I develop a stochastic version of a Townscend turnpike model to study contingent claims equilibria in which the contract must specify the location for delivery. The stochastic endowment is received by an agent at a particular site and goods cannot be moved across sites within a period. Transactions across agents at different sites will depend on the communication across sites and the restrictions on the delivery of the contract. Full communication means that agents can enter into contingent contracts with agents at other sites. Unrestricted delivery means that delivery can be guaranteed even if the counterparties are at different sites on the delivery date. I find that full communication with restricted delivery leads to partial insurance against aggregate risk, even though an agent can enter into a countable infinity of contracts. A model in which there is full communication but partially restricted delivery because of record-keeping constraints or limited netting of transactions is equivalent to a model with borrowing constraints. I also show that the Alvarez-Jermann model of endogenous solvency constraints is related to a particular netting scheme in which there is no default in equilibrium.
The standard contingent claims model doesn’t specify a site or location for delivery of goods, implicitly assuming that all agents are located at the same site or that goods can be costlessly and instantaneously moved across sites at a point in time. Models of incomplete markets sometimes assume that the incompleteness arises because of spatial separation, although many other models of market incompleteness are studied, such as private information or lack of enforcement mechanisms, for example. In this model I assume that at a point in time agents are characterized not only by the aggregate state and their type, but also by their location. The stochastic endowment is received by an agent at a particular site, and this nonstorable consumption good cannot be transported across sites within a time period. The model is based on the Townscend turnpike model, with aggregate risk incorporated.

In this setting, the Arrow-Debreu complete markets allocation can be achieved if contingent claims are pooled across all sites and agents. All contracts must specify the delivery site. If trade is decentralized and takes among agents at different sites, then there be full communication across locations. Specifically, agents located at different sites at time $t$ must be able to communicate and to enter into contingent claims contracts. Second, there must be unrestricted delivery on the contracts. This means that the two counterparties in the contract can be located at different sites at the delivery date. I examine several versions of the model under different assumptions about communication across sites and restrictions on delivery.

As a convenient benchmark, I solve the central planning problem and construct a competitive equilibrium under the assumptions of full communication and unrestricted delivery. When there is no communication across sites, so that agents can enter into exchange only with other agents at that site, then the only solution is autarchy. Next I examine aggregate risk sharing under the assumption that there is full communication but delivery on a contract can take place only if the coun-
terparties are located at the same site on the delivery date. Even though an agent can enter into a countable infinity of contingent claim contracts, there is only partial insurance against aggregate risk. I then allow a clearing house to facilitate delivery on contracts but assume that there are technological constraints on record keeping and netting schemes. I show that these record keeping constraints are equivalent to borrowing constraints. Finally I examine the Alvarez and Jermann model of endogenous solvency constraints to show that it implies a particular netting scheme in which there is no default in equilibrium.

1 Description of the Model

An agent in this model is indexed by his type, his location, the date, and the history of the system. The model is based on Townsend’s turnpike model. There are two types of agents: type $E$ (east-moving) and type $W$ (west-moving). There is a countable infinity of each type of agent. Let $I$ denote the set of all integers (positive and negative) and let $\psi$ denote a finitely additive cumulative distribution function. At time $t$, an agent is characterized by his type $E$ or $W$ and his location $i \in I$. In period $t + 1$, an $E$-type will move to site $i + 1$ while the $W$-type agent will move to site $i - 1$, and the agents $E^i$ and $W^i$ will meet only once.

At each site and in each time period, each type of agent receives a stochastic and exogenous endowment. The exogenous endowments follow a stationary, first-order Markov chain. Let $s_t \in S = \{\epsilon_1, \ldots, \epsilon_n\}$. A type $E$ agent at site $i$ has a nonstorable endowment $y^i_e : S \to Y = [y, \bar{y}]$, where $y \geq 0$. Type $W$ agent at site $i$ has nonstorable endowment $y^i_w : S \to Y$. Denote $\bar{y}^i(s) = y^i_e(s) + y^i_w(s)$ as total endowment in state $s$ at site $i$. A key assumption is that the endowment cannot be moved across sites during the time period it is received and it is nonstorable. Moreover let $y^i_e = y_e$ and
\[ y^i_w = y_w \] for all \( i \), so that type \( E \) agents are identical across sites, as are type \( W \).

