MONITORING EXPERTS†

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Abstract. A decision maker faces a choice problem under uncertainty and may hire experts to collect information regarding the realized state. The experts choose how much (costly) effort to exert, which determines the quality of information they obtain. Efforts and signal realizations are unobservable by the decision maker, and, moreover, payments can’t be contingent on ex-post outcomes. The decision maker thus has to design a contract that induces the experts to ‘monitor each other’ by making payments contingent on the entire vector of reports. We characterize the information structures that the decision maker can implement. In the special case of two states and two signals we characterize the least costly contract that implements a given information structure and study the tradeoff between the value of information and its cost. In particular, we show that discriminating between the experts is a common feature of an optimal contract.

Keywords: Moral hazard, optimal contracts, monitoring, value of information.

JEL Classification: D82, D86.

May 16, 2018

Preliminary and incomplete draft

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1. Introduction

In the classic principal-agent problem, the principal relies on the correlation between the agent’s unobservable effort and the observed output to motivate the agent to work. Consider the case where the principal is a decision maker (DM, henceforth) facing a choice among several alternatives whose attractiveness is uncertain; and where the agent is an expert who has no stake in the decision but is capable of collecting relevant information at a cost that increases with informativeness. The observed ‘output’ in this case can be thought of as the combination of the expert’s investigation outcome and the ex-post realized state which determines the principal’s payoff: If the expert’s recommendation turned out to be a good one then output is high, while a bad advice corresponds to low output. Incentives can then be provided based on the correlation between effort and outcomes in a way that resembles the classic case. See the related literature section below for references to this type of models.

But what if the expert privately observes the data he collects, and, moreover, compensations cannot wait until the DM learns the state or her payoff? To illustrate, suppose that a policy maker contemplates between several proposals to reduce global warming. An expert is tasked with predicting the effectiveness of each of the alternatives. How can the policy maker guarantee that the expert conducts a thorough investigation if the latter has exclusive access to the collected data, and given that uncertainty will be resolved only in the far future? As a second example, consider a political candidate who hires a pollster to estimate public sentiment on a certain issue. Here too it is unreasonable to assume that the contract can depend on the eventual outcome of the politician actions, nor that the politician directly observes the information obtained by the pollster.

A potential solution for the DM, and the one we study in this paper, is to hire several experts and have them ‘monitor each other’. The basic idea goes as follows: When an expert exerts a high effort he gets an accurate signal of the state, so if all experts exert high efforts and truthfully report their signals then (under reasonable assumptions) these signals are likely to be close to each other. Thus, by paying high compensations in the event of matching signals and low compensations when a mismatch occurs the DM can incentivize the experts to work hard and to reveal what they find. A similar idea of peer-monitoring has been recently suggested by Rahman [21] and by Bohren and Kravitz [5], we discuss the relation between the papers below.

In our model, uncertainty is captured by a finite state-space with a common prior belief shared by the DM and all the experts. Everyone is Bayesian, risk-neutral, and maximizes expected-utility. The DM offers a contract which specifies the reward for
each expert as a function of the vector of reported signals she receives from the experts. We assume limited liability – negative rewards are not allowed. Each expert then chooses what experiment to conduct and, upon obtaining the results, what signal to report to the DM. Throughout we assume that experts’ signals are independent conditional on the state of nature. We say that a contract implements a given vector of information structures if in the game it induces it is an equilibrium for the experts to choose their respective structures and to truthfully report their signals.

The contribution of this paper can be divided into two parts. The first is a detailed study of the case in which the state-space and signal-space for each expert are binary, and the information structures available to the experts are symmetric between the two states. The information structure of each expert in this case is summarized by a single number – the probability of the signal matching the state, which we assume to be increasing in effort. We are interested in properties of the optimal contract in this environment. Our analysis is based on the Grossman-Hart [13] approach: As a first step, for each vector of information structure find the least costly way for the DM to implement it. Second, once the cost function is obtained, maximize the difference between the value of information and its cost over all implementable vectors of information structures.

As it turns out, in this simple environment finding the cost function boils down to solving a linear program whose solution can be characterized: The least costly way to implement any vector of information structures involves paying the experts only in the event where they all report the same signal. This gives an expert the maximal incentive to work relative to the expected cost of the contract with that expert. From this we derive an explicit formula for the cost function.

We then move on to study properties of the value of information for the DM. We prove the following result: If two sets of experts have the same average accuracy\(^1\) of signals, and in one of these sets the spread of accuracies is larger than the other, then the former is more informative than the latter in the sense of Blackwell [3]. For example, ignoring the cost, every decision maker in every decision problem prefers two experts with respective accuracies \(\frac{7}{8}\) and \(\frac{5}{8}\) over two experts each with accuracy of \(\frac{3}{4}\), and the latter over 3 experts with accuracy of \(\frac{2}{3}\) each. This result, which is of independent interest and may be useful in other applications,\(^2\) is another expression of non-concavity in the value of information.

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\(^1\)Accuracy is measured as the increase in the probability that the signal matches the state relative to the uninformative structure where this probability is \(\frac{1}{2}\).

