Money and Capital

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Abstract

Recent advances in monetary theory incorporate some decentralized and some centralized trade. These models have an essential role for money and also allow one to easily add key ingredients from more standard macro models. However, existing papers consider only cases that dichotomize: allocations in centralized and decentralized markets are independent, which implies monetary policy has no effect on consumption, investment, employment, or output in the centralized market. We analyze natural generalizations of the model without this special property, and hence with more interesting positive and normative policy implications. We also compare different mechanisms for monetary exchange, including bargaining and competitive markets. Finally, we calibrate two versions of the baseline model using US data and compute the welfare cost of inflation. Since our model does not jump from one steady state to another immediately following a change in inflation, we are able to consider the transition path after the policy change. Our results indicate that the welfare loss of a 10% inflation is between 1% and 6%, depending on the exact specification of the model, which determines the sources and extent of the inefficiencies in the model.
1 Introduction

We believe that much progress has been made over the last 15 years or so in modeling explicitly the microfoundations of monetary exchange. There is now a large literature analyzing models that go beyond previously prominent reduced-form approaches, such as imposing a cash-in-advance constraint, which says people simply “have to” use money to acquire certain goods, or sticking money into preferences or technology, which says people are “happier or more efficient” when they use money. A representative paper in the microfoundations literature provides details about the underlying environment – preferences (over consumption goods, not assets), technology, the pattern of meetings, information, and so on – that give rise to outcomes where agents may choose endogenously to use certain objects as media of exchange, and attempts to derive conditions under which certain institutions, like monetary exchange per se or certain monetary policies, lead to higher output and welfare. Modeling explicitly the frictions in a model that can make money essential seems like progress.

It is still the case, however, that many mainstream macroeconomists continue to use the reduced-form approach. This was clearly understandable in the early days of the microfoundations literature, for a variety of reasons – not least of which was that papers in this literature needed (or at least used) some very strong assumptions about things like the amount of money and goods agents were allowed to inventory, and also because they were so focused on the process of exchange they abstracted from many of the ingredients that more standard macro models routinely incorporate, like physical capital, labor markets, competitive firms, trends or shocks in productivity, etc. These features looked not only unconventional and perhaps aesthetically unpleasing to some economists, but more importantly they seemed to preclude analyses of many macroeconomic issues, including monetary policy as it is usually conceived.¹

¹As Azariadis (1993) describes the situation, “Capturing the transactions motive for holding money balances in a compact and logically appealing manner has turned out to be an enormously complicated task. Logically coherent models such as those proposed by Diamond (1982) and Kiyotaki and Wright (1989) tend to be so removed from neoclassical growth theory as to seriously hinder the job of integrating rigorous monetary theory with the rest of macroeconomics.” And as Kiyotaki and Moore (2001) more recently put it, “The matching models are without doubt ingenious and beautiful. But it is quite hard to integrate them with...
More recent work in monetary theory has gone some way towards reducing the distance between monetary models with microfoundations and mainstream macro. Examples include the models in Shi (1997) and Lagos and Wright (2004) that do away with the artificial restrictions on inventories in the earlier models, with a minimum loss (perhaps a gain) in tractability. Some details in these two models differ a lot – in particular, Shi assumes that the fundamental decision-making unit is a family with a continuum of members that provide intrahousehold insurance against the luck of the trading process, which by the law of large numbers implies the useful result that every household of the same type starts each trading round with the same real balances, while Lagos and Wright assume individuals have periodic access to centralized markets, which by the assumption of quasi-linear utility delivers the same result. But either approach allows us to much more easily analyze standard questions concerning, say, optimal monetary policy and the welfare cost of inflation.

Still, the base-line models in Shi (1997) and Lagos and Wright (2004) do not look much like mainstream macro, as represented by, e.g., the neoclassical growth model and its many applications to business cycles, public finance, development, and so on. One reason is that those models use a very different price-determination mechanism: since the literature on the microfoundations of money has long been based on the notion that bilateral (or at least relatively small group) trade is a key element contributing to the essentiality of a medium of exchange, rather than competitive Walrasian pricing, this literature adopted one of the mechanisms commonly used in search-theory, usually bargaining or price posting. Another reason is that those models are still missing some of the staple ingredients in standard macro models, including labor markets, capital investment, etc. So while these newer models do allow us to address some more conventional issues, they are still pretty far removed from the mainstream, and hence most practitioners continue to ply the reduced-form approach.

The goal of this project is to continue the integration of monetary theory with mainstream macro, in two ways. First, following up on a line in Rocheteau and Wright (2005), we explore the implications of using competitive pricing rather than, say, bargaining in the Lagos-Wright model, not only in the centralized market but in all markets. This allows one to disentangle the rest of macroeconomic theory – not least because they jettison the basic tool of our trade, competitive markets.”
which results come from explicitly incorporating frictions into the physical environment (e.g. from assumptions on specialization, information, etc.) and which come from imposing a particular non-competitive price-determination mechanism. Moreover, it turns out that using competitive pricing dramatically simplifies the workings of the model, and this allows us to pursue our second line – which is that given the basic Lagos-Wright structure, one can without much difficulty add firms, labor, and capital markets, basically integrating a prototypical monetary model with the neoclassical growth model.\footnote{It is also possible to add capital to the basic Shi model, as in Shi (1999) or Faig (2001), e.g., but it seems to us slightly easier and perhaps more natural to do so in the Lagos-Wright version because the centralized markets are already up and running.}

This second line was also pursued in Aruoba and Wright (2003), but the results there are quite special because the way that model was specified implies a very strong \textit{dichotomy}: one can solve independently for the allocations in the centralized and decentralized markets. This dichotomy result is problematic for several reasons. First, in some sense it means that the model has really not integrated monetary theory and standard macro at all – at best, it shows that they may under certain assumptions coexist without getting in each other’s way. Second, it has stark policy conclusions: changing monetary policy affects prices and quantities in the decentralized market, but has no impact on any variable in the centralized market. In particular, aggregate employment and investment are independent of money. We show here that the dichotomy is \textit{not} general: small and natural changes in the specification lead to versions of the model with rich feedback between the centralized and decentralized markets, and hence where monetary policy has interesting implications for aggregate consumption, employment and investment.

The rest of the paper is organized as follows. In Section 2 we describe the basic model and derive the equilibrium under two different pricing structures: bilateral bargaining and competitive pricing. Optimal monetary policy is discussed as is the impact of changes in the money growth rate on consumption, investment and output. Section 3 extends the basic model by introducing market specific capital and by changing the production technology of capital. Section 4 presents our numerical results where we calibrate two versions of our model and compute welfare cost of inflation. Finally, Section 4 concludes.
2 The Basic Model

The environment is similar in spirit to the framework introduced in Lagos and Wright (2004) – hereafter denoted LW. There is a $[0,1]$ continuum of infinite-lived agents. Time is discrete, and there are two rounds of trade within each time period. The trading sequence is as follows. First, agents trade in frictionless markets, followed by a round of trading in markets with various degrees of frictions, depending on the version of the model. One friction that is present in all versions is a double coincidence problem, generated here by taste and technology shocks. Another such friction is that agents are assumed to be anonymous in day markets, which precludes standard credit arrangements, because they cannot be enforced (Kocherlakota 1998; Wallace 2001). These two frictions make money essential. Additionally, while the frictionless market is always perfectly competitive, we will consider two alternative mechanisms for the market with trading frictions: competitive price taking, and bilateral bargaining. For ease of discussion we will refer to the frictionless market as the centralized market (CM) and the market with frictions as the decentralized market (DM).

In the CM, goods can be either consumed or invested as capital, and productive capital and labor services are rented to firms in competitive markets. For trade in the DM labor is not traded because the technology used by firms in the CM does not operate; however, agents’ own labor effort $e$ may be used as an input into an individual technology in the DM. In the base model capital is also not traded in the DM (but it is in one extension considered below). The assumption is that once put in place capital cannot be physically moved to the location where the DM meets. Although capital is not physically present, agents individual technologies for producing during the DM still depend in general on $k$.\footnote{As an example of capital that enters the production function even though it is physically not present and hence not tradable at a given location, think about logging on to your computer from a remote site. The only reason for making capital immobile here is to preclude it from serving as a medium of exchange in the day market; an even simpler alternative would be to interpret $k$ as human capital, but this would obviously change the empirical implications. See Waller (2004) and Lagos and Rocheteau (2002) for models in which capital can be used as money.} We write $q = f(k, e)$ for the individual technology during the DM, and $Q = F(K, H)$ for the production function operated by firms in the CM.
To generate a double coincidence problem we adopt the following specification for tastes and technology during the DM: for each agent, with probability $\sigma$ he wants to consume but cannot produce, with probability $\sigma$ he can produce but does not want to consume, and with probability $1-2\sigma$ he can neither produce nor consume. This is equivalent for many purposes to the standard specification in the search literature of random bilateral matching, where there is a probability $\sigma$ of wanting to consume a good produced by a random partner. We frame things here in terms of random tastes and technology rather than random matching simply because it helps some of the discussion to follow, especially the comparison across the different pricing mechanisms. In any case, due to the double coincidence problem and anonymity, money is essential.\footnote{We mean essential in the technical sense, that (desirable) allocations can be achieved with money that cannot be achieved without money, subject to the relevant resource and incentive feasibility conditions (again see Kocherlakota 1998 or Wallace 2001).}

The supply of money is $M$ and changes according to $M_{t+1} = (1 + \tau)M$, where we use a subscript $+1$ to denote next period.

Instantaneous utility in the CM is $U(x) - Ah$ where $x$ is consumption, $h$ is labor hours and $A$ is a constant. Utility during the DM is random: with probability $\sigma$ an agent wants to consume and has utility $u(q)$ where $q$ is consumption; with probability $\sigma$ an agent is able to produce and has utility $-\eta(e)$ where $e$ is labor effort; and with probability $1-2\sigma$ utility is 0. We assume that $U(x)$, $u(q)$, and $\eta(e)$ have the usual properties. Linearity in $h$ is not important, in principle, but it does generate a huge gain in tractability: as in LW, it allows us to derive nice analytical results.\footnote{Rogerson (1988) shows that having utility linear in $h$ is equivalent having general preferences, indivisible labor, and employment lotteries; the same is true here.} Separability across $(x, q)$ facilitates the presentation somewhat, but is not otherwise important, as we show in the Appendix. For generality, we discount between all trading rounds where $\beta_1$ is the discount rate from CM to DM and $\beta_2$ is the discount rate from DM to CM. Thus the discount factor across time periods is $\beta = \beta_1\beta_2$.

In the analysis below it is convenient to write the agent’s disutility of effort as the utility cost of producing goods using capital. Let $c(q, k)$ denote the cost in terms of utility from producing $q$ units of output using $k$ units of capital. The cost function is obtained as follows: for a given $k$, solve $q = f(e, k)$ for $e = \psi(q, k)$ and let $c(q, k) = \eta[\psi(q, k)]$. Notice $c_q > 0$, $c_k < 0$, $c_{qq} > 0$, and $c_{kk} > 0$ under the usual monotonicity and convexity assumptions on
terms. The government also has real expenditures in the CM given by...

denote holds in the case where ... functions of agents entering the CM and DM, respectively, with money and capital...

money balances let

This is merely for calibration purposes – we are not interested in studying what the optimal fiscal policy is in our framework. To control the non-stationarity in the DM, all nominal variables will be scaled by the aggregate money stock. Thus, define \( p = p'/M \). Letting \( m' \) denote an individual’s money balances let \( m = m'/M \).

We analyze the model by first considering the CM and then the DM. In the CM, if \( r \) is the real rental rate on capital and \( w \) the real wage, profit maximization implies \( r = F_K(K, H) \) and \( w = F_H(K, H) \), and constant returns implies equilibrium profits are zero. Normalize the price of the capital/consumption good to 1. Let \( W(m, k) \) and \( V(m, k) \) denote the value functions of agents entering the CM and DM, respectively, with money and capital \((m, k)\).

\[ G = t_k w H + t_k r K - \delta t_k K + t_x X + \sigma t_d \frac{M}{p'} + T + \frac{\tau M}{p'} \] (1)

where \( p' \) is the nominal price of goods in the CM, \( \tau M \) is the seniorage revenue from money creation and \( \delta \) is the depreciation rate. Note that \( \delta t_k K \) is the tax rebate to the household that returns the capital tax on the depreciated part of capital. So, effectively the government taxes only the undepreciated part of capital with tax revenue being \( t_k (r - \delta) K \). Although the government has access to lump sum taxes, we assume it still uses distortionary taxes.

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6Given \( q = f(k, e) \), where \( f_e, f_k > 0 \), \( f_{ee}, f_{kk} < 0 \) with \( k \) normal. Saying \( k \) is normal means that in the problem min \( w e + r k \) s.t. \( f(k, e) \geq q \), the solution satisfies \( \partial k / \partial q = (f_{ee} - f_{ek}) > 0 \). We then have \( e = \psi(q, k), \partial e / \partial q = \psi_k = 1/f_e > 0 \) and \( \partial e / \partial k = \psi_k = -f_k/f_e < 0 \). Also, \( \psi_{eq} = -f_{ee}/f^2_e > 0 \), \( \psi_{kk} = -(f^2_e f_{kk} - 2f_e f_k f_{ke} + f^2_k f_{ee})/f^3_e > 0 \), and \( \psi_{kk} = -[(f_{ek}(k, e)f_{ee}(k, e) - f_{ee}(k, e)f_{ek}(k, e))]/f_{ee}(k, e)^3 \).

Hence, \( c_q = \eta'/f_e > 0 \), \( c_k = -\eta f_k/f_e < 0 \), \( c_{eq} = \left[ \eta''(\eta')^2 f_e - \eta' f_{ee} \right]/f^3_e > 0 \), \( c_{kk} = -\left[ \eta'' f_{kk} - 2f_{ek} f_{ke} + f^2_k f_{ee} - f_{ee} f^2_k \eta'' \right]/f^3_e > 0 \), and \( c_{qq} = -\left[ \eta'' f_{ee} - \eta' f_{ke} \right]/f^3_e > 0 \).
Then the problem of an agent in the CM is

\[
W(m, k) = \max_{x, h, m, k+1} U(x) - Ah + \beta_1 V(m+1, k+1)
\]

\[
(1 + t_x) x + k_{+1} = w (1 - t_h) h + [1 + (r - \delta) (1 - t_k)] k - T
\]

\[
+ m - (1 + \tau) m_{+1}
\]

where \(\tau\) is the depreciation rate, \((m+1, k+1)\) is the money and capital taken out of the market.