Define \( \pi_{i,j} = \text{prob}(s_{t+1} = x_j \mid s_t = x_i) \) for \( i, j = 1, \ldots, n \). Define \( \Pi \) as the \( n \times n \) matrix of transition probabilities with \( (i,j) \)-element \( \pi(s_j \mid s_i) \), where summation across a row equals one. Finally, let \( \tilde{\pi}(s) \) denote the unconditional probability of being in state \( s \), equal to the sum of a column of the matrix \( \Pi \). Let \( \tilde{\Pi} \) denote the vector of unconditional probabilities. Let \( s^t = (s_1, \ldots, s_t) \) be the history of realizations up to time \( t \) and let \( \pi_t(s^t) \) denote the probability of \( s^t \), where \( s^t \in S^t = S \times \cdots \times S \).

Hence, at time \( t \) an agent is characterized by his type, location \( i \) where \( i \in I \), and the common history \( s^t \).

If at time \( t \) a type \( E \) agent is located at site \( i \) and a type \( W \) agent is located at site \( j \), then the following set of potential interactions are possible. If \( j < i \), then the two agents never meet in the future. If \( i = j \), then the agents are present at the same site at the same point of time but never meet again. If \( j > i \), then the two agents may potentially be at the same site at the same time. If \( j - i \) is an even (and positive) number, then the agents are at the same site at time \( t + \frac{j-i}{2} \). If \( j - i \) is odd, then the two agents are never at the same site at the same point in time. Agents at a location act as price takers.

A type \( E \) agent has preferences over consumption bundles described by

\[
\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t)U(c_t).
\] (1)

Let \( c^i_t(s^t) \) denote the consumption of a type \( E \) agent at time \( t \), location \( i \) when the history is \( s^t \). The type \( W \) agent has preferences over consumption bundles described by

\[
\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t)W(\eta_t).
\] (2)

Let \( \eta^i_t(s^t) \) denote the consumption of a type \( W \) agent in location \( i \) at time \( t \) when history is \( s^t \).
The functions $U, W$ are assumed to be strictly increasing and strictly concave and twice continuously differentiable. Let $U_1, W_1$ denote the first derivatives and assume the Inada conditions hold: $\lim_{c \to 0} U_1(c) = \infty$ and $\lim_{c \to \infty} U_1(c) = 0$ for $U = U, W$.

2 First-Best Solution

It is convenient to start with the central planner’s problem and then construct the competitive equilibrium under the assumptions that there is full communication across sites and no restrictions on the delivery on contracts over different sites. Full communication means that agents located at different sites at a point in time can enter into contracts with agents at different sites. No restrictions on delivery of contracts means that delivery can be guaranteed even if the two counterparties are not present at the same site on the delivery date.

2.1 Central Planning Problem

The central planner allocates resources but is subject to the technology constraint that the consumption good is location-specific and cannot be transported across sites within a period. Let $\phi^i_{e,t}(s^t)$ denote the time $t$ Pareto weight attached to a type e agent, located at site $i$ when $t = 0$, when the state is $s^t$. Let $\phi^i_{w,t}(s^t)$ denote the time-t Pareto weight attached to a type W, located at site $i$ at $t = 0$, when the state is $s^t$. The central planner solves

$$\max \sum_I \sum_t \sum_{s^t} \left[ \phi^i_{e,t}(s^t)U(c^i_{t,t}(s^t)) + \phi^i_{w,t}(s^t)W(\eta^i_{t,t}(s^t)) \right]$$

subject to

$$y^i(s_t) = c^i_1(s^t) + \eta^i_1(s^t)$$
for all $i \in I$. Let $\lambda^i_t(s^t)$ denote the Lagrange multiplier for the resource constraint.