\(^2\)In voting, for example, this result implies that few well-informed voters outperform many little-informed voters, given that the sum of accuracies in the two groups is the same.
The classic result of Radner and Stiglitz [20] (see also Chade and Schlee [8]) expresses a different type of non-concavity.

The above non-concavity of the value of information makes solving for the optimal contract a particularly difficult task, as it implies that first order conditions are not sufficient for optimality. Yet there are some quite general properties of optimal contracts that we can deduce from comparing the cost and value of information. For the case of two experts, we show that arbitrarily close to any decision problem there is a decision problem in which the optimal contract requires uneven compensations to experts. This conclusion holds uniformly across all cost-of-effort functions. The intuition comes directly from the non-concavity result described above. Thus, discrimination between experts naturally follows from optimality considerations and need not be the result of prejudice or bias.\(^3\) With more than two experts, a version of this result continues to hold when the number of experts is even.

The second part of the paper contains a general implementability result that characterizes the information structures that the DM can induce the experts to produce. This result is achieved with minimal structural assumptions about the environment. Instead of modeling the experts as choosing efforts and reporting strategies, we assume that an expert chooses an information structure whose signal the DM will directly observe; in Appendix B we show that the two frameworks are equivalent. The characterization of implementable vectors of information structures follows the footsteps of Theorem 1 in Rahman [21]: An obvious necessary condition for \(m = (m_1, \ldots, m_n)\) to be implementable is that no agent \(i\) can deviate to another structure \(m'_i\) which is less costly than \(m_i\) and generates the same distribution over signal vectors as \(m_i\) (when combined with \(m_{-i}\)). A version of the minmax theorem then implies that this condition is also sufficient.

From this basic characterization, and using the particular structure of our environment (conditional independence, especially), we derive several easy-to-check sufficient conditions for implementability. Suppose that \(m\) is such that, for every expert \(i\), the conditional distributions of \(s_{-i}\) (the signals of all other experts except \(i\)) given the various possible states of nature are linearly independent. Then it is easy to see that every two distinct choices \(m_i, m'_i\) of \(i\) would result in different distributions of the vector of all \(n\) signals. Using the main characterization theorem, this implies that \(m\) is implementable regardless of the cost function. Two immediate corollaries of this result are that (1) if

\[^3\text{A similar point in a very different context is made in Winter [25].}\]
there are only two states then every vector of informative structures is implementable; and (2) if \( m \) is implementable then so is every vector of more informative structures.\(^4\)

1.1. Related literature. The current paper is at the intersection of the literature on monitoring design and the literature on costly information acquisition. The only other paper we are aware of that explicitly considers peer-monitoring of information providers is the recent work of Bohren and Kravitz [5]. In their model there is an infinite stream of identical decision problems, and in each problem the principal gets a fixed positive payoff if her action matches the state and a payoff of 0 otherwise. The principal can hire workers to verify the state at a cost, and the main interest is in the optimal rate of monitoring – how often should two workers (and not just one) be assigned to the same problem to make sure that reports about the state are genuine. In contrast, we consider a single decision problem and are interested in the quality of information that the DM can obtain and in the tradeoff between this quality and the cost of obtaining it. Moreover, we do not restrict attention to a particular decision problem, and instead study general properties of optimal contracts that are satisfied uniformly across all problems.

The debate on the importance of monitoring in organizations goes back at least to Alchian and Demsetz [1], who argue that preventing shirking of employees is one of the principle roles of the owner of a firm. The DM in our model does not directly monitor the experts for lack of appropriate knowledge/technology, and instead relies on the experts to monitor each other. However, our results on optimal contracts (see Proposition 2 and the following discussion) imply that sometimes experts are hired only for the sake of monitoring other experts, i.e., their signal is not taken into account at all for the decision. This can be seen as a form of ‘specialization in monitoring’ that the theory of Alchian and Demsetz [1] suggests. On the other hand, Holmstrom [14] argues that the principle’s role is to break the budget-balance constraint and impose group penalties when output is low in order to incentivize effort. Our DM does exactly that: In the optimal contract no expert is getting paid unless all signals agree.

The work of Rahman [21] emphasizes the ability of a principal to monitor workers by secretly recommend actions and base compensations on reported signals as well as on these recommendations. While the DM in our model does not use this tool, there

\(^4\)These sufficient conditions resemble the results of Crémer and McLean [10] on the full extraction of surplus in auctions. While the results are certainly related, there are important differences. First, the linear independence we use here is of the distributions of others’ signals conditional on the state of nature, while Crémer and McLean use independence of others’ signals distributions conditional on a bidder’s own signal. Second, and more importantly, in our model the experts choose the information structure, while in Crémer and McLean the information structure is exogenously given and agents only choose which signal to report to the auctioneer.
is a close connection between the second part of this paper (Theorem 3, in particular) and Rahman’s characterization of implementable distributions of actions. Specifically, our proof of Theorem 3 uses the same method of applying the minmax theorem as in Theorem 1 of Rahman [21]. Technically, our framework is more restrictive in that an expert’s payoff does not depend directly on other experts’ actions, and is more general in that experts choose from an infinite set of information structures with a non-linear cost. See also Strausz [24] on the connection between Rahman’s paper and the classic revelation principle of Myerson [17].