Eliminating \(h\) using the budget equation, we have

\[
W(m) = \frac{A}{w (1 - t_h)} \left\{ \frac{m}{p} + 1 + (r - \delta) (1 - t_k) k + T \right\}
\]

\[
+ \max_{x, m, k+1} \left\{ U(x) - \frac{A}{w (1 - t_h)} \left[ \frac{(1 + \tau) m_{+1}}{p} + (1 + t_x) x + k_{+1} \right] + \beta_1 V(m+1, k+1) \right\}.
\]

The first order conditions for the choice variables are\(^7\)

\[
x : U'(x) = \frac{A (1 + t_x)}{w (1 - t_h)}
\]

\[
m_{+1} : \frac{A (1 + \tau)}{pw (1 - t_h)} = \beta_1 V_m(m+1, k+1)
\]

\[
k_{+1} : \frac{A}{w (1 - t_h)} = \beta_1 V_k(m+1, k+1).
\]

A key result is that, given prices, \(W\) is linear in \(m\) and \(k\),

\[
W_m(m, k) = \frac{A}{pw (1 - t_h)} \quad (3)
\]

\[
W_k(m, k) = \frac{A [1 + (r - \delta) (1 - t_k)]}{w (1 - t_h)}. \quad (4)
\]

Moreover, it should be clear from the above that the choice of \((m+1, k+1)\) is independent of \((m, k)\), and this makes the distribution of money and capital holdings degenerate in equilibrium. Intuitively, the linearity of utility in \(h\) in an LW environment eliminates wealth

\(^7\)The second order conditions are complicated, and generally ambiguous, since they involve second derivatives of \(V\) which can involve third derivatives of \(u\) and \(c\), at least under the bargaining mechanism. Following the methods in LW, one can show that \(V\) is concave if the bargaining power parameter \(\theta\) is close to 1, or if we impose additional conditions on preferences and technology (in LW \(c\) was normalized to be linear and \(u'\) was assumed log concave). We avoid these details and simply assume \(V\) is concave in the bargaining model, but again this is always true for \(\theta\) close to 1.
effects, and this makes all agents choose the same \((m+1, k+1)\) regardless of \((m, k)\). While models with nondegenerate distributions are worth studying, for some questions it seems reasonable to abstract from distributional issues and study representative agent models first. This is what we get from the linearity of utility in \(h\).

We now proceed to the DM market. The value function is

\[
V(m, k) = \sigma V_b(m, k) + \sigma V_s(m, k) + (1 - 2\sigma)\beta_2W(m, k) \tag{5}
\]

where

\[
V_b(m, k) = u(q_b) + \beta_2 W(m - d_b, k)
\]
\[
V_s(m, k) = -c(q_s, k) + \beta_2 W[m + (1 - t_d) d_s, k]
\]

are the value functions when one is a buyer and seller, respectively, and \(q_b\) and \(d_b\) are the amounts of output and money agents expect to exchange when buying, and \(q_s\) and \(d_s\) are the amounts when selling, to be determined below. The seller must pay \(t_d d_s\) in sales tax to the government thereby leaving him with \((1 - t_d) d_s\) when entering the CM. Using the result in (3) that \(W_m = A/pw (1 - t_h)\), we have

\[
V(m, k) = \sigma \left[ u(q_b) - d_b \frac{\beta_2 A}{pw (1 - t_h)} - c(q_s, k) + d_s \frac{\beta_2 A}{pw (1 - t_h)} \right] + \beta_2 W(m, k).
\]

Differentiating with respect to \(m\) and \(k\) yields the envelope conditions

\[
V_m(m, k) = \sigma \left[ u'q_b \frac{\beta_2 A}{pw (1 - t_h)} \frac{\partial d_b}{\partial m} + \sigma \left[ -c_q \frac{\partial q_s}{\partial m} + \frac{\beta_2 A}{pw (1 - t_h)} \frac{\partial d_s}{\partial m} \right] \right] + \beta_2 A \frac{\partial d_s}{\partial m} \tag{6}
\]
\[
V_k(m, k) = \sigma \left[ u'q_b \frac{\beta_2 A [1 + (r - \delta) (1 - t_k)]}{w (1 - t_h)} \frac{\partial d_b}{\partial k} \right] + \sigma \left[ -c_q \frac{\partial q_s}{\partial k} - c_k \frac{\beta_2 A [1 + (r - \delta) (1 - t_k)]}{w (1 - t_h)} \frac{\partial d_s}{\partial k} \right] + \beta_2 A \frac{[1 + (r - \delta) (1 - t_k)]}{w (1 - t_h)} \frac{\partial d_s}{\partial k} \tag{7}
\]

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8 Actually, in addition to linearity in \(h\), we also require \(V\) strictly concave and an interior solution; see LW for technical assumptions to guarantee these results. The assumptions needed for interiority involve initial conditions: if \((m, k)\) is very disperse across people, then the rich remain rich and the poor remain poor for several periods; if we start with \((m, k)\) not too disperse, however, we converge quickly to a degenerate distribution and stay there.

9 It should be clear how exactly the same equation would emerge from a random matching model (see LW, for example).
It remains to specify how prices are determined in the day market, so that we can substitute for the derivatives in the above expressions. This will differ across the two versions of the model presented below.

Before pursuing equilibrium, however, as a benchmark we begin with the planner’s problem, unconstrained by the assumption that agents are anonymous, so that we can simply enforce whatever exchange we like without using money. The planner’s problem is described by

\[
J(K) = \max_{X,H,q,K+1} \sigma u(q) - \sigma c(q,K) + \beta_2 [U(X) - AH] + \beta J(K+1)
\]

s.t. \[X + G = F(K,H) + (1 - \delta)K - K_{+1}\]

Substituting for \(X\) and differentiating, the first order conditions are

\[
\begin{align*}
H : & \quad A = U'(X)F_H(K,H) \\
K_{+1} : & \quad \beta_2 U'(X) = \beta J'(K_{+1}) \\
q : & \quad u'(q) = c_q(q,K)
\end{align*}
\]

The envelope condition is

\[
J'(K) = \beta_2 U'(X)[F_K(K,H) + 1 - \delta] - \sigma c_k(q,K),
\]

and the Euler equation is

\[
U'(X) = \beta U'(X+1)[F_K(K+1,H+1) + 1 - \delta] - \beta_1 \sigma c_k(q+1,K+1)
\]

It is clear that the solution has \(q = q^*(K)\) where \(q^*(K)\) satisfies \(u'(q) = c_q(q,K)\). Given this, the other control variables \((K_{+1}, H, X)\) satisfy relatively standard conditions, the first equation in (9), (10), and the constraint in (8).

### 2.1 Equilibrium I: Bargaining

Here we consider a mechanism used in much recent work in monetary theory, where agents bargain bilaterally. While the results are more complicated under bargaining than the competitive mechanism presented below, bargaining is arguably a very natural solution concept in models with frictions, and also serves to highlight certain effects that the competitive
mechanism masks. Thus, here each agent with a desire to consume is matched with one who can produce. Since they - in particular, the buyers - are anonymous, trade must be quid pro quo meaning they must pay with cash. The buyer transfers $d$ dollars to the seller in exchange for $q$ units of output, where $(q,d)$ are determined via the generalized Nash solution with the bargaining power of the buyer denoted $\theta$ and threat points given by continuation values. In general, $(q,d)$ depends on the assets of buyer and seller, $(m_b,k_b)$ and $(m_s,k_s)$.

There are two obvious feasibility conditions for the exchange: $q$ cannot exceed the output of the seller, $q \leq f(e,k_s)$, and $d$ cannot exceed the money holdings of the buyer, $d \leq m_b$.

The buyer’s payoff from the trade is $u(q) + \beta_2 W(m_b-d, k_b)$ and his threat point $\beta_2 W(m_b, k_b)$. Thus, his surplus is

$$S_b = u(q) + \beta_2 W(m_b - d, k_b) - \beta_2 W(m_b, k_b)$$

$$= u(q) - \frac{\beta_2 A}{pw (1 - t_h)} d,$$

by virtue of (3). The seller’s payoff is $-c(q,k_s) + \beta_2 W [m_s + (1 - t_d) d, k_s]$ and his threat point $W(m_s, k_s)$. Thus his surplus is

$$S_s = -c(q,k_s) + \beta_2 W(m_s + (1 - t_d) d, k_s) - \beta_2 W(m_s, k_s)$$

$$= -c(q,k_s) + \frac{\beta_2 A (1 - t_d)}{pw (1 - t_h)} d.$$ 

The bargaining problem can be written

$$\max_{q,d} \theta S_b^{\theta} S_s^{1-\theta} \quad \text{s.t.} \quad d \leq m_b.$$ 

As in LW, one can show that in equilibrium with $k_s = K$ for all agents the constraint holds with equality, $d = m_b$. Also as in LW, this further implies $q \leq q^*(k_s)$ where $q^*(k_s)$ is the solution to $u'(q) = c_q(q, k_s)$, typically with strict inequality $q < q^*(k_s)$ (here the inequality is strict unless $\theta = 1$ and we follow the optimal monetary policy). To solve the bargaining problem, insert $d = m_b$ and take the first order condition with respect to $q$ to get

$$\theta S_s u'(q) = (1 - \theta) S_b c_q(q, k_s).$$

Note that while all agents have the same $(m,k)$ in equilibrium, we still need to ask what happens if a given individual deviates off the equilibrium path. Also $d$ is scaled by the aggregate money stock.
Then insert $S_b$ and $S_s$ and rearrange as $m_b = g(q, k_s)pw (1 - t_h) / \beta_2 A$, where

$$g(q, k_s) \equiv \frac{\theta c(q, k_s)u'(q) + (1 - \theta)u(q)c_q(q, k_s)}{(1 - t_d) \theta u'(q) + (1 - \theta)c_q(q, k_s)}. \quad (11)$$

Hence, $q = q(m_b, k_s)$, where the function $q(m_b, k_s)$ is given by the solution to $\beta_2 Am_b / pw (1 - t_h) = g(q, k_s)$ (the dependence on prices $w$ and $\phi$ as well as the parameter $A$ is implicit). This implies the key derivatives we need in (6) and (7) are given by $\partial q / \partial m_b = \beta_2 A / pw (1 - t_h) g_q > 0$ and $\partial q / \partial k_s = -g_k / g_q > 0$, where

$$g_q = \frac{c_q u'[(1 - t_d) \theta u' + (1 - \theta)c_q] + \theta(1 - \theta)[(1 - t_d) u - c][u'c_q - c_q u'']}{(1 - t_d) \theta u' + (1 - \theta)c_q] + [c_q(1 - \theta)u'[(1 - t_d) u - c]} > 0 \quad (12)$$

$$g_k = \frac{\theta c_k u'[(1 - t_d) \theta u' + (1 - \theta)c_k] + c_q(1 - \theta)u'[(1 - t_d) u - c]}{(1 - t_d) \theta u' + (1 - \theta)c_k] + [c_q(1 - \theta)u'[(1 - t_d) u - c]} < 0 \quad (13)$$

(we also have $\partial q / \partial m_s = \partial q / \partial k_b = 0$, $\partial d_b / \partial m_b = 1$, and $\partial d_s / \partial m_b = \partial d_b / \partial k_s = \partial d_s / \partial k_s = 0$).

Thus, if the buyer brings more cash or the seller brings more capital to a meeting, more output gets traded. Notice that in general the price is non-linear: if the buyer brings half as much money, he does not get half as much $q$. For $\theta = 1$, $g(q, k_s) = c(q, k_s)$, which makes things a lot simpler: $g_q = c_q$ and $g_k = c_k$, and so therefore $\partial q / \partial m_b = \beta_2 A / pw (1 - t_h) c_q$ and $\partial q / \partial k = -c_k / c_q$. In this case, if marginal cost $c_q$ is constant, pricing is linear: if you spend another dollar you get another unit of $q$.\footnote{We can also simplify the bargaining solution by setting $\theta = 0$, but then $m_b = 0$ and the monetary equilibrium breaks down. The reason $\theta = 1$ does not symmetrically imply $k_s = 0$ is that the same capital is used in the day and night market in this version of the model.}

Imposing symmetry across agents yields $(m, k) = (1, K)$ and the derivatives, (6) and (7) become

$$V_m(m, K) = \frac{\sigma \beta_2 A u'(q)}{pw (1 - t_h) g_q(q, K)} + \frac{(1 - \sigma) \beta_2 A}{pw (1 - t_h)}$$

$$V_k(m, K) = \frac{\beta_2 A [1 + (r - \delta)(1 - t_k)]}{w(1 - t_h)} - \sigma \gamma(q, K),$$

where $\gamma(q, K) = \frac{c_q(q, K)g_q(q, K) - c_k(q, K)g_k(q, K)}{g_q(q, K)} < 0$. Substitute these into the first order conditions for $m_{+1}$ and $k_{+1}$ in (2). Then setting $x = X$, inserting the equilibrium prices $p = \beta_2 A / w (1 - t_h) g(q, K)$, $r = F_K(K, H)$ and $w = F_H(K, H)$, we arrive at the equilibrium
conditions

\[
g(q, K) (1 + \tau) = \beta g(q+1, K_{+1}) \left[ \sigma \frac{u'(q+1)}{g_q(q+1, K_{+1})} + 1 - \sigma \right] \quad (14)
\]

\[
U'(X) = \beta U'(X_{+1}) \left\{ 1 + [F_K(K_{+1}, H_{+1}) - \delta] (1 - t_k) \right\} - \sigma \beta_1 (1 + t_x) \gamma(q_{+1}, K_{+1})
\]

The other equilibrium conditions come from the first order condition for \(x\) in (2) and the resource constraint on total output

\[
U'(X) = \frac{A (1 + t_x)}{(1 - t_h) F_H(K, H)} \quad (16)
\]

\[
X + G = F(K, H) + (1 - \delta)K - K_{+1}. \quad (17)
\]

A monetary equilibrium is defined as (positive, bounded) paths for \((q, K_{+1}, H, X)\) satisfying (14)-(17), given the initial \(K_0\). A nonmonetary equilibrium also always exists, which satisfies \(q = 0\) instead of (14), (15) with \(\gamma(\cdot) = 0\), and (16)-(17), which are simply the equilibrium conditions for the standard nonmonetary growth model (with \(h\) entering utility linearly). Returning to monetary equilibria, with \(\tau\) constant it makes sense to focus on a steady state, defined as a constant solution \((q, K, H, X)\) to (14)-(17). Defining the rate of time preference \(\rho\) and the nominal interest rate \(i\) such that \(\beta = \frac{1}{1+\rho}\) and \(1+i = (1+\rho)(1+\tau)\), we can simplify the steady state conditions as

\[
1 + \frac{i}{\sigma} = \frac{u'(q)}{g_q(q, K)} \quad (18)
\]

\[
\rho = [F_K(K, H) - \delta] (1 - t_k) - \sigma (1 + t_x) \frac{\gamma(q, K)}{\beta_2 U'(X)} \quad (19)
\]

\[
U'(X) = \frac{A (1 + t_x)}{F_H(K, H) (1 - t_h)} \quad (20)
\]

\[
X + G = F(K, H) - \delta K. \quad (21)
\]

First, one simple special case of our model is the specification in Aruoba and Wright (2003), where capital does not enter the DM technology, \(c(q, K) = c(q)\). In this case

\[\text{This expression for } i \text{ satisfies the Fisher equation, which eliminates arbitrage opportunities from holding nominal versus real assets.}\]
\(g(q, K) = g(q), \gamma(q, K) = 0\), and the equilibrium conditions are

\[
g(q) (1 + \tau) = \beta g(q_{t+1}) \left[ 1 - \sigma + \frac{u'(q_{t+1})}{g'(q_{t+1})} \right]
\]

\[
U'(X) = \beta U'(X_{t+1}) \{1 + [F_K(K_{t+1}, H_{t+1}) - \delta](1 - t_k)\}
\]

\[
U'(X) = A(1 + t_x)/F_H(K, H)(1 - t_h)
\]

\[
X = F(K, H) + (1 - \delta)K - K_{t+1}.
\]

This model displays a strong dichotomy: the first equation determines the path for \(q\) and the other three determine the paths for \((K_{t+1}, H, X)\) independently. An implication of this feature is that \(M\), which enters only the first equation, affects \(q\) but not \((K_{t+1}, H, X)\); that is, investment, employment and consumption in the CM is independent of monetary policy.