The first-order conditions are

$$
\phi^i_{e,t}(s^t)U_1(c^i_{t+i}(s^t)) = \lambda^i_{t+t}(s^t) \tag{4}
$$

$$
\phi^i_{w,t}(s^t)W_1(\eta^i_{t-t}(s^t)) = \lambda^i_{t+t}(s^t), \tag{5}
$$

where $i + t = j - t$. Stationary solutions can be determined by setting $\phi^i_{e,t}(s^t) = \beta^t \phi^i_e(s_t)$ and $\phi^i_{w,t}(s^t) = \beta^t \phi^i_w(s_t)$. Let $c^i : S \to Y$. An optimal allocation at site $i$ satisfies

$$
\frac{U_1(c^i_{t}(s_t))}{W_1(\bar{y}^i(s_t) - c^i_{t}(s_t))} = \frac{\phi^i_e(s_t)}{\phi^i_w(s_t)}
$$

where $i + t = j - t$. If all type e agents are viewed as identical regardless of initial location or over the state $s_t$, then $\phi^i_e = \phi^i_e(s_t)$ and $\phi^i_w = \phi^i_w(s_t)$. Let $K > 0$ be given and determine the solution $c$ to

$$
\frac{U_1(c)}{W_1(\bar{y}(s) - c)} = K
$$

Since $U$ is strictly concave and twice continuously differentiable, the inverse function theorem can be applied to define a function $g$ such that

$$
c = g(\bar{y}(s), K)
$$

where $g_K < 0$, so $c$ is decreasing in $K$. For the central planning problem, the stationary solution is

$$
c^i_*(s) = g(\bar{y}(s), \frac{\phi^i_e}{\phi^i_w}).
$$

### 2.2 Competitive Equilibrium Under Full Communication

In the competitive equilibrium with full communication and unrestricted delivery, agents located at different sites can enter into contracts and guarantee delivery on
those contracts even if the counterparties are at different sites on the delivery date. The delivery of goods when the counterparties are located at different sites on the delivery date requires the existence of a clearing house or a financial intermediary that keeps record of the countable infinity of contracts and state contingent deliveries over time. Although goods can’t be transported across sites at a point in time, under full communication and unrestricted delivery on contracts, contingent claims can be bought and sold in a centralized market meeting at time 0. Let \( q^i_t(s^t) \) denote the time 0 price of a unit of consumption at site \( i \) at time \( t \) contingent on history \( s^t \).

The type-e agent who is located at site \( i \) at \( t = 0 \) has a budget constraint

\[
0 = \sum_{t=0}^{\infty} \sum_{s^t} q^i_{t+t}(s^t)[y^i_{t+t}(s^t) - c^i_{t+t}(s^t)] \tag{6}
\]

Let \( \lambda^i \) denote the Lagrange multiplier, and assume that \( \lambda_i = \lambda \) for all \( i \in I \). The first-order condition is

\[
\beta^i \pi_t(s^t) U_1(c^i_{t+t}(s^t)) = \lambda q^i_{t+t}(s^t). \tag{7}
\]

The budget constraint for a type \( w \) agent who starts at location \( i \) at \( t = 0 \) is

\[
0 = \sum_{t=0}^{\infty} \sum_{s^t} q^i_{t-t}(s^t)[y^i_{t-t}(s^t) - \eta^i_{t-t}(s^t)]. \tag{8}
\]

Let \( \phi \) denote the Lagrange multiplier. The first-order condition is

\[
\beta^i \pi_t(s^t) W_1(\eta^i_{t-t}(s^t)) = \phi q^i_{t-t}(s^t) \tag{9}
\]

At site \( i \), the market clearing condition is

\[
y^i(s_t) - c^i(s^t) - \eta^i_t(s^t) = 0 \tag{10}
\]

The first-order conditions for the two agents located at site \( i \) at time \( t \) can be solved for the price to obtain

\[
\frac{W_1(\eta^i_t(s^t))}{\phi} = \frac{q^i_t(s^t)}{\beta^i \pi_t(s^t)} = \frac{U_1(c^i_t(s^t))}{\lambda} \tag{11}
\]
Consider stationary solutions of the form \( c^i : S \rightarrow Y \) for \( i \in I \). With market clearing at each site, (11) can be rewritten with market-clearing as

\[
\frac{U_1(c^i(s_t))}{W_1(\bar{y}^i(s_t) - c^i(s_t))} = \frac{\lambda}{\phi}.
\]

(12)

Observe that the left side is strictly decreasing in \( c \). The stationary solution is

\[
c(s_t) = g \left( s_t, \frac{\lambda}{\phi} \right).
\]

To determine the value of \( \frac{\lambda}{\phi} \), substitute for the equilibrium price into the type e’s budget constraint

\[
0 = \sum_{t=0}^{\infty} \sum_{s_t} \beta^t \pi_t(s^t) U_1 \left( g(s_t, \frac{\lambda}{\phi}) \right) \left[ y_e(s_t) - g \left( s_t, \frac{\lambda}{\phi} \right) \right].
\]

(13)

The right side is strictly increasing in \( \frac{\lambda}{\phi} \), hence there exists a unique solution \( \lambda^* \).