There are quite a few papers in which a DM hires a single expert to provide (costly) information relevant to the decision she faces. See Osband [18], Zermeño, [26], Rappoport and Somma [22], Chade and Kovrijnykh [7], Carroll [6], and Clark [9], among others. These models differ from each other in various dimensions, especially in terms of what variables are observable and contractible. Since in our model payments can only depend on the unverifiable report of the experts, clearly there is no way for the DM to provide incentives with a single expert.

The experts in our model have no stake in the choice of the DM, which separates our framework from the literature on strategic information transmission. In particular, in models of committee design with costly information acquisition (e.g. Persico [19], Martinelli [16], Gerardi and Yariv [11], Gershkov and Szentes [12]) the voters care about the decision, and incentives to collect information are provided through the choice of a voting rule and not through transfers.

2. A BINARY-BINARY MODEL

2.1. The setup. A decision maker (DM) faces a decision problem under uncertainty. There are two possible states – High (H) or Low (L), with both states being equally likely a-priori.

The DM hires $n \geq 2$ risk-neutral experts to collect information about the realized state, where $N$ denotes the set of experts. Each expert $i$ chooses an effort level $e_i \in [0, \frac{1}{2}]$. The cost of effort is described by the function $c : [0, \frac{1}{2}] \to \mathbb{R}$. We assume that $c$ is strictly increasing, strictly convex, three times continuously differentiable, and that $c(0) = 0$. We denote by $C$ the set of all cost functions with these properties. Each expert privately observes a signal from $S_i = \{h, l\}$, where the distribution over signals conditional on each state depends on the effort level that the expert exerts. Specifically, if $i$ chooses $e_i$ then the information structure is given by
\[
\begin{array}{|c|c|c|}
\hline
& h & l \\
\hline
H & 0.5 + e_i & 0.5 - e_i \\
\hline
L & 0.5 - e_i & 0.5 + e_i \\
\hline
\end{array}
\]

Note that no effort leads to uninformative signal, and that informativeness increases with effort. We assume that signals for different experts are independent conditional on the state. Given the vector of effort levels \( e = (e_1, \ldots, e_n) \), denote by \( b(e) \) the information structure obtained by observing the signals of all the experts.

The experts have no stake in the decision, and the DM may offer monetary compensation for their efforts. We assume however that effort is unobservable. Moreover, compensations occur immediately after the experts report their signals, so transfers can’t be conditional on the true state. We consider ‘direct mechanisms’ in which each expert submits a report \( s_i \in S_i \) and gets compensated based on the entire vector of reports \( s \in S := \times_{i=1}^n S_i \). Thus, a contract is a list \( x = (x_1, \ldots, x_n) \) with each \( x_i : S \to \mathbb{R}_+ \). Note that we assume that payments are non-negative, which captures ‘limited liability’ on the part of the experts.

A contract \( x \) induces a game between the experts. A pure strategy for expert \( i \) in this game is a pair \( (e_i, r_i) \), where \( e_i \in [0, \frac{1}{2}] \) is \( i \)’s effort level and \( r_i : S_i \to S_i \) is the report that \( i \) sends to the DM as a function of the signal he observed. The payoff to expert \( i \) given strategy profile \( (e, r) = ((e_1, \ldots, e_n), (r_1, \ldots, r_n)) \) is

\[
U_i(e, r; x_i) := \mathbb{E}_{(e, r)}[x_i(s)] - c(e_i),
\]

where the distribution of \( s \) used to calculate the expectation is derived from the strategies \( (e, r) \) by

\[
P_{(e, r)}(s) = \sum_{s' \in r^{-1}(s)} \left[ \frac{1}{2} \prod_{\{j : s_j' = h\}} (0.5 + e_j) \prod_{\{j : s_j' = l\}} (0.5 - e_j) + \frac{1}{2} \prod_{\{j : s_j' = l\}} (0.5 + e_j) \prod_{\{j : s_j' = h\}} (0.5 - e_j) \right].
\]

It will be convenient to introduce the following notation. For every subset of experts \( A \subseteq N \) and every efforts’ vector \( e \) let \( e(A) = \prod_{j \in A} (0.5 + e_j) \) and \( \bar{e}(A) = \prod_{j \in A} (0.5 - e_j) \). Given a vector of signals \( s \in S \) denote \( s^h = \{ j : s_j = h \} \) and \( s^l = N \setminus s^h = \{ j : s_j = l \} \). Finally, let \( r^* = (r_1^*, \ldots, r_n^*) \) denote the vector of truthful reporting strategies. Using this notation we have that

\[
P_{(e, r^*)}(s) = \frac{1}{2} [e(s^h) \bar{e}(s^l) + e(s^l) \bar{e}(s^h)].
\]

2.2. The cost of implementation. Say that a contract \( x \) implements the vector of efforts \( e = (e_1, \ldots, e_n) \) if \( (e, r^*) \) is an equilibrium of the game induced by \( x \) with payoff
functions as in (1); using Myerson’s [17] terminology, $x$ implements $e$ if honesty (truthful reporting) and obedience (choosing the desired effort level) is a best response for each expert given that all other experts are honest and obedient. Efforts’ vector $e$ is implementable if there exists a contract $x$ that implements it.