Of course this does not mean policy is super neutral in Aruoba and Wright (2003): the path of \(M\) affects \(q\), and \(q\) is a real variable. For example, in steady state \(q\) satisfies

\[
1 + \frac{i}{\sigma} = \frac{u'(q)}{g'(q)}.
\]

From this it follows that \(\partial q/\partial i < 0\) as long as the steady state \(q\) is unique (which is true under certain conditions addressed in LW). Furthermore, since \(g'(q) > c'(q)\) when \(\theta < 1\), \(q\) is lower than it would be under buyer-take-all bargaining. This is because of a holdup problem on money identified in LW. Since the buyer bears the full cost of acquiring an additional unit of money but must share the surplus generated by the additional unit, he underinvests in money balances.

For \(i > 0\), \(q < q^\ast\) in any equilibrium, where \(q^\ast\) is the efficient quantity defined by \(u'(q^\ast) = c'(q^\ast)\). Hence, we maximize welfare by making \(i\) as small as is consistent with equilibrium. This turns out to be the Friedman Rule, \(i = 0\), which requires the money growth rate \(\tau^F\) to satisfy \((1 + \tau^F)(1 + \rho) = 1\) (for any \(\tau < \tau^F\) equilibrium does not exist; see LW). Hence, the optimal policy is \(\tau = \tau^F\) and it implies \(u'(q) = g'(q)\). However, \(\tau^F\) does not yield the first best outcome unless \(\theta = 1\), since in this case, \(g(q) = c(q)\) and so \(\tau = \tau^F\) implies \(u'(q) = c'(q)\). When \(\theta < 1\) the Friedman Rule corrects the dynamic wedge associated with impatient agents holding non-interest-bearing money, but cannot correct a second distortion on money demand arising from the bargaining game itself when \(\theta < 1\).\(^{13}\)

\(^{13}\)Rocheteau and Waller (2005) have shown that this result is driven by the monotonicity properties of the
The dichotomy in Aruoba and Wright is very special, and does not hold in the generalization where \( k \) enters the cost function since \( K \) and \( q \) both appear in (14) and (15), and there is no way to solve independently first for \( q \) and then the other variables. Naturally, the efficient investment decision not only takes into account the fact that \( K \) affects productivity in the CM, but also productivity in the DM. A change in the growth rate of \( M \) affects \( q \) and this in turn affects the return to \( K \). Intuitively, an increase in inflation (nominal interest rates) reduces the return to trading in the DM, which affects the value of capital in that market and hence investment. But the same capital is used in both DM and CM production, and so an increase in inflation affects productivity and hence employment and output in the CM.

However, in the case \( \theta = 1 \), notice that \( \gamma(q, K) = 0 \). This means that, although the model is not dichotomous, it is recursive: (15)-(17) can be solved for \( (x, K_{t+1}, H) \) independently of \( q \), and the solution is exactly the path from the standard (nonmonetary) model; then, given the path for capital, (14) determines the path for \( q \). In this case, anything that affects capital affects the value of money, but there is no feedback in the other direction from \( q \) to \( K \). For example, in steady state we have

\[
\frac{\partial q}{\partial K} = \frac{c_{qk}}{c_{q'u''} - u'c_{qq}} > 0
\]

(anything that increases \( K \) raises the value of money). An implication is that monetary policy affects \( q \), but not investment, employment or consumption in the CM. Intuitively, what happens when \( \theta = 1 \) is that sellers get none of the gains from trade, so they realize none of the cost savings from bringing extra capital into the DM (a holdup problem) and hence the investment decision is based solely on returns from CM production.

This holdup problem in the demand for capital is general (it does not only apply in the extreme case \( \theta = 1 \)) and will cause \( K \) to diverge from its efficient level. This represents an additional distortion over and above the usual inefficiency that arises when \( \tau > \tau^{F} \), and the holdup problem in money demand that arises when \( \theta < 1 \). Normally these problems are resolved if one sets \( \theta \) correctly (this is the insight of Hosios (1990) and others), but here it
cannot be done: \( \theta = 1 \) is required to resolve the holdup problem in the demand for money, but this is the worst possible case for the holdup problem in the demand for capital.\(^{14}\) When capital reduces the cost of producing DM goods, this should be taken into account when investing in \( K \), but whenever \( \theta > 0 \) the investor has to share the cost savings with the buyer and hence under-invests. There is obviously no way to set \( \theta \) to both 1 and 0 to eliminate the holdup problems on capital and money demand. In the next section we consider an alternative pricing mechanism that does.\(^{15}\)

### 2.2 Equilibrium II: Competitive Pricing

The idea of using competitive (Walrasian) price-taking behavior as an alternative to bargaining in search-type monetary models was explored in Rocheteau and Wright (2005). There it was assumed that agents were randomly allocated trade opportunities in the sense of access to markets but in these markets, rather than having agents bargain bilaterally, there is an auctioneer who sets prices to equate supply and demand. It is legitimate to consider this pricing mechanism and still assume anonymous traders so as to rule out credit and maintain an essential role for money.\(^{16}\) In fact, this mechanism can be reinterpreted in terms of “competitive search equilibrium” – an equilibrium concept used by others in nonmonetary search theory. In Rocheteau and Wright (2005), this mechanism actually dominates Walrasian pricing due to a “search externality” at the entry decision; since we do not have an entry decision here the allocations are the same under the two mechanisms - Walrasian pricing and competitive search - we present things in terms of the simpler story.

The value function for the DM market before the shocks are realized has the same form as in (5) except now \( V_b(m, k) \) and \( V_s(m, k) \) are different. The buyer’s problem is

\[
V_b(m, k) = \max_{q_b, d} u(q_b) + \beta_2 W(m - d, k) \\
\text{s.t. } \tilde{p} q_b = d \text{ and } d \leq m
\]

\(^{14}\)When \( \theta = 0 \), we have \( \gamma(q, K) = c_b(q, K) \), which yields the efficient investment decision, given \( q \) but also yields \( q = 0 \).

\(^{15}\)In addition to LW, see Rauch (2000), and Camera, Reed and Waller (2003) for discussions of holdup problems in monetary models.

\(^{16}\)See also Levine (19xx), Kocherlakota (2003), and Temzilides (19xx) for related models.
where $\tilde{p}$ is the nominal price of goods in the DM market scaled by the aggregate money stock. The seller’s problem is

$$V_s(m, k) = \max_{q_s} -c(q_s, k) + \beta_2 W(m + \tilde{p}q_s, k).$$

These are standard competitive demand and supply problems with $\tilde{p}$ taken parametrically. In equilibrium $q_b = q_s = q$ because we have conveniently assumed there are the same number $\sigma$ of buyers and sellers.

The buyer’s choice satisfies $u'(q) = p\beta_2 W_m(M - pq, k) = \tilde{p}\beta_2 A/pw (1 - t_h)$ if the constraint is not binding and $q = 1/\tilde{p}$ if it is, where we have inserted the equilibrium condition $m = 1$, and $W_m = A/pw (1 - t_h)$ (which we can do because the CM here is exactly the same as before). The seller’s choice satisfies $c_q(q, k) = \tilde{p}\beta_2 W_m(M + (1 - t_d) \tilde{p}q, k) = \tilde{p} (1 - t_d) \beta_2 A/pw (1 - t_h)$. If the buyer’s constraint is not binding, the equilibrium $q$ solves $(1 - t_d) u'(q) = c_q(q, k)$; if the constraint is binding, the equilibrium solves $c_q(q, k) = \beta_2 (1 - t_d) A/pqw (1 - t_h)$. It is again easy to show that the constraint will be binding in equilibrium. Not surprisingly, the tax on trade in the DM leads to a distortion since the planner’s choice is $u'(q) = c_q(q, k)$.

The next step is to differentiate (??) with respect to $m$ to get

$$V_m(m, k) = \sigma \left[ u'(q) - \frac{\beta_2 \tilde{p}A}{pw (1 - t_h)} \right] \frac{\partial q}{\partial m} + \frac{\beta_2 A}{pw (1 - t_h)}$$

$$= \sigma \frac{u'(q)}{\tilde{p}} + (1 - \sigma) \frac{\beta_2 A}{pw (1 - t_h)}$$

where we have used $\partial q/\partial m = 1/\tilde{p}$ since the buyer’s constraint is binding. Similarly,

$$V_k(m, k) = -\sigma c_k(q, k) + \beta_2 A \frac{1 + (r - \delta) (1 - t_k)}{w (1 - t_h)}.$$

Inserting $V_m$ and $V_k$ into the first-order conditions in (2) and rearranging yields the analogs to (14)-(15) for this model:\textsuperscript{17}

$$(1 + \tau) c_q(q, K) q = \beta c_q(q_{+1}, K_{+1}) q_{+1} \left[ \frac{\sigma (1 - t_d) u'(q_{+1})}{c_q(q_{+1}, K_{+1})} + 1 - \sigma \right]$$

$$U'(X) = \beta U'(X_{+1}) \{1 + [F_K(K_{+1}, H_{+1}) - \delta] (1 - t_k)\} - \sigma \gamma_1 (1 + \tau_x) c_k(q_{+1}, K_{+1})$$

\textsuperscript{17}In this model it is easy to verify the second order conditions must hold; the difference is that now pricing is linear so we do not need any conditions on third derivatives the way we do in the bargaining model with $\theta < 1$. 

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The other equilibrium conditions are the same, and we repeat them here for convenience:

\[ U'(X) = A(1 + t_x) / F_H(K, H) (1 - t_h) \]  
\[ X + G + K_{+1} = F(K, H) + (1 - \delta)K. \]  

Monetary equilibrium is now defined by (positive, bounded) paths for \((q, X, K_{+1}, H)\) satisfying (22)-(25) given the initial \(K_0\). The difference between the bargaining and competitive pricing models is in the difference between (14)-(15) and (22)-(23). They differ because \(g(q, K) \neq c_q(q, K)q\) and \(g_q(q, K) \neq c_q(q, K)\) in the first pair of equations and because \(\gamma(q, K) \neq c_k(q, K)\) in the second pair. Suppose we concentrate for now on steady states.\(^{18}\) Then in the competitive pricing model we have

\[ 1 + \frac{i}{\sigma} = \frac{(1 - t_d) u'(q)}{c_q(q, K)} \]  
\[ \rho = [F_K(K, H) - \delta] (1 - t_k) - \sigma (1 + t_x) \frac{c_k(q, K)}{\beta_2 U'(X)} \]  

while in the bargaining model we have

\[ 1 + \frac{i}{\sigma} = \frac{u'(q)}{g_q(q, K)} \]  
\[ \rho = [F_K(K, H) - \delta] (1 - t_k) - \sigma (1 + t_x) \frac{\gamma(q_{+1}, K_{+1})}{\beta_2 U'(X)}. \]

Competitive pricing significantly alters the model: (26) and (28) are the same iff \(\theta = 1\); and (27) and (29) are the same iff \(\theta = 0\). In this way, competitive pricing is able to eradicates the bargaining distortion affecting money demand and the holdup problem on investment decisions. The idea is that in the competitive model agents take the price as given; their individual choices have no effect on the terms of trade. Since both distortions are eliminated under Walrasian pricing, the only distortion remaining is the dynamic wedge associated with discounting, and under the Friedman rule \(i = 0\) with \(t_d = 0\) we get the first best.

With \(t_d = t_k = 0\), comparing (10) with (23), the investment decision is not distorted in the competitive monetary equilibrium except to the extent that \(q\) is wrong. The first order

\(^{18}\)In steady state the difference between \(g(q, K)\) and \(c_q(q, K)q\) across the two models is irrelevant. This is not so out of steady state. For example, even if \(\theta = 1\), so that \(g(q, K) = c(q, K)\), (14) and (22) differ as long as \(c(q, K) \neq c_q(q, K)q\) – i.e. as long as \(c\) is nonlinear in \(q\).
condition for \( q \) in (9) says that the efficient solution is \( q = q^*(k) \). From (22), for this to be true in the competitive monetary equilibrium we require

\[
1 + i = \frac{c_q(q_{+1}, K_{+1})q_{+1}}{c_q(q, K)q};
\]
in particular, in a steady state we require the Friedman rule. Hence, the steady state of the competitive monetary equilibrium achieves the first best outcome at \( i = 0 \): the value of money is given by \( q = q^*(k) \), and then investment, employment and consumption are all efficient. By comparison, in the bargaining model, even at \( i = 0 \), \( q \) was too low due to the bargaining distortion affecting money demand that occurs whenever \( \theta < 1 \), and \( k \) is too low due to the holdup problem in investment that occurs whenever \( \theta > 0 \).

To close this section, we mention that even though the above equations determine the aggregate variables \((q, x, H, K_{+1})\), the individual values of these variables differs across agents. First, only a measure \( \sigma \) of the population consume \( q \) and have \( m = 0 \) when they enter the night market. A group also of measure \( \sigma \) are sellers each period and enter the night market with \( m = M + (1 - t_d)M \), while a group of measure \( 1 - 2\sigma \) did not trade and enter with \( m = M \). These agents all choose the same \( x, k' \) and \( m' \), but supply different amounts of labor,

\[
h = \begin{cases} 
H - \frac{\sigma t_d}{w(1-t_h)} \frac{M}{p} + \frac{1}{w(1-t_h)} \frac{M}{p} & \text{for buyers} \\
H - \frac{\sigma t_d}{w(1-t_h)} \frac{M}{p} - \frac{(1-t_d)}{w(1-t_h)} \frac{M}{p} & \text{for sellers} \\
H - \frac{\sigma t_d}{w(1-t_h)} \frac{M}{p} & \text{otherwise}
\end{cases}
\]

where \( H \) is aggregate hours.

### 2.3 Example

To obtain more insight on how inflation affects the steady state of the economy, we construct an example using explicit functional forms. Analysis of the general model is contained in the appendix. For ease of presentation, we focus on the competitive pricing equilibrium and ignore taxes and government spending.
Consider the following functional forms:\(^19\)

\[
F(K, H) = K^\alpha H^{1-\alpha} \quad 0 < \alpha < 1
\]

\[
U(x) = \ln x
\]

\[
u(q) = \frac{q^{1-\eta}}{1-\eta} \quad 0 < \eta < 1
\]

\[
c(q, K) = q^{\varphi} K^{1-\varphi} \quad \varphi > 1.
\]

Let \( k = K/H \) denote the capital-labor ratio. Then equations (20), (21), (26) and (27) can be solved to obtain

\[
X = \frac{(1-\alpha) \cdot k^\alpha}{A} \quad (31)
\]

\[
K = \frac{(1-\alpha) \cdot k}{A(1 - \delta k^{1-\alpha})} \quad \text{with } K > 0, \frac{\partial K}{\partial k} > 0 \text{ for } k < \left( \frac{1}{\delta} \right)^{\frac{1}{1-\alpha}} \quad (32)
\]

\[
q = \left[ \frac{\sigma}{\varphi (i + \sigma)} \right]^{\frac{1}{\varphi-1+\eta}} \left[ \frac{(1-\alpha) \cdot k}{A(1 - \delta k^{1-\alpha})} \right]^{\frac{\varphi-1}{\varphi+\eta-1}} \quad (33)
\]

\[
\rho + \delta = \frac{\alpha}{k^{1-\alpha}} + \sigma \left[ \frac{\sigma}{\varphi (i + \sigma)} \right]^{\frac{\varphi}{\varphi+\eta-1}} \left[ (1-\alpha) / A \right]^{1-\mu} (\varphi - 1) (1 - \delta k^{1-\alpha})^\mu \quad \equiv \quad N(k) \quad (34)
\]

where \( \mu = \frac{\varphi \eta}{\varphi+\eta-1} < 1 \).