As is well known, this solution has the property that the marginal rate of substitution (MRS) between the two agents at time \( t \) is equal across states for all states \( s_t \in S \). Moreover, the MRS is equal to a constant \( \lambda^* \), regardless of history \( s^{t-1} \).

To see that unrestricted delivery and full communication allow borrowing and lending, define

\[
A^i_{1,t} \equiv q^i_{t}(s^t)y^i_e(s_t)
\]

and more generally define

\[
A^i_{n,t} = q^i_{t}(s^t)y^i_e(s_t) + \sum_{t+1} A^i_{n-1,t+1}(s^{t+1}),
\]

which equals the discounted present value of endowment for \( n \) periods, including the current period, measured in time 0 prices. Similarly for the type W agent define

\[
B^i_{1,t} = q^i_{t}(s^t)y^i_w(s_t)
\]

and

\[
B^i_{n,t} = q^i_{t}(s^t)y^i_w(s_t) + \sum_{t+1} B^i_{n-1,t+1}(s^{t+1}).
\]
For the type $E$ agent located at site $i$ at time $t$, the recursive formulation of the problem is

$$V_e(i, D_e, s_t) = \max[U(c^i_t) + \beta \sum_{s_{t+1}} \pi(s_{t+1} | s_t)V_e(i + 1, D_{t+1,e}(s_{t+1}))] \quad (14)$$

subject to

$$y_e(s_t) + D_e = c^i_t + \sum_{s_{t+1}} \hat{q}_{t+1}(s_{t+1})D_{t+1,e}(s_{t+1}) \quad (15)$$

and the no-Ponzi scheme condition

$$-D_{e,t+1}(s^i_t) \leq A_{e,\infty}(s^i_t) \quad (16)$$

Let $\lambda_t$ denote the Lagrange multiplier for the budget constraint and $\mu_{t+1}$ denote the multiplier on the no-Ponzi scheme. The first-order conditions are

$$U'(c) = \lambda_t \quad (17)$$

$$\lambda_t q(s') = \mu_{t+1} \beta \pi V' \quad (18)$$

$$V' = \lambda_t \quad (19)$$

Since marginal utility tends to infinity as consumption goes to zero, the agent will never choose to borrow the full expected discounted present value of his future income, so that $\mu_{t+1} = 0$. But notice that the agent will choose to set

$$\sum_{s_{t+1}} \hat{q}_{t+1}(s_{t+1})D_{t+1,e}(s_{t+1})$$

equal to non-zero values. Since these trades are settled with type $W$ agents located at site $i + 2$ at time $t$, the positive or negative portfolio requires borrowing or lending relative to the type $W$ agent at site $i$ at time $t$. 
3 Limited Communication and Restricted Delivery

The centralized trading of claims in the first-best solution allowed agents to share completely aggregate risk, as summarized in (). The other extreme, no communication across sites at a point in time, leads to no trading and no sharing of aggregate risk, a point demonstrated in the section below. Between the two extreme cases, there are a variety of intermediate cases of economic interest. I first examine an economy in which there is full communication, but delivery is restricted to counterparties that are located at the same site on the delivery date. While some risk-sharing occurs, the arrangement falls short of full risk sharing because no intermediation occurs. Next I introduce a clearing house that can guarantee delivery on contracts between counterparties not present at the same site on the delivery date. This requires intermediation on the part of the clearing house. The clearing house may use different procedures for netting across the various transactions, because of technology constraints on record keeping for example, and these netting processes lead to borrowing constraints.