If $e$ is implementable then there would typically be many contracts $x$ that implement it. Let $\psi_i(e)$ be the minimal expected payment that the DM would need to make to expert $i$ in a contract that implements $e$. More formally, $\psi_i(e)$ is given by

$$\psi_i(e) = \min_{x_i \geq 0} \{ \mathbb{E}_{(e, r^*)}[x_i(s)] \}$$

s.t. $(e_i, r^*_i) \in \arg \max_{(e_i', r'_i)} \{ U_i ((e_i', e_{-i}, r^*); x_i) \}$.

**Theorem 1.** Let $0 < e < 0.5$. Then $e$ is implementable, and if $x_i$ is a solution to the program (2) then $x_i(s) = 0$ for every $s \notin \{(h, \ldots, h), (l, \ldots, l)\}$. Furthermore,

$$\psi_i(e) = \frac{e(N) + \bar{e}(N)}{e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})} c'(e_i),$$

and the cost function of the decision maker is therefore

$$\psi(e) = \sum_{i=1}^{n} \psi_i(e) = [e(N) + \bar{e}(N)] \sum_{i=1}^{n} \frac{c'(e_i)}{e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})}.$$ 

**Proof.** Fix $e$ as in the theorem and suppose that all experts other than $i$ are honest and obedient. We will find an optimal solution to (2), i.e., a least costly contract $x_i$ such that $(e_i, r^*_i)$ is $i$’s optimal choice in the game induced by $x_i$.

**Step 1:**
We first consider the following relaxation of problem (2):

$$\min_{x_i \geq 0} \{ \mathbb{E}_{(e, r^*)}[x_i(s)] \}$$

s.t. $e_i \in \arg \max_{e_i'} U_i ((e_i', e_{-i}, r^*); x_i)$.

In the relaxed problem $i$ is required to be honest (report according to $r^*_i$) and only deviations from the designated effort $e_i$ are allowed.

To solve (3) note that the distribution $\mathbb{P}_{(e_i', e_{-i}, r^*)}$ over $S$ is linear in $e_i'$, and hence so is the expectation $\mathbb{E}_{(e_i', e_{-i}, r^*)}[x_i(s)]$. It follows that the payoff function $U_i ((e_i', e_{-i}, r^*); x_i)$ is concave in $e_i'$. Thus, a necessary and sufficient condition for $x_i \geq 0$ to be feasible for (3) is that $\frac{\partial U_i((e_i', e_{-i}, r^*); x_i)}{\partial e_i'} = 0$. A straightforward calculation shows that this first-order

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5The inequality $0 < e < 0.5$ means that $0 < e_i < 0.5$ for every $i = 1, \ldots, n$.  
condition is equivalent to

$$\frac{1}{2} \sum_{\{s \in S : s_i = h\}} \left[ e\left(s^h \setminus \{i\}\right) \bar{e}\left(s^l\right) - e\left(s^l\right) \bar{e}\left(s^h \setminus \{i\}\right) \right] x_i(s) +$$

$$\frac{1}{2} \sum_{\{s \in S : s_i = l\}} \left[ e\left(s^l \setminus \{i\}\right) \bar{e}\left(s^h\right) - e\left(s^h\right) \bar{e}\left(s^l \setminus \{i\}\right) \right] x_i(s) = c'(e_i).$$

Now, since the objective is to minimize

$$\frac{1}{2} \sum_{s \in S} \left[ e\left(s^h\right) \bar{e}\left(s^l\right) + e\left(s^l\right) \bar{e}\left(s^h\right) \right] x_i(s),$$

it follows that a feasible $x_i$ is an optimal solution to (3) if and only if $x_i(s) > 0$ only for those $s$ for which the ratio between the coefficient in the constraint and in the objective is maximal.

Suppose first that $s$ is such that $s_i = h$. Then the ratio between the coefficients is given by

$$\frac{e\left(s^h \setminus \{i\}\right) \bar{e}\left(s^l\right) - e\left(s^l\right) \bar{e}\left(s^h \setminus \{i\}\right)}{e\left(s^h\right) \bar{e}\left(s^l\right) + e\left(s^l\right) \bar{e}\left(s^h\right)} - \frac{1}{0.5 + e_i + (0.5 - e_i) \frac{e\left(s^h \setminus \{i\}\right) \bar{e}\left(s^l\right)}{e\left(s^h\right) \bar{e}\left(s^l\right)}} = \frac{1}{(0.5 + e_i) \frac{e\left(s^h \setminus \{i\}\right) \bar{e}\left(s^l\right)}{e\left(s^l\right) \bar{e}\left(s^h \setminus \{i\}\right)} + 0.5 - e_i}.$$ 

Note that the fraction in the first denominator is the likelihood of observing $s_{-i}$ in state $L$ over the likelihood of observing it in state $H$, and that the fraction in the second denominator is the inverse likelihood ratio. Thus, the difference is maximized at $s_{-i} = (h, \ldots, h)$. If $s$ is such that $s_i = l$ then a similar calculation shows that the ratio is maximized at $s_{-i} = (l, \ldots, l)$. Furthermore, from the symmetry between the two states we get that the ratios for $s = (h, \ldots, h)$ and $s = (l, \ldots, l)$ are equal.

