1991, 1997), and Danziger (1999).

This distinction between jump prices and predetermined prices is valid regardless of whether there is a price adjustment cost ($\gamma > 0$) or there is no price adjustment cost ($\gamma = 0$). In fact, the idea of modeling prices as preset in models of monopolistic competition in discrete time goes back at least to the paper by Svensson (1986), which does not feature any price adjustment cost. In the continuous-time literature, this idea can be traced back to an even earlier data to Dornbush’s (1976) model with no price adjustment cost. Interestingly, later papers in continuous-time setups that do model price adjustment costs almost entirely follow this convention about the timing of price-setting [see, besides Dupor (2001), Benhabib, et al. (2001a&b) and Kimball (1995), among others]. In contrast, later papers in discrete-time setups that do model price adjustment costs mostly choose to use the alternative timing convention of modeling prices as jump variables [e.g., Rotemberg (1982), Hairault and Porter (1993), Ireland (2000), and Kim (2000)]. Although the discrete-time setup of Dupor (2003) models prices as preset, it does not feature any price adjustment cost. To our knowledge, the present paper provides the first discrete-time model with a price adjustment cost that features prices as predetermined variables. It in fact models the two timing conventions.
in a unified framework.

We adopt here the convention of end-of-period real money balances in the utility function. Were we to assume beginning-of-period real money balances in the utility function, the result obtained herein for the case with \( J = 1 \) would hold in its exact form, but it would be quite easy to get real indeterminacy for the case with \( J = 0 \) under the assumption of a Ricardian fiscal policy, since in this case the initial-period real money balances might not be determinant even when the entire paths for consumption and the expected inflation rate were.

The specification of the flow budget constraint (4) implies that nominal bond carried from period \( t-1 \), \( B_{t-1} \), matures at the beginning of period \( t \), so that \( B_{t-1}R_{t-1} \) becomes available in period \( t \). An alternative specification is that \( \frac{M_{t+1}+B_{t+1}}{P_{t+1}}/\bar{r} = \frac{M_{t}+B_{t}}{P_{t}} + \cdots [e.g., \text{Ljungqvist and Sargent (2000)}] \). These two specifications lead to identical first-order conditions for bonds.

It could be assumed alternatively that firms directly own capital in a fixed stock. Since there is no relative price distortion in a symmetric equilibrium in our present setting, it does not matter whether it is assumed that each firm has the same amount of capital never being relocated, or it is assumed that capital in a fixed stock on aggregate can be relocated among firms. Either way, \( k_{t}, \tilde{k}_{t}, \) and \( r_{t} \) would drop all together out of the budget constraint (4). The result to be presented below for the case with exogenous capital supply would hold in its exact form under these alternative assumptions.

Given the assumption of a Ricardian fiscal policy, neither the government’s public budget constraint nor the household’s private budget constraint is of any use for our equilibrium determinacy analysis.

In deriving (22), we have used a equilibrium condition, \( \beta r_{t} k_{t} = \alpha c_{t} n_{t} \), which is implied by (7), (10), (11), as well as the capital demand equation (12). We have also used this condition in writing (25).

Under an alternative, non-linear policy rule, such as \( R_{t} = R^{*}(\pi_{t}/\pi^{*})^{q} \), Propositions 1 and 2 continue to hold with the only modification being \( \rho q - 1 \) replaced with \( q - 1 \). The linear specification adopted in the text is in the spirit of Leeper (1991), who considers linear backward-looking interest rate policy in a discrete-time flexible price model.

One can show that \( B_{1} \) is larger than \( \bar{B}_{2} \) if and only if the quadratic concave function

\[
f(\beta) \equiv \phi(1 + \rho - \rho\delta)\beta^{2} + [(1 + \phi)\rho\delta - 2\phi]\beta + [\phi(1 - \rho) - \rho\delta] > 0.
\]

It can be verified that equation \( f(\beta) = 0 \) has two real distinct roots, equal to 1 and \( \beta^{**} \), respectively.

Note that, with \( J = 1 \), prices are preset by only one period. In the literature, the case that prices are set multi-periods in advance has also been considered. We conjecture that Propositions 1 and 2 would continue to hold for \( J > 1 \) and determinacy could be even more likely to occur if prices are set more than one period
in advance than if prices are set just one period in advance.

13 Through conducting a series of numerical experiments in a Calvo variable-capital model with moderate to large capital adjustment costs (that ensure determinacy), Woodford (2003, Chapter 5) finds that the predictions of constant-capital and variable-capital models on output and inflation dynamics in response to the same kind of monetary policy shock can be very similar.