3.1 No Communication and Restricted Delivery

Full information and unrestricted delivery leads to the first-best solution, described above. At the other extreme, is an economy in which there is no communication among agents located at different sites and delivery on contracts is restricted. No communication means that contracts can be written only when the two counterparties are located at the same site on the date the contract is written. By restricted delivery, I mean that delivery on a contract can occur only if the counterparties are located at the same site at the delivery date. Hence the only available trading opportunities are with agents located at the same site in the current period. Under these assumptions,
the only equilibrium is the autarchy solution.

An agent at site $i$ at time $t$ with history $s^t$ can only trade contingent contracts with other agents located at site $i$ at time $t$. Moreover, assume that delivery on the contract can occur only if the two agents are located at the same site on the delivery date. Then clearly agents will not trade because delivery on any contract negotiated with the other agent at site $i$ cannot occur. In this case, the only solution is the autarchy solution because the two types of agents at site $i$ are only present in the market at the same site in the current period. It is useful to determine the pricing system in the autarchy solution.

The budget constraint for type E at site $i$ in period $t$ is

$$0 = q^i_t(s^t)[y^i_e(s_t) - c^i_t(s^t)]$$

(20)

Let $\lambda^i_{e,t}(s^t)$ denote the Lagrange multiplier for the type $e$ agent at site $i$ on date $t$, given history $s^t$. The type $e$ agent solves

$$\max_c \sum_t \sum_{s^t} \beta^t \pi_t(s^t)U(c^i_{t+t}) + \sum_t \sum_{s^t} \lambda^i_{t+t}(s^t)q^i_{t+t}(s^t)[y^i_{t+t}(s_t) - c^i_{t+t}]$$

(21)

The first-order condition is

$$\beta^t \pi_t(s^t)U_1(c^i_{t}(s^t)) = \lambda^i_t(s^t)q^i_t(s^t).$$

(22)

The type W maximizes his objective function subject to the budget constraint

$$0 = q^i_t(s^t)[y^i_w(s_t) - \eta^i_t(s^t)].$$

Let $\phi^i_t(s^t)$ denote the Lagrange multiplier. The first-order condition is

$$\beta^t \pi_t(s^t)W_1(\eta^i_t(s^t)) = \phi^i_t(s^t)q^i_t(s^t).$$

(23)

In this case, no contingent contracts will be traded because the two types of agents present at site $i$ on date $t$ will not meet again at any future date. The equilibrium price associated with no trade is

$$\frac{U'(y^i_e(s_t))}{\lambda^i_t(s^t)} = \frac{q^i_t(s^t)}{\beta^t \pi_t(s^t)} = \frac{W'(y^i_w(s_t))}{\psi^i_t(s^t)}.$$
Essentially, the infinite-lived agents are unable to trade because writing contracts and delivery on those contracts can occur only if the agents are at the same site on the initial trading day and on the delivery date. Since the agents meet only once, there is no possibility of delivery on a contract and so no contracts are written.

3.2 Full Information with Restricted Delivery

Agents can freely communicate with agents at other sites but delivery on a contract is guaranteed only if the counterparties are located at the same site on the delivery date. If trading of contingent claims is to occur, then an agent located at site \( i \) at time must be able to communicate with sites other than the adjacent sites \( i - 1, i + 1 \). Suppose that type \( E \) agents at site \( i \) can enter into contingent contracts with type \( W \) agents at site \( i + 1 \). Under the restriction on the delivery site, the two agents will not enter into a contract because the type \( W \) will be located at site \( i \) at \( t + 1 \) while the type \( E \) will be located at site \( i + 1 \). Hence the type \( E \) at site \( i \) at time \( t \) will enter into a contract with a type \( W \) at site \( i + 2 \) at time \( t \). Similarly, the type \( W \) at site \( i \) will enter into a contract with a type \( E \) at site \( i - 2 \) at time \( t \). In general, each trader can enter into a countable infinity of contracts with agents of the other types who will be located at the same site at some point in the future.

While a type \( E \) agent at site \( i \) may enter into a contract with a type \( W \) at site \( i + 2 \), the agents are unable to lend or borrow from one another, because resources can’t be moved across sites and there is, by assumption, no intermediary to facilitate borrowing and lending or delivery on contracts.