To conclude, a feasible $x_i$ is an optimal solution to (3) if and only if $x_i(s) > 0$ only for $s = (h, \ldots, h)$ and $s = (l, \ldots, l)$. With this, the first-order condition (4) becomes

$$x_i(h, \ldots, h) + x_i(l, \ldots, l) = \frac{2c'(e_i)}{e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})},$$

and the objective is

$$\frac{1}{2} \left( e(N) + \bar{e}(N) \right) (x_i(h, \ldots, h) + x_i(l, \ldots, l)).$$

It follows that the value of the relaxation (3) is

$$\frac{e(N) + \bar{e}(N)}{e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})} c'(e_i).$$
Step 2:
Consider the contract $x_i$ given by
\[ x_i(h, \ldots, h) = x_i(l, \ldots, l) = \frac{c'(e_i)}{e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})}, \]
and $x_i(s) = 0$ otherwise. By the previous step this $x_i$ is an optimal solution to the relaxed problem (3). We now argue that $x_i$ is feasible also in the original problem (2), and hence that it is optimal for that problem as well.

Suppose to the contrary that $i$ has a profitable deviation $(e'_i, r'_i)$ with $r'_i \equiv h$, i.e., $i$ sends the message $h$ regardless of his signal. Then the deviation $(0, r'_i)$ is profitable as well, since it gives $i$ the same expected transfer as $(e'_i, r'_i)$ at a minimal cost. The payoff to $i$ under $(0, r'_i)$ is given by
\[ U_i((0, r'_i), (e_{-i}, r^*_r); x_i) = \frac{1}{2} \left[ e(N \setminus \{i\}) + \bar{e}(N \setminus \{i\}) \right] \frac{c'(e_i)}{e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})}, \]
while his payoff under honesty and obedience $(e_i, r_i^*)$ is
\[ U_i((e_i, r_i^*), (e_{-i}, r^*_r); x_i) = \frac{e(N) + \bar{e}(N)}{e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})} c'(e_i) - c(e_i). \]
Thus,
\[ U_i((0, r'_i), (e_{-i}, r^*_r); x_i) - U_i((e_i, r_i^*), (e_{-i}, r^*_r); x_i) = \frac{c'(e_i)}{2e(N \setminus \{i\}) - 2\bar{e}(N \setminus \{i\})} \left[ e(N \setminus \{i\}) + \bar{e}(N \setminus \{i\}) - 2e(N) - 2\bar{e}(N) \right] + c(e_i) = -e_i c'(e_i) + c(e_i) \leq 0, \]
where the inequality follows from convexity of $c$ and $c(0) = 0$. This shows that $(e'_i, r'_i)$ is not a profitable deviation.

A similar argument applies for deviations $(e'_i, r'_i)$ with $r'_i \equiv l$. Finally, if $(e'_i, r'_i)$ is such that $r'_i(h) = l$ and $r'_i(l) = h$ then it is immediate to check that the strategy $(e'_i, r_i^*)$ gives $i$ a higher expected payment (with the same cost of effort). But from the first step of the proof we know that $(e'_i, r_i^*)$ is not a profitable deviation, and hence $(e'_i, r'_i)$ is not profitable as well. This completes the proof. □

Remark. If the cost function $c$ satisfies $c'(0.5) < +\infty$ then the theorem remains true for any vector $0 < e \leq 0.5$, i.e., even if some experts fully reveal the state. The same proof applies.

Remark. If $e_i = 0$ for some expert $i$ then $x_i \equiv 0$ solves (2) and $\psi_i(e) = 0$. In addition, the signals obtained from zero-effort experts are uninformative. We can therefore
restrict attention only to experts that exert strictly positive effort. However, for $e$ to be implementable it is necessary (and sufficient) that at least two experts exert effort. See Section 3 for a general characterization of implementability.

**Remark.** The proof of Theorem 1 heavily relies on the uniform prior assumption. In Section 4 below we show that the structure of the least costly contract in the binary-binary model with a non-uniform prior is similar to the one obtained in Theorem 1 above. However, with a non-uniform prior the non-verifiability of the signals becomes an issue, and truth-telling constraints bind. This is not the case when the prior is uniform, as can be seen in Step 2 of the above proof.