Equation (34) determines the solution for \( k \) which can then be used to determine the steady state values of \( x, q, K \) and \( H \). It is straightforward to show that for \( \mu \geq \alpha \), \( N(k) \) is a monotonically decreasing function in \( k \) that approaches infinity as \( k \to 0 \) and approaches zero as \( k \to +\infty \). Thus, a unique equilibrium value of \( k \) exists. For \( \sigma = 0 \), we obtain the non-stochastic steady state corresponding to Hansen’s (1985) RBC model. With \( \sigma > 0 \), capital creates additional value in production during the DM which leads agents to accumulate more capital on the margin. An increase in the money growth rate decreases \( N(k) \) for any given value of \( k \). Consequently, greater money growth raises \( i \) and reduces the steady state value of \( k \) which in turn lowers \( X, K, \) and \( q \). Furthermore, from (32), \( H = (1-\alpha) / [A (1 - \delta k^{1-\alpha})] \) which is increasing in \( k \). So agents also work less in the CM when money growth is higher.\(^{20}\)

\(^{19}\)The cost function below is obtained when \( \eta(e) = e \) and \( q = e^\Phi k^{1-\Phi} \) where \( 0 < \Phi < 1 \). As a result, \( \varphi = 1/\Phi > 1 \).

\(^{20}\)For \( \alpha > \mu \), \( N(k) \) can be U-shaped implying that multiple equilibria may exist.
The intuition for these results is the following. An increase in inflation lowers the value of money and the quantity of goods traded in the DM. Since production is lower, the marginal value of capital in the DM falls and so agents accumulate less capital in the CM. The reduction in capital reduces the real wage and so agents work less in the CM. Since the planner’s problem is replicated only under the Friedman rule, \( i = 0 \), then any \( i > 0 \) is clearly welfare reducing.

3 Alternative Specifications

3.1 Two Capital Goods

So far, the same stock of physical capital \( k \) was an input to both day and night production. However, it would also seem reasonable to assume that different types of capital are needed to produce each good. In this section we modify the baseline model to allow for two types of capital: \( k \) is used to produce goods in the CM and a new type of capital \( z \) is used to produce in the DM. Production of both capital stocks requires an investment in the CM; \( k \) and \( z \) are both traded solely in the CM and are not mobile. The two capital stocks can also depreciate at different rates, \( \delta \) for \( k \) and \( \omega \) for \( z \).

The problem in the CM is now

\[
W(m, k, z) = \max_{x, h, m_{+1}, k_{+1}, z_{+1}} U(x) - Ah + \beta_1 V(m_{+1}, k_{+1}, z_{+1})
\]

s.t. \((1 + t_x) x = (1 - t_h) wh + (1 - \omega) z + [1 + (r - \delta)(1 - t_k)] k - k_{+1} - z_{+1}\)

\[+ \left( \frac{m - (1 + \tau) m_{+1}}{p} \right)\]

Eliminating \( h \), this can be written as

\[
W(m, k, z) = A \frac{m}{w(1 - t_h)} \left\{ \frac{m}{p} + [1 + (r - \delta)(1 - t_k)] k + (1 - \omega) z \right\}
\]

\[+ \max_{x, m_{+1}, k_{+1}, z_{+1}} \frac{A}{w(1 - t_h)} \left[ (1 + t_x) x + \frac{(1 + \tau) m_{+1}}{p} + k_{+1} + z_{+1} \right]
\]

\[+ \beta_1 V(m_{+1}, k_{+1}, z_{+1}).\]
The first order conditions are

\[ x : \quad U'(x) = \frac{A(1 + t_x)}{w(1 - t_h)} \]  \hspace{1cm} (35)\]

\[ m_{+1} : \quad \frac{A(1 + t_x)(1 + \tau)}{pw(1 - t_h)} = \beta_1 V_m(m_{+1}, k_{+1}, z_{+1}) \]

\[ k_{+1} : \quad \frac{A}{w(1 - t_h)} = \beta_1 V_k(m_{+1}, k_{+1}, z_{+1}) \]

\[ z_{+1} : \quad \frac{A}{w(1 - t_h)} = \beta_1 V_z(m_{+1}, k_{+1}, z_{+1}) \]

and the envelope conditions are given by

\[ W_m(m, k, z) = \frac{A}{pw(1 - t_h)} \]

\[ W_k(m, k, z) = \frac{A[1 + (r - \delta)(1 - t_k)]}{w(1 - t_h)} \]

\[ W_z(m, k, z) = \frac{A(1 - \omega)}{w(1 - t_h)} \]

As with \( k \), (35) shows that agents take the same amount of \( z \) out of the CM. Hence the distribution of \((m, k, z)\) will be degenerate in equilibrium. In the DM, everything is as before except we replace \( c(q, k) \) with \( c(q, z) \). The bargaining solution is still given by (11) with the substitution of \( z \) for \( k \),

\[ \frac{\beta_2 Am}{pw(1 - t_h)} = g(q, z_s) = \frac{\theta c(q, z_s)u'(q) + (1 - \theta)u(q)c_q(q, z_s)}{\theta(1 - t_d)u'(q) + (1 - \theta)c_q(q, z_s)}. \]

As before it can be shown that buyers spend all of their money balances so that \( d = m \).

The value function in the DM is the same as before except there is an extra state variable, and \( z \) replaces \( k \). The envelope conditions are

\[ V_m(m, k, z) = \frac{\beta_2 A}{pw(1 - t_h)} \left[ 1 - \sigma + \sigma \frac{u'(q)}{g_q(q, z)} \right] \]

\[ V_k(m, k, z) = \frac{\beta_2 A}{w(1 - t_h)} \left[ 1 + (r - \delta)(1 - t_k) \right] \]

\[ V_z(m, k, z) = \frac{\beta_2 A(1 - \omega)}{w(1 - t_h)} - \sigma \gamma(q, z) \]

where \( \gamma(q, z) = \frac{c_q(q, z)}{g_q(q, z)} \frac{\partial g_q(q, z)}{\partial g_q(q, z)} < 0 \). Again, if \( \theta = 1 \) then \( \gamma(q, z) = 0 \), and if \( \theta = 0 \),

\[ \gamma(q, z) = -c_z(q, z). \]
The same methods used above to close the model with bargaining reduces the equilibrium conditions to

\[(1 + \tau) g(q, Z) = \beta g(q_{+1}, Z_{+1}) \left[ 1 - \sigma + \sigma \frac{u'(q_{+1})}{g_q(q_{+1}, z_{+1})} \right] \quad (37)\]

\[U'(X) = \beta U'(X_{+1}) \left[ \{1 + [F_K(K_{+1}, H_{+1}) - \delta](1 - t_k)\} \right] \quad (38)\]

\[U'(X) = \beta U'(X_{+1}) \left[ 1 - \omega - \frac{(1 + t_x) \sigma g(q_{+1}, Z_{+1})}{\beta_2 u'(x_{+1})} \right] \quad (39)\]

\[A(1 + t_x) = U'(x) F_H(K, H) (1 - t_h) \quad (40)\]

\[x + K_{+1} + Z_{+1} + G = F(K, H) + (1 - \delta) K + (1 - \omega) Z \quad (41)\]

Equation (37) is equivalent to (14) with Z replacing K. Equation (38) is the standard equilibrium condition for \(k_{+1}\) in the one-sector growth model. Equation (39) is the equilibrium condition for \(z_{+1}\).

In steady state we get

\[
1 + \frac{i}{\sigma} = \frac{u'(q)}{g_q(q, Z)} \quad (42)
\]

\[
\rho + \omega = -\sigma \frac{\gamma(q, Z) F_H(K, H) (1 - t_h)}{\beta_2 A} \quad (43)
\]

\[
\rho + \delta = F_K(K, H) \quad (44)
\]

\[
A(1 + t_x) = U'(x) F_H(K, H) (1 - t_h) \quad (45)
\]

\[
X + G = F(K, H) - \delta K - \omega Z \quad (46)
\]

This model also does not display the dichotomy in Aruoba-Wright, even though \(k\) has no direct effect on \(q\) production. Since investment in \(z\) is done in the CM, it has to be financed by changes in \(x\), \(h\) or \(k_{+1}\)\(^{21}\)

For \(\theta = 1\), \(g_q(q, z) = c_q(q, z)\) and \(\gamma(q, z) = 0\). Then from (42) we see that the Friedman rule generates the efficient quantity, conditional on \(z\), \(q^* = q^*(z)\). However, when \(\theta = 1\), \(z = 0\). The reason is that \(z\) only has value in \(q\) production, and when \(\theta = 0\) sellers get no surplus from selling \(q\). Since \(z\) is costly, agents do not accumulate any. This is an extreme

\(^{21}\) However, when \(z\) does not depreciate, \(\omega = 0\), the model is recursive since \(k\), \(h\) and \(x\) are determined by (44), (45) and (46) independently of \(q\) and \(z\). Changes in \(k\), \(h\) and \(x\) will affect \(q\) and \(z\) but not vice-versa. Since monetary policy changes \(q\), this will change the steady state level of \(z\) but will have no effect on \(k\), \(h\), and \(x\) in the night market. In this sense, when \(\omega = 0\) the dichotomy reappears.
outcome of the holdup problem; if $z$ is a necessary input for $q$ production, then for $\theta = 0$ the holdup problem causes $q$ production and the monetary equilibrium to collapse.

With Walrasian pricing, once again the holdup problems on money and capital are eliminated and we get

\[
1 + \frac{i}{\sigma} = \frac{u'(q)}{c'(q, Z)} \tag{47}
\]

\[
\rho + \omega = -\sigma \frac{c_z(q, Z)F_H(K, H)(1 - t_h)}{\beta_2 A} \tag{48}
\]

As with bargaining, the dichotomy is broken. Consequently, changes in the money growth rate will affect the choice of $z$ which affects $x$, $h$ and $k_{+1}$. Intuitively, we expect that an increases in the money growth rate $\tau$ raise $i$, which lowers $q$ thereby reducing the incentive to invest in $z$.

### 3.2 Example

Again, we use explicit functional forms to gain insight as to how monetary policy affects the economy. We use the same functional forms as before except that $Z$ now replaces $K$ in the cost function. For presentation purposes we look at the equilibrium with Walrasian pricing and no taxes.

Using the specified functional forms as before, (44), (45) and (47) yield

\[
K = H \left( \frac{\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}}
\]

\[
x = \frac{1 - \alpha}{A} \left( \frac{\alpha}{\rho + \delta} \right)^{\frac{\alpha}{1-\alpha}}
\]

\[
q = Z \frac{\varphi-1}{\varphi^{1+\gamma}} \left\{ \frac{\sigma}{\varphi(i + \sigma)} \right\} \frac{1}{\varphi^{1+\gamma}}
\]

implying that the capital-labor ratio is uniquely pinned down which in turn determines the equilibrium level of consumption. Using these expressions (48) yields

\[
Z = \left[ \frac{\sigma}{\varphi(i + \sigma)} \right]^{\frac{1}{\gamma}} \left[ \frac{\sigma(\varphi - 1)(1 - \alpha)}{A(\rho + \omega)} \left( \frac{\alpha}{\rho + \delta} \right)^{\frac{\alpha}{1-\alpha}} \right]^{\frac{1}{\mu}}
\]

where $\mu = \frac{\varphi \gamma}{\varphi^{\gamma+1}} < 1$ as before. So $Z$ is pinned down. Finally, (46) yields

\[
H = \Theta \left\{ 1 - \alpha + A \omega \left[ \frac{\sigma}{\varphi(i + \sigma)} \right]^{\frac{1}{\gamma}} \left[ \frac{\sigma(\varphi - 1)(1 - \alpha)}{A(\rho + \omega)} \right]^{\frac{1}{\mu}} \left( \frac{\alpha}{\rho + \delta} \right)^{\frac{\alpha(1-\mu)}{\mu(1-\alpha)}} \right\}
\]
where $\Theta = \frac{\rho + \delta}{A(\rho + (1 - \alpha)\delta)}$.

How does policy affect this economy? An increase in the money growth rate above the Friedman rule increases the nominal interest rate. An increase in $i$ again lowers the value of money and thus the quantity of goods produced in the DM. As before, this reduces the marginal value of a unit of $Z$ so there is less investment. Since agents need fewer resources for investment, they work less in the night market and so there is less $K$. Aggregate output in the CM falls however, the capital-labor ratio is unaffected which leaves the real wage and consumption unchanged. Since aggregate output falls but consumption stays the same, the saving rate declines.

3.3 Capital Produced in the Day

In the previous models, all investment occurred in the CM, and so money is not needed to pay for capital goods. It is known that in reduced form models it makes a difference if one has to pay for capital goods with cash; e.g. Stockman (1981). To consider this effect in our model, we modify things by assuming that investment occurs in the DM where agents are anonymous and therefore money is essential for trade. Suppose that agents do not consume the output of the DM but use it as an intermediate input that can be transformed into capital $k$ for production in the CM, where without loss of generality we assume $q$ can be transformed one for one into $k$.\(^\text{22}\) As in the previous sections, a fraction $\sigma$ have the ability to produce the intermediate input, and the same fraction have the ability to transform it into capital, but no agent can do both. Once capital is produced it is immobile, as in the other models, and so it cannot serve as a medium of exchange.

Capital is productive in the CM, where it will be rented to competitive firms, but not in the DM – i.e. $c(q, k) = c(q)$. Since trade is anonymous, money is needed to buy capital, as in Stockman’s model. The night market problem is

$$W(m, k) = \max_{x, h, m_{+1}} U(x) - Ah + \beta_1 V(m_{+1}, k)$$

subject to

$$st \quad (1 + t_x)x = w(1 - t_h)h + \left[1 + (r - \delta)(1 - t_k)\right]k + \frac{m - (1 + \tau)m_{+1}}{p}$$

\(^{22}\)See Shi (1999) for a related model.
We assume for now that $k$ is not traded in this market. Substituting for $h$ we obtain

\[
W(m, k) = \frac{A}{w(1-t_h)} \left\{ \frac{m}{p} + [1 + (r - \delta)(1 - t_k)] k \right\}
\]

\[
\max_{x, m+1} U(x) = \frac{A}{w(1-t_h)} \left[ (1 + t_x) x - (1 + \tau) \frac{m+1}{p} \right] + \beta_1 V(m+1, k).
\]

The first-order conditions are given by

\[
x : \quad U'(x) = \frac{A(1 + t_x)}{w(1-t_h)}
\]

\[
m+1 : \quad \frac{A(1 + \tau)}{pw(1-t_h)} = \beta_1 V(m+1, k).
\]

Note that since individual $k$ is obtained in the DM in this model, individual capital holdings depend on the idiosyncratic trading shocks. Hence, there is a distribution of $k$ across agents. Since the first-order condition for $m+1$ is not independent of one’s capital holdings it is not obvious at this stage if the distribution of money holdings is degenerate. We demonstrate below that it is. The envelope conditions are still given by (3) and (4).