Let \( q_{t+1}^{i+1}(s^{t+1}) \) denote the time 0 price of a unit of consumption delivered to site \( i + 1 \) at time \( t + 1 \) in state \( s^{t+1} \). Agent \( E \) may with to hold a positive or negative portfolio of contingent claims at the end of the period, but to do so requires transactions with the type \( W \) agent present at site \( i \) at time \( t \). This agent will never enter into a contract
with the type E agent at $i$ because delivery on a contract can’t be guaranteed. Hence the end of period value of the portfolio must sum to zero. Specifically, agent E at site $i$ in state $s^t$ is restricted such that

$$0 = \sum_{s^t+1} q^{i+1}_{t+1}(s^t+1)[y^{i+1}_{t}(s_{t+1}) - c^{i+1}_{t+1}(s^{t+1})].$$

The type E agent at site $i$ at time 0 solves

$$\max_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) U(c^{i+t}_t)$$

subject to

$$0 = \sum_{t} \sum_{s^t} \lambda^{i+t}_t(s^t) \sum_{s^t+1} q^{i+1+t}_{t+1}(s^{t+1})[y^{i+1}_{t}(s_{t+1}) - c^{i+1}_{t+1}(s^{t+1})].$$

(24)

The first-order conditions are

$$\beta^t \pi_t(s^t) U’(c^{i}_t(s^t)) = \lambda^{i-1}_t(s^{t-1})q^{i}_t(s^t).$$

(25)

The type W agent solves

$$\max_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) W(\eta_t)$$

subject to

$$0 = \sum_{t} \sum_{s^t} \psi^{-t}_t(s^t) \sum_{s^t+1} q^{i-1-t}_{t+1}(s^{t+1})[y^{i-1-t}_{t}(s_{t+1}) - \eta^{i-1-t}_{t+1}(s^{t+1})].$$

(26)

The first-order conditions are

$$\beta^t \pi_t(s^t) W’(\eta^{i}_t(s^t)) = \psi^{j+1}_{t-1}(s^{t-1})q^{i}_t(s^t).$$

(27)

In equilibrium, at site $j$ in time $t$ and history $s^t$,

$$\frac{q^{i}_t(s^t)}{\beta^t \pi_t(s^t)} = \frac{U’(c^{i}_t(s^t))}{\lambda^{i-1}_t(s^{t-1})} = \frac{W’(\eta^{i}_t(s^t))}{\psi^{j+1}_{t-1}(s^{t-1})}.$$ 

(28)

The type E agent at site $i$ can also enter into contracts with other type W agents, besides $w(i+2)$. Since agent $E(i)$ will cross paths with $W(i+4), W(i+6)$ and into
the infinite future, $E(i)$ may enter into contracts with the restriction that the end-of-period portfolio with delivery on a particular date and specific site have zero value. Hence an agent is able to enter into a countable infinity of contracts into the future with delivery at different sites, but each period is constrained in terms of borrowing (or lending) against future income in that the value of the portfolio of contingent claims for a specific delivery site held at the end of any period must have zero value.

This problem can be expressed in sequential form. Let $V_e(i, s_t, D)$ denote the value function of a type $e$ agent at location $i$ in state $s_t$ who holds contingent claims $D$ at the beginning of the period. The type $e$ agent solves

$$V_e(i, s_t, D) = \max[U(c_t) + \beta \sum_{s_{t+1}} \pi(s_{t+1} \mid s_t)V_e(i + 1, s_{t+1}, D(s_{t+1})] \quad (29)$$

subject to

$$y_e^i(s_t) + D = c_t^i + \sum_{s_{t+1}} \hat{q}^i(s_{t+1}, s_t)D(s_{t+1}) \quad (30)$$

and

$$0 = \sum_{s_{t+1}} \hat{q}^i(s_{t+1}, s_t)D(s_{t+1}) \quad (31)$$

Let $\mu_t$ denote the multiplier on the second constraint. The first-order conditions and envelope condition reduce to

$$U_1(c_t^i)\hat{q}^i(s_{t+1}, s_t) + \mu(s_t) = \beta \pi(s_{t+1} \mid s_t)U_1(c_{t+1}) \quad (32)$$