An immediate corollary of Theorem 1 is that the expected payment to an expert in the least costly contract increases in that expert’s own effort and decreases in other experts’ efforts. Formally,

**Corollary 1.** For any $e > 0$ and $i \neq j$ we have

$$\frac{\partial \psi_i(e)}{\partial e_i} = c'(e_i) + \frac{e(N) + \bar{e}(N) - e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})}{e(N \setminus \{i\} - \bar{e}(N \setminus \{i\})}c''(e_i) > 0,$$

and

$$\frac{\partial \psi_j(e)}{\partial e_i} = -e(N \setminus \{i, j\}) \bar{e}(N \setminus \{i, j\}) \frac{e'(e_j)}{[e(N \setminus \{j\} - \bar{e}(N \setminus \{j\})]_2} < 0.$$

2.3. **Value of information.** Finding the cost of obtaining information, we now consider the value of information for the DM. The decision problem she faces is modeled as a set of alternatives she can choose from and a utility function that maps each alternative-state pair to the reals. After receiving the vector of signals $s$ from the information structure $b(e)$, the DM updates her belief using Bayes rule and chooses the alternative that maximizes her expected utility. Any decision problem induces a convex and continuous function $v : \Delta(\{H, L\}) \rightarrow \mathbb{R}$ that assigns to each (posterior) belief the maximal achievable expected utility for the DM given that belief. The value of information structure $b(e)$, denoted $V(e)$, is the expectation of $v$ relative to the distribution over posterior beliefs that $b(e)$ induces. We denote by $\mathcal{V}$ the set of all convex and continuous functions $v : \Delta(\{H, L\}) \rightarrow \mathbb{R}$.\(^6\)

**Example 1.** Suppose that the set of available alternatives is $\{H, L\}$ (same as the set of states) and that the DM gets a utility of 1 if her choice matches the state and a utility

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\(^6\)Every convex and continuous $v \in \mathcal{V}$ corresponds to some decision problem (i.e., a set of available actions and a state-dependent utility function). See for example Azrieli and Lehrer [2].
of 0 otherwise. Then if \( q \in [0, 1] \) denotes the DM’s belief that the state is \( H \) then the function \( v \) is given by \( v(q) = \max\{q, 1 - q\} \).

**Example 2.** Suppose that the DM needs to choose between a safe alternative \( S \) and a risky alternative \( R \). Choosing \( S \) yields a sure utility of 0, while choosing \( R \) yields a utility of 1 in state \( H \) and a utility of -1 in state \( L \). The corresponding \( v \) is then \( v(q) = 0 \) for \( 0 \leq q \leq 0.5 \) and \( v(q) = 2q - 1 \) for \( 0.5 < q \leq 1 \).

**Example 3.** Let the set of alternatives be the unit interval \([0, 1]\), and the utility function be \( u(a, H) = -(1 - a)^2 \) and \( u(a, L) = -a^2 \) for every alternative \( a \in [0, 1] \). Then it is well-known and easy to check that when the DM’s belief is \( q \) her optimal choice is \( a = q \). This gives \( v(q) = -q(1 - q) \).

While the above examples appear in many applications, we are also interested in universal properties of the value of information that hold for every decision problem; this will later allow us to draw more general conclusions about which effort vectors may be optimal for the DM. More specifically, our next result formulates a condition on a pair of efforts’ vectors \( e, e' \) which guarantees that \( V(e) \geq V(e') \) for every decision problem (i.e., for every \( v \in \mathcal{V} \)). As is well-known, this is equivalent to saying that \( b(e') \) is a garbling of \( b(e) \) in the sense of Blackwell [3].

Since \( V \) is symmetric in its arguments, we may (and will) assume without loss that efforts are ordered in decreasing order from highest to lowest. Consider two efforts’ vectors \( e = (e_1 \geq e_2 \geq \ldots \geq e_n) \) and \( e' = (e'_1 \geq e'_2 \geq \ldots \geq e'_m) \). Say that \( e \) dominates \( e' \) if \( e_i \geq e'_i \) for every \( i = 1, \ldots, \max\{m, n\} \), and that \( e \) majorizes \( e' \) if \( \sum_{i=1}^{k} e_i \geq \sum_{i=1}^{k} e'_i \) for every \( k = 1, \ldots, \max\{m, n\} \), where, in case \( m \neq n \), the shorter of the two vectors is appended with zeroes. Domination clearly implies majorization, but the converse is not true: \( e = (3/8, 1/8) \) majorizes \( e' = (1/4, 1/4) \) but does not dominate it.

A classic result of Blackwell and Girschick [4, Theorem 12.3.1 on page 332] says that if information structure \( P' \) is a garbling of \( P \) and \( Q' \) is a garbling of \( Q \), and if each of the pairs \((P, Q)\) and \((P', Q')\) are independent conditional on the state, then the combined information \((P', Q')\) is a garbling of the combined information \((P, Q)\). This implies that if \( e \) dominates \( e' \) then \( b(e') \) is a garbling of \( b(e) \). Thus, for any decision problem, \( V \) is non-decreasing in each expert’s effort. The next Theorem 2, which may be of independent interest, strengthen this conclusion by showing that if \( e \) majorizes \( e' \) then \( V(e) \geq V(e') \) in any decision problem; in other words, \( V \) is a Schur-convex function. Schur-convexity is closely related to, though weaker than, convexity.\(^7\) We note that a different kind of

\(^7\)To be more precise, convexity together with symmetry implies Schur-convexity. Schur-convexity implies symmetry but not convexity.
non-concavity in the value of information has been shown by Radner and Stiglitz [20] (see also Chade and Schlee [8]).