One can assume agents bargain just as in the earlier model, but the surpluses are different. The buyer gives up $d$ units of money and acquires $q$ units of intermediate goods which is transformed into $k = d$ units of capital. Hence his surplus is $S_b = W(m-d, k+q) - W(m, k) = q [1 + (r - \delta)(1 - t_k)] \beta_2 A/w(1-t_h) - d\beta_2 A/pw(1-t_h)$. Similarly, the seller’s surplus is $S_s = -c(q) + W[m + (1-t_d)d, k] - W(m, k) = -c(q) + (1-t_d)d\beta_2 A/pw(1-t_h)$. Notice these surpluses of the individuals’ capital holdings and the seller’s money holdings. Hence $(q, d)$ are independent of them as well. Again one can show $d = m_b$. Then the first-order condition for $q$ can be written

\[
\frac{m_b}{p} = g(q, r, w) = \frac{\theta c(q) + (1-\theta)qc'(q)}{\theta [1 + (r - \delta)(1 - t_k)] \beta_2 A/w + (1-\theta)c'(q) [1 + (r - \delta)(1 - t_k)]}
\]

and $\partial q/\partial m_b = 1/pg_q(q, r, w)$.

The value function in the day market is now

\[
V(m, k) = \sigma \int \left\{ \beta_2 W[m - d, k + q(m)] + \beta_2 W[m + \tilde{d}(\tilde{m}), k] - c[q(\tilde{m})] \right\} dF(\tilde{m})
\]

\[
+ (1 - 2\sigma) \beta_2 W(m, k)
\]

\[
= \sigma \left[ q(m) [1 + (r - \delta)(1 - t_k)] \frac{\beta_2 A}{w(1-t_h)} - \frac{d\beta_2 A}{pw(1-t_h)} \right]
\]

\[
+ \sigma \int \left\{ -c[q(\tilde{m})] + \frac{\tilde{d}(\tilde{m}) \beta_2 A}{pw(1-t_h)} \right\} dF(\tilde{m}) + \beta_2 W(m, k)
\]

26
where \( F(\tilde{m}) \) is the distribution of money holdings across agents and \( \tilde{d}(\tilde{m}) \) is the money received by a randomly encountered buyer holding \( \tilde{m} \) units of money. The integration is only with regards to \( \tilde{m} \) since capital holdings are irrelevant for the payoffs in bargaining.

The envelope condition is

\[
V_m(m, k) = \sigma \int \left\{ \frac{[1 + (r - \delta)(1 - t_k)]}{w(1 - t_h)} \frac{\beta_2 A}{\partial m} - \frac{\beta_2 A}{pw(1 - t_h)} \frac{\partial d}{\partial m} \right\} dF(\tilde{m}) + \frac{\beta_2 A}{pw(1 - t_h)}
\]

\[
= \sigma \frac{[1 + (r - \delta)(1 - t_k)]}{w(1 - t_h)} \frac{\beta_2 A}{pg_q(q, r, w)} - \sigma \frac{\beta_2 A}{pw(1 - t_h)} + \frac{\beta_2 A}{pw(1 - t_h)}
\]

\[
= \frac{\beta_2 A}{pw(1 - t_h)} \left[ 1 - \sigma + \frac{[1 + (r - \delta)(1 - t_k)]}{pg_q(q, r, w)} \right].
\]

Since \( V_m(m, k) \) is independent of the buyer’s capital holdings, then it must be the case that the choice of money taken out of the CM according to (49) is the same for everyone – the distribution of \( m \) is again degenerate regardless of whether or not the distribution of capital is degenerate.

The first-order condition for \( m_{+1} \) implies

\[
g(q, r, w) = \beta \frac{g(q_{+1}, r_{+1}, w_{+1})}{w_{+1}} \left[ 1 - \sigma + \sigma \frac{1 + (r_{+1} - \delta)(1 - t_{k})}{g_q(q_{+1}, r_{+1}, w_{+1})} \right].
\]

(50)

It is apparent that this model does not dichotomize – we cannot solve for \( q \) without knowing \( r = F_K(K, H) \) and \( w = F_H(K, H) \). In steady state, we have

\[
1 + \frac{i}{\sigma} = \frac{1 + [F_K(K, H) - \delta](1 - t_k)}{g_q[q, F_K(K, H), F_H(K, H)]}.
\]

If we set \( \theta = 1 \) then \( g(q, r, w) = c(q)w(1 - t_h)/A \), and \( g_q(q, r, w) = c'(q)w(1 - t_h)/A = c'(q)(1 - t_h)F_H(K, H)/A \), which reduces the steady state condition to

\[
1 + \frac{i}{\sigma} = A \frac{1 + [F_K(K, H) - \delta](1 - t_k)}{c'(q)F_H(K, H)(1 - t_h)}.
\]

(51)

Using (20)-(21) and the steady-state condition \( \sigma q = \delta K \), a steady state with \( \theta = 1 \) is a pair \((K, H)\) solving

\[
1 + \frac{i}{\sigma} = A \frac{1 + [F_K(K, H) - \delta](1 - t_k)}{c'(\delta K/\sigma)F_H(K, H)}.
\]

(52)

Using (51) and (52), it is straightforward to show that \( \partial K/\partial i < 0 \). The intuition behind this result is that an increase in the money growth rate lowers the value of money acquired.
by sellers of intermediate goods and so they produce less. Since intermediate goods are used to produce capital, it follows immediately that aggregate $K$ is lower. Thus, we get a similar result to Stockman but for a different reason.

What if agents were allowed to trade $k$ in the CM? Notice that it is merely a secondary market – no investment occurs, only reallocation of $k$. Let $\lambda$ denote the price of existing capital. Then the agent’s value function in the night market satisfies

$$W(m, k) = \frac{A}{w(1-t_h)} \left\{ \frac{m + \tau}{p} + \lambda [1 + (r - \delta)(1-t_k)] k \right\} + \max_{x,m+1,k+1} U(x) - \frac{A}{w(1-t_h)} \left[ (1 + t_x) x + \lambda k + \frac{(1 + \tau) m_{+1}}{p} \right] + \beta V(m_{+1}, k_{+1}).$$

The first order condition for $k_{+1}$ is

$$\frac{A}{w(1-t_h)} \lambda = \beta V_k(m_{+1}, k_{+1}).$$

Since wealth is linear in capital holdings and capital does not affect the value of intermediate good trades, $V_k(m_{+1}, k_{+1}) = \beta W_k(m_{+1}, k_{+1}) = \frac{\beta A}{w(1-t_h)} \lambda_{+1} [1 + (r_{+1} - \delta)(1-t_k)]$ which gives

$$\frac{\lambda}{F_H(K, H)} = \frac{\beta \lambda_{+1}}{F_H(K_{+1}, H_{+1})} \left[ 1 + (F_k(K_{+1}, H_{+1}) - \delta)(1-t_k) \right]$$

This expression is independent of individual $k$ and merely pins down the path for the price of capital in the secondary market such that no arbitrage opportunities exist. Agents are indifferent between buying or selling capital at this price and so the distribution of capital is not pinned down without further assumptions on agents’ behavior.

With competitive pricing, buyers choose how much of the intermediate good to purchase. As before, $d = m$ so buyers spend all of their money and acquire $q_b = m/\tilde{p}$ units of goods. Sellers set marginal cost equal to the value of a marginal unit of money received in payment, $c'(q_s) = \beta_2 A\tilde{p}/pw(1-t_h)$. In equilibrium, $q_b = q_s = q$ which solves $c'(q) = \beta_2 Am/pqw(1-t_h)$. Following the same methods as before, the first-order condition for money becomes

$$(1 + \tau) c'(q) q = \beta c'(q_{+1}) q_{+1} \left[ 1 - \sigma + \frac{\sigma A}{c'(q_{+1})w_{+1}(1-t_h)} \right]$$

Using $\frac{\beta A}{pw(1-t_h)} = c'(q)/\tilde{p}$ and $\tilde{p} = m/q$ this can be written as

$$(1 + \tau) c'(q) q = \beta c'(q_{+1}) q_{+1} \left[ 1 - \sigma + \sigma A \frac{1 + (r_{+1} - \delta)(1-t_k)}{c'(q_{+1})w_{+1}(1-t_h)} \right]$$

(53)
Comparing (50) and (53) note that the dynamics of the model under bargaining and Walrasian pricing will differ if \( g(q,r,w) \neq c'(q)q \) and \( g_q(q,r,w) \neq c'(q)w(1-t_h)/A \). In steady state, (53) becomes

\[
1 + \frac{i}{\sigma} = A \frac{1 + [F_k(K,H) - \delta](1-t_h)}{c'(q)F_H(K,H)(1-t_h)}
\]

which is the same steady-state expression that arises under bargaining when \( \theta = 1 \). So an equilibrium with Walrasian pricing is a pair \((K,H)\) solving (51) and (52). Once again, there is no dichotomy and excessive money growth, creates inflation, raises the nominal interest rate and lowers the equilibrium capital stock.

4 Numerical Results

In this section we calculate the welfare cost of inflation using the basic model discussed in detail in Section 2. In particular, for each pricing version of the model, we calibrate the model, solve for the decision rules in the stationary equilibrium, which is different from the steady state, unlike AW or LW, and compute the welfare loss of inflation, along with the transition paths for variables of interest. Similar welfare computations were carried out in LW as well. However, the calibration in this model will be much more realistic due to the existence of capital. Moreover we are able to consider the transition path of a change in inflation for the first time in this literature since in LW and AW, the economy would jump to the new steady state immediately following the change.

4.1 Accounting

Before turning to calibration, we need to do some accounting. The price of a special good in the DM is \( P_D = \frac{M}{q} \) (\( M \) units of money buys \( q \) units of special goods) and the price of a general good in the CM is defined as \( P_C = p' \). As a convention, we define all real objects in terms of CM prices.

We can define the real GDP in terms of the general good as

\[
Y = \underbrace{\frac{(\sigma q)}{p'}}_{\text{DM output}} + \underbrace{F(K,H)}_{\text{CM output}} = \frac{\sigma}{p} + F(K,H) \quad (54)
\]
where \( \frac{M}{p'q} \) is the relative price of special good in terms of the general good.\(^{23}\) The nominal GDP of this economy is

\[
pY = \sigma + pF(K, H)
\]

where we substitute for \( 1/p \) from

\[
\frac{1}{p} = \frac{(1 - t_h) g(q, K) w}{A \beta_2}
\]

(55)

for the bargaining version and from

\[
\frac{1}{p} = \frac{(1 - t_h) c_q(q, K) qw}{(1 - t_d) A \beta_2}
\]

(56)

for the competitive pricing version.

4.2 Calibration

4.2.1 Functional Forms

We choose the following functional forms:

- **Utility Function (DM)**
  \[
u(q) = \frac{(b + q)^{1-\eta} - b^{1-\eta}}{1-\eta}\]

- **Utility Function (CM)**
  \[
U(X) = B \frac{X^{1-\gamma} - 1}{1 - \gamma}
\]

- **Production Function (CM)**
  \[
F(K, H) = K^\alpha H^{1-\alpha}
\]

- **Cost Function (DM)**
  \[
c(q, K) = q^\psi K^{1-\psi}
\]

which gives us a total of 18 parameters, grouped as follows:

- **Preference Parameters** \( \Omega_p = (\beta_1, \beta_2, A, \gamma, \eta, b, B) \)
- **Technology Parameters** \( \Omega_t = (\delta, \alpha, \psi) \)
- **Search Parameters** \( \Omega_s = (\sigma, \theta) \)
- **Policy Parameters** \( \Omega_g = (G, \tau, t_h, t_k, t_x, t_d) \)

\(^{23}\)Note that we defined \( p = \frac{K'}{M} \) as the scaled price in the DM.
We can observe or otherwise set arbitrarily some of these parameters, independent from other parameters.

<table>
<thead>
<tr>
<th>Directly Observed</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau, t_h, t_k, t_x, t_d$</td>
<td>Relevant rates</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Match real interest rate ($\bar{r}$)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Match investment-capital ratio ($\bar{IK}$)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Match labor share of output ($\bar{LY}$)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Set Arbitrarily</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>Log utility</td>
</tr>
<tr>
<td>$b$</td>
<td>Small</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>No discounting between DM and CM</td>
</tr>
</tbody>
</table>

where the expressions in the parentheses are values observed from the data.

We are left with the following parameters ($A, \eta, B, \psi, \sigma, \theta, G$). We will use the following calibration targets to calibrate these parameters in the steady state:

- Average Velocity: $\sigma + pF(K, H)$
- Average Markup ($\bar{AM}$): See Appendix C.1
- Interest Elasticity of Money Demand: See Appendix C.2
- Capital-Output Ratio ($\bar{KY}$): $\frac{K}{F(K, H) + \sigma/p}$
- Consumption-Output Ratio ($\bar{CY}$): $\frac{X + (1-t_d)\sigma/p}{F(K, H) + \sigma/p}$
- Share of Government Expenditures in Output ($\bar{GY}$): $\frac{F(K, H) + \sigma/p}{G}$
- Average Time Spent Working ($\bar{AH}$): $\frac{H}{H}$

All told, we have 4 equations that define the steady state of the economy, given the parameters and we have 7 calibration targets for calibrating 7 parameters.

We will follow the following procedure. As will be shown below for each version of the model, we can impose the last two conditions directly on the steady state and solve for $A$ and $24$We assume that the labor share of output in the DM is equal to the aggregate labor share, which in turn equals, by definition, the labor share in the CM.
$G$ as functions of other unknowns in the problem. This insight leaves us with 5 calibration targets. Instead of trying to match these targets exactly, we will minimize the sum of squared percentage deviations from them. Therefore, we will look for a set of parameters $(\eta, B, \psi, \sigma, \theta)$ that give the minimum deviations from the calibration targets in the steady state.

Define $k \equiv K^r_H$ as the capital-labor ratio in the steady state. Using the assumption of constant-returns-to-scale production function for $F(K, H)$, we can define

\[ f(k) \equiv \frac{F(K, H)}{H} \]

\[ f_K(k) = F_K(k, 1) \]

\[ f_H(k) = F_H(k, 1) \]

**Bargaining Version**  Holding parameters $(\eta, B, \psi, \sigma, \theta)$ constant for the time being, and using the transformation above, the steady state conditions for the bargaining case reduces to

\[ X + G = H \left[f(k) - \delta_k\right] \]

\[ (1 + \tau)(1 + \rho) = \frac{\sigma u'(q)}{g(q, k)} + (1 - \sigma) \]

\[ U'(X) = \frac{A (1 + t_x)}{f_H(k) (1 - t_h)} \]

\[ 1 = \beta \{1 + [f_K(k) - \delta] (1 - t_k)} - \beta_1 \frac{(1 + t_x) \sigma \gamma(q, k)}{U'(X)} \]

where $g(q, k) \equiv g(q, kH)$ and $\gamma(q, k) \equiv \gamma(q, kH)$. To these conditions we add the two calibration targets we match exactly

\[ H = \frac{\hat{A}H}{\hat{A}} \]

\[ \frac{\bar{K}Y}{GY} = \frac{kH}{G} \]

where the latter is obtained by combining two targets to obtain the capital-government expenditure ratio.

We can simplify this system of six equations in six unknowns $(X, H, k, q, A, G)$ significantly by making some substitutions. First, since we are fixing the value of $H$, we can
substitute this value in to the remaining equations and remove it from the list of variables. Also (61) can be explicitly solved for $G(k)$. Next, (57) can be solved explicitly for $X(k)$ and (59) can be explicitly solved for $A(k)$. All these substitutions define the remaining variables as a function of $q$ and $k$. We can solve (58) and (60) for $q$ and $k$, which would be a system of two equations in two unknowns.