Multiply both sides by $D(s_{t+1})$ and then sum over $s_{t+1}$ Observe that $c_t^i = D + y_e^i(s_t)$ so that

$$U_1(c_t^i)\sum_{s_{t+1}} \hat{q}^i(s_{t+1}, s_t)D(s_{t+1}) + \mu(s_t) \sum_{s_{t+1}} D(s_{t+1}) = \beta \sum_{s_{t+1}} \pi(s_{t+1} \mid s_t)U_1(c_{t+1})D(s_{t+1}) \quad (33)$$

or

$$\mu(s_t) \sum_{s_{t+1}} D(s_{t+1}) = \beta \sum_{s_{t+1}} \pi(s_{t+1} \mid s_t)U_1(y_e(s_{t+1}) + D(s_{t+1}))D(s_{t+1})$$
Hence under full communication, contracts in which both counterparties are at the same site at the delivery date can be arranged privately. Moreover, each type of agent at any site can enter into a countable infinity of such contracts. Regardless, the restriction on the delivery prevents full risk sharing and there is a role for a financial intermediary or similar institution to intermediate loans, or equivalently in this model, to guarantee delivery of goods between counterparties located at different sites on the delivery date.

4 Introduction of a Clearing House

In the example above in which delivery on a contract is guaranteed only if the counterparties are present at the same site on the delivery date, the contingent claims provide only partial insurance against aggregate risk. Inefficient risk sharing can be eliminated only if there is an intermediary or a clearing house that facilitate delivery when agents are at different sites on the delivery date. The introduction of a clearing house allows borrowing and lending among agents at different sites.

To achieve full communication and unrestricted delivery equilibrium described earlier, the intermediary must keep records on all trades at all sites, contingent on the history of the state. This is the limiting case of a model in which the clearing house intermediates loans between agents that are located a fixed number of sites apart at time $t$.

4.1 Borrowing Constraints as Netting Schemes

Suppose there is a clearing house that facilitates delivery on contracts between counterparties located at different sites on the delivery date. To prevent Ponzi schemes from occurring, the clearing house will need to maintain records all of the contingent
claims contracts entered into by agents \(W(i)\) or \(E(i)\), \(i \in I\). As discussed earlier, there are a countable infinity of such contracts. In this section I examine a clearing house which maintains records but is technologically constrained from maintaining records and deliveries on all contracts at all sites over time. In the example above, agent \(E(i)\) will enter into a contract with \(w(i+2)\), and more generally any \(W(i+2j)\), \(j = 1, \ldots, \infty\).

Suppose that the clearing house nets deliveries at site \(i + 1\) at time \(t + 1\) with deliveries at site \(i - 1\). If \(E(i)\) borrows from \(W(i+2)\) at time \(t\), resources are shifted from \(W(i)\) to \(E(i)\) for the loan to take place. Agent \(W(i)\) will be repaid at site \(i - 1\) from payments made by \(E(i-2)\) and \(W(i+2)\) will be repaid at site \(i + 2\) by \(E(i)\) for resources that \(W(i+2)\) shifted to \(E(i+2)\). Although multiple agents are involved in the transaction, the initial loan takes place at two sites \((i, i+2)\) and the repayment takes place at two sites \((i - 1, i+1)\). If the clearing house requires that the budget constraint for agent \(E(i)\) must satisfy

\[
0 = q_i^t(s^t)[y_i^t(s_t) - c_i^t(s^t)] + \sum_{t+1} q_{i+1}^{t+1}(s^{t+1})y_{e}^{t+1}(s_{t+1}).
\]

The type \(E\) agent solves

\[
\max \left[ \sum \sum \beta^t \pi_t(s')U(c_t^{i+t}(s')) \right] + \lambda^e \left[ \sum \sum q_t^{i+t}(s')[y_e^{i+t}(s_t) - c_t^{i+t}(s^t)] \right] + \sum \sum \mu_{2,i}^{i+t}(s')[A_{2,i}^{i+t}(s') - q_{i}^{i+t}(s')c_{i}^{i+t}] (33)
\]

The first-order condition is

\[
\beta^t \pi_t(s')U'(c_t^{i+t}(s')) = [\lambda^e + \mu_{2,i}^{i+t}(s')]q_t^{i+t}(s')
\] (34)

The borrowing constraint in this problem corresponds to a netting scheme used by the
clearing house in which the sum of the contingent claims delivered at sites \( i + 1, i - 1 \) at time \( t + 1 \) must equal zero.