**Theorem 2.** If \( e \) majorizes \( e' \) then \( b(e') \) is a garbling of \( b(e) \) in the sense of Blackwell.

**Proof.** Suppose that \( e \) majorizes \( e' \). First, we may assume without loss of generality that both have the same number \( n \) of experts; otherwise, add zero-effort experts to the shorter of the two. Second, it is without loss to assume that \( \sum_i e_i = \sum_i e'_i \): If \( e \) majorizes \( e' \) and \( \sum_i e_i > \sum_i e'_i \) then there exists \( e'' \) such that (i) \( e'' \) majorizes \( e' \), (ii) \( \sum_i e''_i = \sum_i e'_i \), and (iii) \( e \) dominates \( e'' \) (Marshal et al. [15, Proposition A.9 on page 177]). By Blackwell and Girschick’s result \( b(e) \) is more informative than \( b(e'') \), so the case of unequal total effort follows from the case of equal total effort.

Now, for two vectors \( z, z' \in \mathbb{R}^n \) say that \( z' \) is obtained from \( z \) by a Pigou-Dalton (PD) transfer if there are coordinates \( i, j \) with \( z_i \geq z_j \) and \( 0 \leq \delta \leq z_i - z_j \) such that \( z'_i = z_i - \delta \) and \( z'_k = z_k \) for every \( k \neq i, j \). Also, say that \( z' \) can be obtained from \( z \) by a sequence of PD transfers if there are \( L \) and vectors \( z_1, \ldots, z_L \) such that \( z_1 = z \), \( z_L = z' \), and \( z_l \) is obtained from \( z_{l-1} \) by a PD transfer for every \( l = 2, \ldots, L \). It is well known (see, e.g., Marshal et al. [15, Proposition A.1 on page 155]) that if \( z \) majorizes \( z' \) and \( \sum_i z_i = \sum_i z'_i \) then \( z' \) can be obtained from \( z \) by a sequence of PD transfers.\(^8\)

Therefore, to complete the proof we only need to show that if \( e' \) is obtained from \( e \) by a PD transfer then \( b(e) \) is more informative than \( b(e') \). In fact, since a PD transfer changes the efforts of only two experts, it follows from Blackwell and Girschick’s result that it is enough to prove this for the case \( n = 2 \). This is established in the following lemma, whose proof appears in the appendix.

**Lemma 1.** Suppose that \( e_1 \geq e_2 \) and \( e'_1 \geq e'_2 \) are such that \( e_1 + e_2 = e'_1 + e'_2 \) and \( e_1 \geq e'_1 \) (i.e., \( e' \) is obtained from \( e \) by a PD transfer). Then \( b(e_1, e_2) \) is more informative than \( b(e'_1, e'_2) \).

\( \square \)

2.4. **Optimal contracts.** We now consider the DM problem of maximizing the difference between the value of information and its cost. Recall that the primitives of the model are the cost of effort function \( c \in \mathcal{C} \) from which \( \psi \) is derived, and the value function \( v \in \mathcal{V} \) from which \( V \) is derived. For any \( e \) let \( \pi_{c,v}(e) = V_v(e) - \psi_c(e) \) be the net expected utility of the DM given efforts’ vector \( e \), where, by convention, \( \pi_{c,v}(e) = -\infty \) when \( e \) is

\(^8\)The converse of this statement is true as well, but is not needed for our purposes.
not implementable. We sometimes omit the subscripts $c$ and $v$ when no confusion may arise.

Even though the arguments of $\pi$ are efforts’ vectors, we refer to a maximizer of this function as an optimal contract. One can think of a contract as specifying both the required efforts $e$ and the payments $x$ that implement $e$ in the least costly way.

2.4.1. Two experts. We start with the case in which the DM is able to hire only two experts ($n = 2$). From Theorem 1 we know that every $0 < e = (e_1, e_2) < 0.5$ is implementable, and that the cost of implementing any such $e$ is

$$\psi(e_1, e_2) = \left(\frac{1}{2} + 2e_1e_2\right)\left(\frac{c'(e_1)}{2e_2} + \frac{c'(e_2)}{2e_1}\right).$$

If $c'(0.5) < +\infty$ then this formula remains true for $0 < e_1, e_2 \leq 0.5$. If $e_i = 0$ and $e_j > 0$ then $e$ is not implementable, while the no-effort vector can be implemented at zero cost: $\psi(0, 0) = 0$.