**Competitive Pricing Version** The methodology is exactly as before and the steady state conditions for the competitive pricing version after the transformations are

$$X + G = H[f(k) - \delta k]$$

$$(1 + \tau) (1 + \rho) = \frac{(1 - t_d) \sigma u'(q)}{c_q(q, k)} + (1 - \sigma)$$

$$U'(X) = \frac{A(1 + t_x)}{f_H(k) (1 - t_h)}$$

$$1 = \beta \left\{ 1 + \left[ f_K(k) - \delta \right] (1 - t_k) \right\} - \beta_1 \frac{(1 + t_x) \sigma c_k(q, k)}{U'(X)}$$

4.2.2 Calibration Targets and Calibrated Parameters

We consider two calibrations. First, we calibrate a version of the economy with no proportional taxes which we will label Calibration 1. The results from this calibration may be compared to other results from search-based models such as LW. Next, we calibrate the economy with all the fiscal tools, which we will label Calibration 2. Both calibrations will cover the period 1951-2004 and will have an annual frequency.

The calibration targets and the corresponding observed parameters are reported in the table below.
### Observed Parameters

<table>
<thead>
<tr>
<th>Observed Value</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Labor Income Tax</td>
<td>0.150</td>
</tr>
<tr>
<td>Capital Income Tax</td>
<td>0.150</td>
</tr>
<tr>
<td>Consumption Tax</td>
<td>0.122</td>
</tr>
<tr>
<td>Real Interest Rate</td>
<td>0.035</td>
</tr>
<tr>
<td>Investment-Capital Ratio</td>
<td>0.067</td>
</tr>
<tr>
<td>Labor Share of Output</td>
<td>0.712</td>
</tr>
</tbody>
</table>

Labor and capital income taxes are arbitrarily fixed at 15% for now. The consumption tax is computed as the average of the sum of excise and sales taxes over personal consumption expenditures excluding durables. The real interest rate is obtained from the average yield on a Aaa-rated corporate bond and the average inflation over the period. Capital is measured as the sum of private fixed assets and stock of private inventories and investment refers to the appropriate flow of this sum. Labor share of output is computed using the breakdown of national income, dividing the ambiguous income appropriately between capital and labor. The remaining calibration targets are reported in the table below.

### Other Calibration Targets

<table>
<thead>
<tr>
<th>Other Calibration Targets</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity</td>
<td>5.655</td>
</tr>
<tr>
<td>Interest Elasticity of Money Demand</td>
<td>-0.332</td>
</tr>
<tr>
<td>Markup</td>
<td>0.100</td>
</tr>
<tr>
<td>Capital-Output Ratio</td>
<td>2.404</td>
</tr>
<tr>
<td>Consumption-Output Ratio</td>
<td>0.636</td>
</tr>
<tr>
<td>Share of Government Expenditures</td>
<td>0.205</td>
</tr>
<tr>
<td>Time Spent Working</td>
<td>0.330</td>
</tr>
<tr>
<td>Nominal Interest Rate</td>
<td>0.072</td>
</tr>
<tr>
<td>Inflation Rate</td>
<td>0.036</td>
</tr>
</tbody>
</table>

The money measure we use is M1. We measure output as GDP excluding net exports. Velocity therefore refers to the ratio of money supply to output. Consumption is private
consumption expenditures. Time spent working is chosen to be a third of discretionary
time, which is from Juster and Stafford. The inflation rate is based on the GDP deflator.
See the appendix for the sources of the data and additional details.

4.3 Stationary Equilibrium

The stationary equilibrium is defined as the time-invariant functions \( X(S) \), \( H(S) \), \( K'(S) \)
and \( q(S) \) that solve (14)-(17) for the bargaining version and (22)-25) for the competitive
pricing version. In the original statement of the problem, we used \((m, k)\) as an individual’s
state variables where \( m \) is the individual’s money holdings relative to the aggregate money
supply. In equilibrium, \( m = 1 \) and \( k = K \) must hold. Therefore, the only state variable that
we need to consider is \( S = K \).

Solving these sets of functional equations analytically is virtually impossible. As such, we
use a numerical method, namely the Weighted Residual Method with Chebyshev Polynomials
and Orthogonal Collocation\(^{25}\) to solve the set of functional equations

\[
\mathcal{F}[X(K), H(K), q(K), K'(K), K] = 0
\]

defined by the equilibrium conditions.

We use the same method to solve for the value function \( V(K) \) using the Bellman’s
equation given by (5).

4.4 Welfare

For expositional purposes, let us use the notation \( q(K; \tau) \), \( K'(K; \tau) \), \( H(K; \tau) \), \( X(K, \tau) \)
for the decision rules and \( V(K; \tau) \) for the value function with the inflation rate \( \tau \). The steady
state at inflation rate \( \tau \) is defined as

\[
K_\tau = K'(K_\tau; \tau)
\]

\(^{25}\)See Judd (1992) for the explanation of the method and Aruoba et al. (2003) for a comparison of different
solution methods for solving the stochastic Neoclassical growth model.
Since, unlike AW and LW, there will be a non-trivial transition path in response to a change in inflation rate, we can consider two different welfare questions, which will have different answers.

1. **What is the difference in welfare of two economies with different inflation rates?** For the answer to this question we can simply compare the steady state values of the two economies. That is, the comparison is between

\[ V(K_{\tau_1}; \tau_1) \text{ and } V(K_{\tau_2}; \tau_2) \]

2. **If we implement a policy change while the economy is at its steady state, what is the difference in welfare of the policy, compared with no change?** The answer to this question requires comparing the initial steady state with the new steady state plus the transition period while the economy reaches the new steady state. Thus, the comparison is between

\[ V(K_{\tau_1}; \tau_1) \text{ and } V(K_{\tau_1}; \tau_2) \]

where now the second value function is evaluated at the old steady state (but the new inflation rate) since by definition \( V(K, \tau) \) provides the value to the agent starting at some \( K \) with inflation rate \( \tau \) ad infinitum. Note that

\[ V(K_{\tau_2}; \tau_2) - V(K_{\tau_1}; \tau_2) \]

is the welfare loss in the transition which might be of some interest.

For computing welfare, we will use the consumption-equivalent variation. Consider the following value function which is the value when both consumption streams are increased by \( (\Delta - 1)\% \)

\[
V(K; \tau, \Delta) = \sigma \{ u[\Delta q(K_{\tau}; \tau)] - c[q(K_{\tau}; \tau), K_{\tau}] \} + \beta_2 U[\Delta X (K_{\tau}; \tau)] - \beta_2 AH (K_{\tau}; \tau) + \beta V[K(K_{\tau}; \tau); \tau, \Delta]
\]

(64)

where the decision rules are still the optimal decision rules for inflation rate \( \tau \).
For comparing two economies with two different inflation rates $\tau_1$ and $\tau_2$, where without loss of generality $\tau_1 > \tau_2$, we look for $\Delta$ that solves

$$V(K_{\tau_1}; \tau_1, \Delta) = V(K_{\tau_2}; \tau_2, 0)$$

(65)

On the other hand, for assessing the effects of a policy change in an economy, we look for $\Delta$ that solves

$$V(K_{\tau_1}; \tau_1, \Delta) = V(K_{\tau_1}; \tau_2, 0)$$

(66)

4.5 Results

We first consider Calibration 1, the calibration with no proportional taxes. The calibrated parameters along with the calibration targets using these parameters are reported in Table 1.
For comparison purposes, in the first two columns we report results from the special case considered in AW where we fix $\psi = 1$ to get $c(q, K) = c(q) = q$. The first column reports results for $\theta = 1$, which also correspond to the competitive pricing model in this case, and second column reports results where $\theta$ is calibrated to match the markup. The remaining three columns report results for the more general case where $\psi$ is calibrated. Column 3 and 4 have the bargaining version with $\theta = 1$, and calibrated $\theta$, respectively, while the last column reports results for the competitive pricing version. There are a few things to point out. First,
all calibration targets are reasonably met by the parametrizations.\textsuperscript{26} Second, when $\psi$ is left free, our calibration chooses a value away from unity, which means the generalization we consider in this paper is supported by the data. Finally, for all parametrizations considered, the size of the DM is about 10\% of the economy, which lead to markups in the DM of about 100\% to achieve the aggregate markup of 10\%.

In Table 2, we report the comparison of steady states of the economy under 10\% inflation versus the Friedman rule along and the solution to the social planner’s problem. We also report the welfare results on this table.

First, as we discussed earlier, we see the Neoclassical Dichotomy of AW in the first two columns. Change in the inflation rate changes only the output in the DM and therefore the aggregate output, but leaves CM variables unchanged. Moreover, we see that when $\theta = 1$ in AW, Friedman rule achieves the first-best allocation, as expected. When we look at the general model with $\theta = 1$ (column 3), we see that the dichotomy is still at work since steady state CM variables can still be determined independent from DM variables and the inflation rate. However, Friedman rule no longer yield the first-best allocation. If we lower $\theta$, or consider competitive pricing, the dichotomy is broken.\textsuperscript{27}

Turning to the welfare results, we note that the two welfare questions posed above has the same answer for the first three columns, since there is no transition in these models as capital is not affected from the inflation rate. With $\theta = 1$ or with competitive pricing in AW, the welfare loss of a 10\% inflation, compared to 0\% is around 1\% of consumption. In these versions of the model, the only source of inefficiency is the $\beta$-wedge, as it is the case for many reduced-form models. It is not surprising, therefore that our results match, with those of Cooley and Hansen (1989) or Lucas (2000). More interesting results are obtained in the full model with free $\theta$ and competitive pricing (columns 4 and 5). In the former, all three inefficiencies are at work, namely the $\beta$-wedge, the capital hold-up problem and the money hold-up problem. This increases the welfare gain of going from a 10\% inflation to

\textsuperscript{26}Remember that we are using percentage deviations and not absolute deviations as the objective function. Therefore even though the deviation for $K/Y$ seems large, in percentage terms it is small and in the same order as other errors.

\textsuperscript{27}On columns 4 and 5, the 1.00 terms are rounded up to unity, even though they are numerically lower than unity.
the Friedman rule (the optimal policy) to 5.5% and the gain of going to a 0% inflation to 3.9%. Furthermore, we see that during the transition there are nonnegligible welfare losses, in the order of 0.2%. In other words, had we ignored the transition, we would over-state the welfare gain since the agent is temporarily worse off during the transition.

### Table 2 - Steady State with 10% Inflation for Calibration 1

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
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<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AW</td>
<td>AW</td>
<td>AWW-B</td>
<td>AWW-B</td>
<td>AWW-C</td>
</tr>
<tr>
<td><strong>Compared to the Friedman Rule</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q(\tau)/q(F)$</td>
<td>0.53</td>
<td>0.46</td>
<td>0.72</td>
<td>0.70</td>
<td>0.24</td>
</tr>
<tr>
<td>$y(\tau)/y(F)$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.98</td>
</tr>
<tr>
<td>$Y(\tau)/Y(F)$</td>
<td>0.94</td>
<td>0.94</td>
<td>0.95</td>
<td>0.95</td>
<td>0.84</td>
</tr>
<tr>
<td>$K(\tau)/K(F)$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.99</td>
<td>0.95</td>
</tr>
<tr>
<td>$H(\tau)/H(F)$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$X(\tau)/X(F)$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.99</td>
</tr>
<tr>
<td><strong>Compared to the Solution to the Social Planner’s Problem</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q(\tau)/q^*$</td>
<td>0.53</td>
<td>0.21</td>
<td>0.51</td>
<td>0.21</td>
<td>0.24</td>
</tr>
<tr>
<td>$y(\tau)/y^*$</td>
<td>1.00</td>
<td>1.00</td>
<td>0.85</td>
<td>0.79</td>
<td>0.98</td>
</tr>
<tr>
<td>$Y(\tau)/Y^*$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K(\tau)/K^*$</td>
<td>1.00</td>
<td>1.00</td>
<td>0.64</td>
<td>0.53</td>
<td>0.95</td>
</tr>
<tr>
<td>$H(\tau)/H^*$</td>
<td>1.00</td>
<td>1.00</td>
<td>0.95</td>
<td>0.93</td>
<td>1.00</td>
</tr>
<tr>
<td>$X(\tau)/X^*$</td>
<td>1.00</td>
<td>1.00</td>
<td>0.89</td>
<td>0.85</td>
<td>0.99</td>
</tr>
<tr>
<td><strong>Welfare Gains (%)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>From 10% to Friedman Rule</td>
<td>1.19</td>
<td>5.74</td>
<td>0.89</td>
<td>6.06</td>
<td>2.71</td>
</tr>
<tr>
<td>During Transition</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>−0.22</td>
<td>−0.92</td>
</tr>
<tr>
<td>From 10% to 0%</td>
<td>1.19</td>
<td>4.07</td>
<td>0.80</td>
<td>4.17</td>
<td>2.31</td>
</tr>
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<td>During Transition</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>−0.12</td>
<td>−0.51</td>
</tr>
</tbody>
</table>

To understand the source of the welfare loss during the transition, consider Figure 1, where we plot the transition path of the variables of interest following a reduction of the inflation rate from 10% to the Friedman rule in period 1. Unity on the y-axis corresponds
Figure 1: Transition Path of Variables Following a Reduction of Inflation Rate from 10% to the Friedman Rule
to the initial steady state. All lines converge to the new steady state in about 40 periods. We see that DM output immediately jumps up by about 40% and quickly converges to its new steady state value. Hours in the CM, on the other hand jump up by about 1% initially and slowly go down the new steady state value which is only 0.1% above the initial one. Similarly, consumption in the CM jump down by a small amount on impact, before slowly converging to the new and higher steady state value. We see that in order to accumulate the extra capital the agents work more and consume less initially, and this leads to a welfare loss of about 0.2%.