For more general netting schemes, the problem solved by a type \( E \) is

\[
\max \left[ \sum_t \sum_{s'} \beta^t \pi_t(s') U(c^i_{t+1}(s')) \right] \\
+ \lambda_e \left[ \sum_t \sum_{s'} q^{i+t}_t(s') [y^i_{t+t}(s_t) - c^i_{t+t}(s_t)] \right] \\
+ \sum_t \sum_{s'} \mu^{i+t}_{n,t}(s') [A^{i+t}_{n,t}(s') - q^{i+t}_t(s') c^i_{t+t}(s')] (35)
\]

The first-order condition is

\[
\beta^t \pi_t(s') U'(c^i_{t+1}(s')) = [\lambda_e + \mu^{i+t}_{n,t}(s')] q^{i+t}_t(s') (36)
\]

If \( n = 1 \), then the clearing house will not intermediate any loans between the agents at site \( i \) at time \( t \). If \( n = 2 \), then the clearing house will intermediate loans but only for agents that are no more than 2 sites apart at time \( t + 1 \), which is referred to as bilateral netting.

5 Endogenous Solvency Constraints

Alvarez and Jermann constraint a constrained efficient equilibrium based on endogenous solvency constraints. Their work builds on earlier work by Kocherlakota [1996] and Kehoe and Levine [1993], who construct equilibria in endowment economies where there are participation constraints. Agents can always opt to revert to the autarchy solution and so any efficient allocation with market participation must take this into account. Alvarez and Jermann show that the participation constraints can be interpreted as endogenous solvency constraints. Agents can choose to default and revert to the autarchy solution. They derive endogenous borrowing constraints such that the agent, while having the option of default, will in equilibrium never choose default.
Define $V_e^a(s_t)$ as
\begin{equation}
V_e^{i,a}(s_t) = U(y_e^i(s_t) + \beta \sum_{s_{t+1}} \pi(s_{t+1} \mid s_t)V_e^{i+1}(s_{t+1})
\end{equation}
so that $V$ is the value of the endowment under autarchy.

Alvarez and Jermann state the problem as
\begin{equation}
V_e^i(s_t, D_e) = \max \left[ U(c_i^t) + \beta \sum_{s'} \pi(s' \mid s)V_e^{i+1}(s', D_e') \right]
\end{equation}
subject to
\begin{equation}
y_e^i(s_t) + D_e = c_i^t + \sum_{s'} \hat{q}(s', s)D_e'(s', s)
\end{equation}
and
\begin{equation}
D_e'(s', s) \geq B_e^{i+1}(s', s)
\end{equation}
Let $\lambda_t$ denote the multiplier on the budget constraint and let $\mu$ denote the multiplier on the borrowing constraint. The first-order conditions and the envelope condition are
\begin{align*}
U'(c) &= \lambda_t \\
\lambda_t \hat{q}(s', s) &= \mu_t + \beta \pi V'
\end{align*}
\begin{equation}
V' = \lambda_t
\end{equation}
They show that the solvency constraint is not too tight if
\begin{equation}
V_e^i(s_t, B^i(s_t, s_t^{t-1}) = V_e^{i,a}(s_t).
\end{equation}
The borrowing constraint can be interpreted as a particular netting scheme by the clearing house. Under this netting scheme, there will be no default in equilibrium.
6 Conclusion

In the absence of an outside asset, or some mechanism by which contingent claims can be traded with agents at other sites, the only equilibrium in the stochastic turnpike model is the autarchy solution. If there is full communication so that agents at different sites can enter into contracts, then the location of delivery must be specified because goods cannot be transferred across sites within a period. If delivery is restricted such that the counterparties must be located at the same site when delivery is made, then there is partial risk sharing, even though each agent enters into a countable infinity of contracts. When there is a clearing house facilitating delivery on contracts, then the clearing house will necessarily require some netting scheme over transactions to prevent Ponzi schemes. I examine various netting schemes, and show that these schemes (which are related to record keeping) are equivalent to borrowing constraints. Finally I examine the Alvarez-Jermann model of endogenous solvency constraints and examine the associated netting scheme in which there is no default in equilibrium.
References


Kocherlakota,
Kehoe and Levine
Townscend
Wallace and Kocherlakota