Next we turn to the results from Calibration 2, which are reported in Table 3 and 4.
### Table 3 - Results from Calibration 2 (With Proportional Taxes)

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
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<td>AW</td>
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<td>AWW-B</td>
<td>AWW-C</td>
</tr>
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<td><strong>Calibrated Parameters</strong></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.49</td>
<td>0.50</td>
<td>0.49</td>
<td>0.49</td>
<td>0.50</td>
</tr>
<tr>
<td>$B$</td>
<td>1.96</td>
<td>1.97</td>
<td>1.10</td>
<td>1.05</td>
<td>0.78</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.39</td>
<td>0.59</td>
<td>0.20</td>
<td>0.50</td>
<td>0.13</td>
</tr>
<tr>
<td>$\psi$</td>
<td>-</td>
<td>-</td>
<td>1.52</td>
<td>1.67</td>
<td>1.08</td>
</tr>
<tr>
<td>$A$</td>
<td>5.38</td>
<td>5.40</td>
<td>2.93</td>
<td>2.89</td>
<td>2.18</td>
</tr>
<tr>
<td>$G$</td>
<td>0.11</td>
<td>0.12</td>
<td>0.11</td>
<td>0.11</td>
<td>0.12</td>
</tr>
<tr>
<td>$\theta$</td>
<td>1.00</td>
<td>0.50</td>
<td>1.00</td>
<td>0.45</td>
<td>-</td>
</tr>
<tr>
<td><strong>Calibration Targets from the Benchmark Economy</strong></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$\bar{AM}$ (10.00)</td>
<td>0.00</td>
<td>10.00</td>
<td>-2.95</td>
<td>10.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$\bar{KY}$ (2.40)</td>
<td>2.43</td>
<td>2.43</td>
<td>2.43</td>
<td>2.43</td>
<td>2.49</td>
</tr>
<tr>
<td>$\bar{CY}$ (0.64)</td>
<td>0.62</td>
<td>0.62</td>
<td>0.62</td>
<td>0.62</td>
<td>0.61</td>
</tr>
<tr>
<td>$\bar{GY}$ (0.21)</td>
<td>0.21</td>
<td>0.21</td>
<td>0.21</td>
<td>0.21</td>
<td>0.21</td>
</tr>
<tr>
<td>Velocity (5.66)</td>
<td>5.64</td>
<td>5.64</td>
<td>5.65</td>
<td>5.64</td>
<td>5.64</td>
</tr>
<tr>
<td>Interest Elasticity (-0.33)</td>
<td>-0.33</td>
<td>-0.33</td>
<td>-0.33</td>
<td>-0.33</td>
<td>-0.33</td>
</tr>
<tr>
<td><strong>Miscellaneous</strong></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>Share of DM</td>
<td>8.70</td>
<td>8.86</td>
<td>8.63</td>
<td>8.67</td>
<td>9.78</td>
</tr>
<tr>
<td>Markup in DM</td>
<td>0.00</td>
<td>112.81</td>
<td>-34.17</td>
<td>115.34</td>
<td>0.00</td>
</tr>
</tbody>
</table>

One thing to note is that the model with proportional taxes is more flexible than the previous model in matching the calibration targets.\(^{28}\) However, it must be noted that the tax rates used here are arbitrary for now.

The steady state and welfare results for the calibration with proportional taxes are reported on Table 4.

\(^{28}\)What we mean by this is that the deviations from the targets are smaller (an order of magnitude smaller).
The results show that, now the social planner’s problem’s allocation cannot be achieved by any inflation rate, due to the existence of distortionary taxes. When we compare the numbers on Table 4 with those on Table 2, we see that the steady state values of the economy with 10% inflation relative to the Friedman rule are virtually unchanged with the introduction of proportional taxes. Welfare results are quite similar, although the loss during transitions are smaller with proportional taxes. The most interesting specification (free $\theta$ and $\psi$, column 5), shows that the introduction of taxes lower the gain of going to Friedman rule.

The Friedman Rule

\[ q(\tau)/q(F) = 0.53 \quad y(\tau)/y(F) = 1.00 \quad Y(\tau)/Y(F) = 0.94 \quad K(\tau)/K(F) = 1.00 \quad H(\tau)/H(F) = 1.00 \quad X(\tau)/X(F) = 1.00 \]

\[ q(\tau)/q^* = 0.38 \quad y(\tau)/y^* = 0.79 \quad Y(\tau)/Y^* = 0.74 \quad K(\tau)/K^* = 0.74 \quad H(\tau)/H^* = 0.81 \quad X(\tau)/X^* = 0.74 \]

Welfare Gains (%)

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>From 10% to Friedman Rule</td>
<td>2.08</td>
<td>5.65</td>
<td>1.59</td>
<td>5.41</td>
<td>4.57</td>
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<tr>
<td>During Transition</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.08</td>
<td>-0.79</td>
</tr>
<tr>
<td>From 10% to 0%</td>
<td>1.62</td>
<td>3.93</td>
<td>1.25</td>
<td>3.72</td>
<td>3.21</td>
</tr>
<tr>
<td>During Transition</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.04</td>
<td>-0.45</td>
</tr>
</tbody>
</table>

The Friedman Rule

\[ q(\tau)/q(F) = 0.53 \quad y(\tau)/y(F) = 1.00 \quad Y(\tau)/Y(F) = 0.94 \quad K(\tau)/K(F) = 1.00 \quad H(\tau)/H(F) = 1.00 \quad X(\tau)/X(F) = 1.00 \]

\[ q(\tau)/q^* = 0.38 \quad y(\tau)/y^* = 0.79 \quad Y(\tau)/Y^* = 0.74 \quad K(\tau)/K^* = 0.74 \quad H(\tau)/H^* = 0.81 \quad X(\tau)/X^* = 0.74 \]

Welfare Gains (%)

<table>
<thead>
<tr>
<th></th>
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<tr>
<td>During Transition</td>
<td>0.00</td>
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<td>0.00</td>
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</tr>
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<td>3.93</td>
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<td>3.72</td>
<td>3.21</td>
</tr>
<tr>
<td>During Transition</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.04</td>
<td>-0.45</td>
</tr>
</tbody>
</table>
rule by about 0.6 percentage points.

5 Conclusions

In this paper we have taken another step towards closing the gap between search models of money and standard macro models. We have shown how deriving the demand for money from first principles can be incorporated in the neoclassical growth model and how monetary policy affects aggregate output, employment and consumption. The key point of our paper is that there are many links by which changes in the value of money in the search market spill over to affect real variables in markets that do not require the use of money for exchange.

In this paper, we also took the first step of taking this model to the data and answering some interesting questions. In ongoing research, we add technology and money growth shocks in to our model and study important issues such as business cycle properties of the model as well as the welfare cost of inflation.
A Nonseparable Utility

Here we consider the model with utility nonseparable in \((x,q,e)\), but still linear in \(h\), say \(\hat{U}(x,q,e) - Ah\). Since \(q\) and \(e\) are determined during the day, they are state variables in the CM. For this section we assume that capital is not used for production during the day so \(q = f(e)\). Furthermore, for presentation purposes, we eliminate fiscal policy and assume monetary injections occur via lump-sum injections in the CM. Let \(W(m,k,q,e)\) now denote the value function at night

\[
W(m,k,q,e) = \max_{x,h,m+1,k+1} \hat{U}(x,q,e) - Ah + \beta_1 V(m+1,k+1)
\]

s.t. \(x = wh + \frac{m + \tau - (1 + \tau) m + 1}{p} + (1 + r - \delta) k - k + 1\).

Substituting for \(h\) yields

\[
W(m,k,q,e) = \frac{A}{w} \left[ \frac{m + \tau}{p} + (1 + r - \delta) k \right] + \max_{x,m+1,k+1} \left[ \hat{U}(x,q,e) - \frac{A}{w} \left[ x + \frac{(1 + \tau) m + 1}{p} + k + 1 \right] + \beta_1 V(m+1,k+1) \right].
\] (67)

The first-order conditions are given by:

\[
x : \hat{U}_x(x,q,e) = \frac{A}{w}
\] (68)

\[
k+1 : \frac{A}{w} = \beta_1 V_k(m+1,k+1)
\] (69)

\[
m+1 : \frac{A}{pw} = \beta_1 V_m(m+1,k+1)
\] (70)

Hence we again have a degenerate distribution of \((m,k)\). More importantly for this section, the choice of \(x\) in the night market is affected by how much the agent consumed or produced in the day market. The envelope conditions are

\[
W_m(m,k,q,e) = \frac{A}{pw}
\] (71)

\[
W_k(m,k,q,e) = \frac{A}{w} (1 + r - \delta)
\] (72)

\[
W_q(m,k,q,e) = \hat{U}_q(x,q,e)
\] (73)

\[
W_e(m,k,q,e) = \hat{U}_e(x,q,e).
\] (74)

46
Suppose that during the day agents meet and bargain bilaterally. The bargaining problem is \( \max S_b^{a} S_s^{1-\theta} \) subject to \( q = f(e) \) and \( d \leq m \), where now we have
\[
S_b = W(m_b - d, k_b, q, 0) - W(m_b, k_b, 0, 0) \\
S_s = W(m_s + d, k_s, 0, e) - W(m_s, k_s, 0, 0)
\]

By the usual logic, one can show \( d = m_b \). Using this and \( e = \psi(q) = f^{-1}(q) \), the first order condition with respect to \( q \) can be written
\[
\theta S_s \dot{U}_q(x_b, q, 0) + (1 - \theta) S_b \dot{U}_e[x_s, 0, \psi(q)] \psi'(q) = 0. \quad (75)
\]

Agents generally choose different values of \( x \) in the night market. Letting \( x_s, x_b, \) and \( x_0 \) be the quantities purchased by day market sellers, buyers and non-traders, we have
\[
S_b = \dot{U}(x_b, q, 0) - \dot{U}(x_0, 0, 0) - \frac{A}{w} \left( x_b - x_0 + \frac{m_b}{p} \right) \\
S_s = \dot{U}[x_s, 0, \psi(q)] - \dot{U}(x_0, 0, 0) - \frac{A}{w} \left( x_s - x_0 - \frac{m_b}{p} \right).
\]

From the FOC for \( x \),
\[
\dot{U}_x[x_s, 0, \psi(q)] = \frac{A}{w} \\
\dot{U}_x(x_b, q, 0) = \frac{A}{w} \\
\dot{U}_x(x_0, 0, 0) = \frac{A}{w}
\]

From these we get the equilibrium choices \( x_s = x_s[\psi(q), \frac{A}{w}] \), \( x_b = x_b(q, \frac{A}{w}) \) and \( x_0 = x_0(\frac{A}{w}) \).

Then we can solve (75) to obtain
\[
\frac{A}{pw} m_b = g \left( q, \frac{A}{w} \right)
\]

where
\[
g \left( q, \frac{A}{w} \right) = \frac{(1 - \theta) \left\{ U \left[ x_0(\frac{A}{w}), 0, 0 \right] - U \left[ x_b(\frac{A}{w}), q, 0 \right] \right\} U_e \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} \psi'(q)}{\theta U_q \left[ x_b(q, \frac{A}{w}), q, 0 \right] - (1 - \theta) U_e \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} \psi'(q)} + \frac{\theta \left\{ U \left[ x_0(\frac{A}{w}), 0, 0 \right] - U \left[ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right] \right\} U_s \left[ x_b(q, \frac{A}{w}), q, 0 \right]}{\theta U_q \left[ x_b(q, \frac{A}{w}), q, 0 \right] - (1 - \theta) U_e \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} \psi'(q)} + \frac{(1 - \theta) \frac{A}{w} \left[ x_b(q, \frac{A}{w}) - x_0(\frac{A}{w}) \right] U_e \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} \psi'(q)}{\theta U_q \left[ x_b(q, \frac{A}{w}), q, 0 \right] - (1 - \theta) U_e \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} \psi'(q)} + \frac{\theta \frac{A}{w} \left\{ x_s \left[ \psi(q), \frac{A}{w} \right] - x_0(\frac{A}{w}) \right\} U_q \left[ x_b(q, \frac{A}{w}), q, 0 \right]}{\theta U_q \left[ x_b(q, \frac{A}{w}), q, 0 \right] - (1 - \theta) U_e \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} \psi'(q)}.
\]
The key observation here is that $A/w$ enters $g$. If $U = U(x) + u(q) - \eta(e) - Ah$ is separable, then $g(q, \frac{A}{w}) = g(q)$ reduces to the model in the text with no capital used in the DM – that is, to a model that dichotomizes. Also, for any $U$, if $\theta = 1$ then the previous equation reduces to

$$\frac{A}{pw}m_b = U \left[ x_0(\frac{A}{w}), 0, 0 \right] - U \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} + \frac{A}{w} \left\{ x_s \left[ \psi(q), \frac{A}{w} \right] - x_0(\frac{A}{w}) \right\}.$$ 

Notice

$$g_q(q, \frac{A}{w}) = -U_e \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} \psi'(q) > 0$$

since $U_x \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} = \frac{A}{w}$ from the first order condition for $x$. If $U = U(x, q) - \eta(e) - Ah$, then the first order conditions imply $x_s \left[ \psi(q), \frac{A}{w} \right] = x_0(\frac{A}{w})$, and this becomes

$$\frac{A}{pw} m = \eta[\psi(q)] = c(q).$$

The value function in the DM is given by

$$V(m, k) = \beta_2 \{ \sigma W(m, k, q, 0) + \sigma W(m, k, 0, \psi(q)) + (1 - 2\sigma) W(m, k, 0, 0) \} \quad (76)$$

By the usual methods the first order condition for $m$ is

$$(1 + \tau) \frac{A}{w} = \beta \frac{A}{w+1} \left[ 1 - \sigma + \frac{U_q \left[ x_b(q, \frac{A}{w+1}, q+1, 0) \right]}{g_q(q, \frac{A}{w+1})} \right]$$

or

$$(1 + \tau) g(q, \frac{A}{w}) = \beta g(q+1, \frac{A}{w+1}) \left[ 1 - \sigma + \frac{U_q \left[ x_b(q+1, \frac{A}{w+1}, q+1, 0) \right]}{g_q(q+1, \frac{A}{w+1})} \right]$$

It is clear from this expression that $q$ cannot be determined independently of $w$ which in turn is a function of $K$ via $w = F_H(K, H)$. A steady-state satisfies

$$1 + \frac{i}{\sigma} = \frac{U_q \left\{ x_b \left[ q, \frac{A}{F_H(K, H)} \right], q, 0 \right\}}{g_q \left[ q, \frac{A}{F_H(K, H)} \right]}$$

$$\rho + \delta = F_K(K, H)$$

$$x = F(K, H) - \delta K$$

$$H = \frac{x - [F_K(K, H) - \delta] K}{F_H(K, H)}$$
and

\[ H = \sigma h_s + \sigma h_b + (1 - 2\sigma) h_0 \]
\[ x = \sigma x_b \left( \psi(q), \frac{A}{w} \right) + \sigma x_s \left[ \psi(q), \frac{A}{w} \right] + (1 - 2\sigma) x_0 \left( \frac{A}{w} \right) \]

where

\[ h_s = H + \frac{1}{F_H(K, H)} \left( x_s \left[ \psi(q), \frac{A}{w} \right] - x \right) - \frac{M}{p} \frac{A}{F_H(K, H)} \]
\[ h_b = H + \frac{1}{F_H(K, H)} \left( x_b \left( q, \frac{A}{w} \right) - x \right) + \frac{M}{p} \frac{A}{F_H(K, H)} \]
\[ h_0 = H + \frac{1}{F_H(K, H)} \left( x_0 \left( \frac{A}{w} \right) - x \right) \]

with \( h_s, h_b, h_0 \) denoting the hours worked in the night market by day market sellers, buyers and non-traders respectively. It is clear from this equation that unless \( q \) disappears from \( x \) when aggregating over \( x_b, x_s \) and \( x_0 \), the dichotomy is broken and changes in \( i \) affect \( x, H \) and \( K \). So monetary policy affects \( q \) and \( e \) and this spills over to affect consumption, hours worked and capital accumulation in the night market. For \( \theta = 1 \), we have

\[ 1 + \frac{i}{\sigma} = \frac{U_q \left\{ x_b \left[ q, \frac{A}{F_H(K, H)} \right], q, 0 \right\}}{-U_e \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} \psi'(q)} \] \hspace{1cm} (77)

Under the Friedman rule, this reduces to

\[ U_q \left\{ x_b \left[ q, \frac{A}{F_H(K, H)} \right], q, 0 \right\} = -U_e \left\{ x_s \left[ \psi(q), \frac{A}{w} \right], 0, \psi(q) \right\} \psi'(q) \]

which is the efficiency condition for producing \( q \) in the day market.

Under Walrasian pricing, buyers in the day market solve the following problem

\[ \max_{q_b} W(m_b - \tilde{p} q_b, k_b, q_b, 0) \]
\[ \text{s.t.} \quad \tilde{p} q_b \leq m_b \]

where \( \tilde{p} \) is the money price of goods. The seller’s problem is

\[ \max_{q_s} W \left[ m_s + \tilde{p} q_s, k_s, 0, \psi(q_s) \right] \]

The seller’s first-order condition is

\[ W_{m} \left[ m_s + pq_s, k_s, 0, \psi(q_s) \right] \tilde{p} + W_{e} \left[ m_s + pq_s, k_s, 0, \psi(q_s) \right] \psi'(q_s) = 0 \]
or
\[
\frac{p_A}{p_w} = -U_e [x_s, 0, \psi(q_s)] \psi'(q_s)
\]

By the usual methods, the first-order condition for \( m \) becomes
\[
(1 + \tau) g \left[ q, x_s \left( \frac{A}{w} \right) \right] = \beta g \left[ q_{+1}, x_s \left( \frac{A}{w_{+1}} \right) \right] \left\{ 1 - \sigma + \sigma \frac{U_q \left[ x_b(q_{+1}, \frac{A}{w_{+1}}), q_{+1}, 0 \right]}{-U_e \left\{ x_s \left[ \psi(q_{+1}), \frac{A}{w_{+1}} \right], 0, \psi(q_{+1}) \right\} \psi'(q_{+1})} \right\}
\]

In steady state
\[
1 + \frac{i}{\sigma} = \frac{U_q \left\{ x_b \left[ q, \frac{A}{F_H(K,H)} \right], q, 0 \right\}}{-U_e \left\{ x_s \left[ \frac{A}{F_H(K,H)} \right], 0, \psi(q) \right\} \psi'(q)} \tag{78}
\]

Equation (78 is equal to the bargaining steady state under bargaining with \( \theta = 1 \), (77).
B Existence and Uniqueness of Equilibrium

B.1 Baseline Model

Assume constant returns to scale production function for general goods. So

\[ \frac{F(K, H)}{H} = F(K/H, 1) = F(\kappa) \]

where \( \kappa = K/H \). In the steady state of the baseline model with Walrasian pricing we have the following four equations:

\[ 1 + \frac{i}{\sigma} = \frac{u'(q)}{c_q(q, K)} \]  
(79)

\[ \rho = [F_K(k) - \delta] (1 - t_k) - (1 + t_x) \frac{\sigma c_K(q, K)}{\beta^2 U'(X)} \]  
(80)

\[ A (1 + t_x) = U'(X) F_H(k) (1 - t_h) \]  
(81)

\[ X + G = H [F(\kappa) - \delta \kappa] \]  
(82)

From (81) we get

\[ X = U'^{-1} \left[ \frac{A (1 + t_x)}{F_H(k) (1 - t_h)} \right] \]  
(83)

which combined with (82) yields

\[ H = \frac{U'^{-1} \left[ \frac{A (1 + t_x)}{F_H(k) (1 - t_h)} \right]}{\kappa \left[ F(\kappa)/\kappa - \delta \right]} + G \]  
(84)

which implies

\[ K = \frac{K}{H} \frac{U'^{-1} \left[ \frac{A (1 + t_x)}{F_H(k) (1 - t_h)} \right]}{F(\kappa)/\kappa - \delta} + G \]  
(85)

From (85) we obtain

\[ \frac{dK}{d\kappa} = \frac{-A (1 + t_x) F_{HK}(k) - \left\{ U'^{-1} \left[ \frac{A (1 + t_x)}{F_H(k) (1 - t_h)} \right] + G \right\} [F_K(\kappa) \kappa - F(\kappa)]}{[F(\kappa)/\kappa - \delta]^2 \kappa^2} \]
Equation (79) yields
\[
\frac{\partial q}{\partial K} = \frac{(1 + \frac{i}{\sigma}) c_{qK}(q, K)}{u''(q) - (1 + \frac{i}{\sigma}) c_{qq}(q, K)} > 0
\]
\[
\frac{\partial q}{\partial i} = \frac{c(q, K)}{\sigma [u''(q) - (1 + \frac{i}{\sigma}) c_{qq}(q, K)]} < 0
\]

Using (85) and (79) we get
\[
u'(q) = \left(1 + \frac{i}{\sigma}\right) c_q \left\{ q, \frac{U'^{-1} \left[ \frac{A(1+t_x)}{F_H(k)(1-t_h)} \right] + G}{F(k)/k - \delta} \right\}
\]
\[
\Rightarrow \quad q = q(k, i)
\]

Finally, we can rewrite (80) as
\[
\rho = [F_K(k) - \delta] (1 - t_k) - \frac{\sigma F_H(k) (1-t_h)}{A\beta_2} c_K \left\{ q(k, i), \frac{U'^{-1} \left[ \frac{A(1+t_x)}{F_H(k)(1-t_h)} \right] + G}{F(k)/k - \delta} \right\} \equiv N(k)
\]
(86)

A steady state for the baseline model is a value \( k \) that solves (86). From this expression we have that
\[
\frac{\partial N(k)}{\partial K} = F_{KK}(k) (1 - t_k)
\]
\[
- \frac{\sigma (1 - t_h)}{A\beta_2} F_{HK}(k) c_K \left\{ q(k, i), \frac{U'^{-1} \left[ \frac{A(1+t_x)}{F_H(k)(1-t_h)} \right] + G}{F(k)/k - \delta} \right\}
\]
\[
- \frac{\sigma F_H(k) (1-t_h)}{A\beta_2} \left( c_{qK} \frac{\partial q}{\partial K} + c_{KK} \frac{\partial K}{\partial K} \right)
\]

The first term is negative. The second term is positive if \( F_{HK}(k) \) is positive. The third term is ambiguous. Thus without further restrictions on the properties of the cost function, it is not possible to say anything about existence or uniqueness of the equilibrium.
B.2 Two Types of Capital

For this model with Walrasian pricing, replace \( c(q, K) \) with \( c(q, Z) \) where \( Z \) is special capital. The steady-state conditions are

\[
1 + \frac{i}{\sigma} = \frac{u'(q)}{c'(q, Z)} \tag{87}
\]

\[
\rho = [F_K(\kappa) - \delta] (1 - t_k) \tag{88}
\]

\[
\rho = - \left[ \omega + (1 + t_x) \sigma \frac{c_z(q, Z)}{\beta_2 U'(X)} \right] \tag{89}
\]

\[
A (1 + t_x) = U'(x) F_H(\kappa) (1 - t_h) \tag{90}
\]

\[
X = H [F(\kappa) - \delta \kappa] - \omega Z \tag{91}
\]

From (88), the steady state value of \( \kappa \) is given by

\[
\kappa = F_K^{-1} \left( \frac{\rho}{1 - t_k} + \delta \right) \tag{92}
\]

As before (90) gives us (83) which in conjunction with (92) gives

\[
X = U'^{-1} \left\{ \frac{A (1 + t_x)}{F_H \left[ F_K^{-1} \left( \frac{\rho}{1 - t_k} + \delta \right) \right]} \right\} \tag{93}
\]

Equation (87) can be written to obtain

\[
u'(q) = \left( 1 + \frac{i}{\sigma} \right) c_q(q, Z)
\]

\[
\Rightarrow q = q(Z, i), \text{ with } q_z(Z, i) > 0 \text{ and } q_i(Z, i) < 0
\]

where \( q \) is unique given \( Z \). Consequently, (89) becomes

\[
\rho + \omega = - \frac{\sigma}{\beta_2 A} F_H \left[ F_K^{-1} \left( \frac{\rho}{1 - t_k} + \delta \right) \right] c_z [q(Z, i), Z] \tag{94}
\]

which pins down \( Z \) if a solution exists. Finally, using (91), (93) and (94) we get

\[
X = H \left\{ F \left[ F_K^{-1} \left( \frac{\rho}{1 - t_k} + \delta \right) \right] - \delta F_K^{-1} \left( \frac{\rho}{1 - t_k} + \delta \right) \right\} - \omega Z
\]

which reduces to

\[
\bar{H} = \frac{x + \omega Z}{\left\{ F \left[ F_K^{-1} \left( \frac{\rho}{1 - t_k} + \delta \right) \right] - \delta F_K^{-1} \left( \frac{\rho}{1 - t_k} + \delta \right) \right\}} \tag{95}
\]

Thus, if a solution to (94) exists, then \( q, x, K, \) and \( H \) are all uniquely determined.
C Computational Issues

C.1 Markup

The markup is defined as the ratio of price and marginal cost. The markup in the CM market is zero since it is a competitive market. Similarly, the markup in the DM market under competitive pricing is also zero. In case of bargaining, the marginal cost for production in the DM market in terms of utility is given by

\[ MC_D = c_q (q, K) \]  

Due to the quasi-linearity of the utility function, one can convert one unit of money in to \( \frac{A \beta_2}{p (1 - t_h) w} \) units of utilis,\(^{29}\) which means the marginal cost in terms of money is

\[ MC_D = \frac{c_q (q, K)}{A \beta_2} \frac{1}{p (1 - t_h) w} = \frac{p (1 - t_h) wc_q (q, K)}{A \beta_2} \]  

and combining the price (for the seller) of \( (1 - t_d) \frac{M}{q} \), yields the markup

\[ 1 + \mu_D = \frac{P_D}{MC_D} = \frac{(1 - t_d) \frac{M}{q}}{p (1 - t_h) wc_q (q, K)} = \frac{(1 - t_d) M A \beta_2}{p (1 - t_h) wc_q (q, K) q} = \frac{(1 - t_d) g (q, K)}{c_q (q, K) q} \]  

where the last equality follows from the solution of the bargaining problem.

\(^{29}\)There are two ways to see this. One unit of money corresponds to \( \frac{1}{p (1 - t_h) w} \) units of labor from the budget constraint and this has a utility of \( \frac{A \beta_2}{p (1 - t_h) w} \) since it is discounted from DM to CM. Alternatively, with one unit of money, the agent can purchase \( 1 / p (1 + t_x) \) units of the general good and this extra good brings in \( \beta_2 U' (X) \) units extra utility. From the first order conditions \( U' (X) = \frac{A (1 + t_x) \beta_2}{w (1 - t_h) w} \) which means the additional utility from giving up one unit of money is \( \frac{\beta_2 A}{p (1 - t_h) w} \).
Combining the two markups with the weights of each market, we get

\begin{align}
\mu &= s_D \mu_D + (1 - s_C) \mu_C \\
&= s_D \left[ \frac{(1 - t_d) g (q, K)}{c_q (q, K) q} - 1 \right] 
\end{align}

(102)

where \( s_D \) is the share of the decentralized market.

### C.2 Money Demand Estimation

A generally accepted way of estimating money demand (See, for example, Goldfeld and Sichel, 1990) consists of estimating the following equation

\begin{align}
\log \left( \frac{M_t}{P_t} \right) &= \omega + \beta_y \log Y_t + \beta_i \log i_t + \lambda \log \left( \frac{M_{t-1}}{P_{t-1}} \right) + u_t \\
&= \rho u_{t-1} + \xi_t \text{ where } \xi_t \sim N \left( 0, \sigma^2_{\xi} \right) 
\end{align}

(103)

We include an AR(1) specification for the residuals in order to have a consistent estimate of money demand, following relevant empirical literature and its existence will not alter the interpretation of the other parameters.

The long-run interest elasticity of money demand is given by

\[ \xi_{M,i} = \frac{\beta_i}{1 - \lambda} \]

(104)

and this will be one of the targets of our calibration. Below we explain how we obtain the counterpart of \( \xi_{M,i} \) from our model.

#### C.2.1 Bargaining Version

The interest elasticity of money demand is defined as

\[ \xi_{M,i} = \frac{\partial (M/P)}{\partial i} \frac{i}{M/P} \]

(106)

Using (55), we can express \( \frac{\partial (M/P)}{\partial i} \) as

\[ \frac{\partial (M/P)}{\partial i} = \frac{(1 - t_h)}{\beta_2 A} \left[ \frac{\partial g (q, K)}{\partial i} f_h (q) + \frac{\partial f_h (q)}{\partial i} g (q, K) \right] \]

(107)
We can further write the components of (108) as
\[ \frac{\partial g(q, K)}{\partial i} = g_q(q, K) \frac{\partial q}{\partial i} + g_K(q, K) \frac{\partial K}{\partial i} \] (109)
\[ \frac{\partial f_h(\|k\|)}{\partial i} = \frac{\partial f_h(\|k\|)}{\partial k} \frac{\partial k}{\partial i} \] (110)

Totally differentiating (18) we get
\[ di = \sigma g_q(q, K) u''(q) dq - u'(q) [g_{qq}(q, K) dq + g_{qk}(q, K) dK] \]
\[ = \frac{\sigma}{[g_q(q, K)]^2} \left\{ [g_q(q, K) u''(q) - u'(q) g_{qq}(q, K)] dq - u'(q) g_{qk}(q, K) dK \right\} \] (112)
which yields partial derivatives
\[ \frac{\partial q}{\partial i} = \frac{[g_q(q, K)]^2}{\sigma} \left[ \frac{1}{g_q(q, K) u''(q) - u'(q) g_{qq}(q, K)} \right] \] (113)
\[ \frac{\partial K}{\partial i} = -\frac{[g_q(q, K)]^2}{\sigma u'(q) g_{qk}(q, K)} \] (114)

Note that
\[ \|k\| \equiv \frac{K}{H} \] (115)
where $H$ is a fixed number in our calibration. So
\[ \frac{\partial k}{\partial i} \equiv \frac{1}{H} \frac{\partial K}{\partial i} \] (116)
and we can compute the interest elasticity of money demand using these expressions.

### C.2.2 Competitive Pricing Version

The interest elasticity of money demand is defined as
\[ \xi_{M,P,i} = \frac{\partial (M/P)}{\partial i} \frac{i}{M/P} \] (117)

Using (56), we can express $\frac{\partial (M/P)}{\partial i}$ as
\[ \frac{\partial (M/P)}{\partial i} = \frac{(1 - t_h)}{(1 - t_a) \beta p A} \left\{ \frac{\partial c_q(q, K)}{\partial i} q f_h(\|k\|) + \left[ \frac{\partial q}{\partial i} f_h(\|k\|) + \frac{\partial f_h(\|k\|)}{\partial i} q \right] c_q(q, K) \right\} \] (118)

We can further write the components of (118) as
\[ \frac{\partial c_q(q, K)}{\partial i} = c_{qq}(q, K) \frac{\partial q}{\partial i} + c_{qk}(q, K) \frac{\partial K}{\partial i} \] (119)
\[ \frac{\partial f_h(\|k\|)}{\partial i} = \frac{\partial f_h(\|k\|)}{\partial k} \frac{\partial k}{\partial i} \] (120)
Totally differentiating (42) we get

\[ di = \sigma (1 - t_d) \frac{c_q(q, K) u''(q) dq - u'(q) [c_{qq}(q, K) dq + c_{qk}(q, K) dK]}{[c_q(q, K)]^2} \]

which yields partial derivatives

\[ \frac{\partial q}{\partial i} = \frac{[c_q(q, K)]^2}{\sigma (1 - t_d)} \left[ \frac{1}{c_q(q, K) u''(q) - u'(q) c_{qq}(q, K)} \right] \]

\[ \frac{\partial K}{\partial i} = -\frac{[c_q(q, K)]^2}{\sigma (1 - t_d) u'(q) c_{qk}(q, K)} \]
References