Our main result here is that ‘typically’ the optimal contract involves discriminating between the experts. That is, in the optimal contract the DM will offer different compensations to the two experts, and this will result in the experts exerting different effort levels. The intuition for this result comes directly from Theorem 2: Getting two signals of the same accuracy from the experts is less valuable than getting one more accurate and one less accurate signals, subject to the two combinations having the same average accuracy. On the other hand, the cost of inducing the former option is lower than the cost of inducing the latter, which pushes towards equal efforts.\(^9\) We show however that near any decision problem $v$ there is another decision problem $\tilde{v}$ such that, for every cost function $c$, the maximizer of $\pi_{c, \tilde{v}}$ is not on the diagonal $e_1 = e_2$.

**Proposition 1.** Fix $v \in \mathcal{V}$ and $\epsilon > 0$. Then there is $\tilde{v} \in \mathcal{V}$ such that

(i) $|\tilde{v}(q) - v(q)| \leq \epsilon$ for all $q \in [0, 1]$; and

(ii) For every $c \in \mathcal{C}$, if $e = (e_1, e_2) > 0$ is a maximizer of $\pi_{c, \tilde{v}}$ then $e_1 \neq e_2$.

**Proof.** Given $v \in \mathcal{V}$ and $\epsilon > 0$, let $v' : [0, 1]$ be given by $v'(q) = \epsilon \max\{q, 1 - q\}$, and let $\tilde{v} = v + v'.$\(^{10}\) Then $\tilde{v} \in \mathcal{V}$ as the sum of two convex and continuous functions, and $|\tilde{v}(q) - v(q)| = |v'(q)| \leq \epsilon$ for all $q \in [0, 1]$.

\(^9\)Under quite general conditions the cost function $\psi(e_1, e_2)$ is convex, for example one sufficient condition is that the ratio $\frac{c''}{c'}$ is increasing on $[0, 1/2]$.

\(^{10}\)Recall that $v'$ is obtained from a decision problem with two alternatives as in Example 1.
Fix some $c \in C$, and let $0 < e_1 = e_2 < 0.5$. Let $\delta > 0$ be a small number and define the function $g : [-\delta, \delta] \to \mathbb{R}$ by $g(\delta) = \pi_{c,\tilde{v}}(e_1 + \delta, e_2 - \delta)$. We show that $g$ has a strict local minimum at $\delta = 0$, which implies that $(e_1, e_2)$ is not a maximizer of $\pi_{c,\tilde{v}}$.

By linearity of expectation we have

$$g(\delta) = V_{\nu}(e_1 + \delta, e_2 - \delta) + V_{\nu'}(e_1 + \delta, e_2 - \delta) - \psi_c(e_1 + \delta, e_2 - \delta).$$

From Theorem 2 we know that for all $\delta \in [-\bar{\delta}, \bar{\delta}]$

$$V_{\nu}(e_1 + \delta, e_2 - \delta) \geq V_{\nu}(e_1, e_2),$$

which implies

$$g(\delta) \geq V_{\nu}(e_1, e_2) + V_{\nu'}(e_1 + \delta, e_2 - \delta) - \psi_c(e_1 + \delta, e_2 - \delta).$$

Now, a direct calculation gives

$$V_{\nu'}(e_1 + \delta, e_2 - \delta) = \begin{cases} 
\epsilon(0.5 + e_1 + \delta) & \text{if } 0 \leq \delta \leq \bar{\delta} \\
\epsilon(0.5 + e_2 - \delta) & \text{if } -\bar{\delta} \leq \delta \leq 0.
\end{cases}$$

Also, since $\psi_c$ is symmetric and differentiable, the derivative $\frac{d\psi_c(e_1 + \delta, e_2 - \delta)}{d\delta}$ is zero at $\delta = 0$. It follows that right-derivative at $\delta = 0$ of the difference

$$V_{\nu'}(e_1 + \delta, e_2 - \delta) - \psi_c(e_1 + \delta, e_2 - \delta)$$

is $+\epsilon$ and the left-derivative of this difference at $\delta = 0$ is $-\epsilon$. Thus for all $\delta > 0$ sufficiently small we get

$$g(\delta) > V_{\nu}(e_1, e_2) + V_{\nu'}(e_1, e_2) - \psi_c(e_1, e_2) = g(0).$$

Finally, $e_1 = e_2 = 0.5$ is not optimal (when implementable) since the DM can learn the state at a lower cost by choosing $e_1 = 0.5$ and $e_2 < 0.5$ (see the next proposition). □

In the next proposition we derive an additional property of optimal contracts, namely that there is a lower bound on the effort that an expert should be asked to exert. More precisely, given the cost function $c$, if expert 1 is asked to exert effort $e_1 > 0$ then there is a positive number $f(e_1)$ such that it is never (i.e., for no decision problem) optimal to ask expert 2 to exert effort less than $f(e_1)$. The intuition is that if $e_2$ is close to zero then the report of expert 2 is almost independent of the state, and hence the distribution over pairs of signals $(s_1, s_2)$ (that determines the expected payment) does not change much when expert 1 increases his effort; in the extreme case where $e_2 = 0$ the distribution over $(s_1, s_2)$ is uniform for every $e_1$. Thus, in order to induce expert 1 to exert $e_1$ the payment needs to be very large when the reports match. It follows that for small $e_2$ the
increase in the expected payment needed to incentivize expert 2 to increase his effort is overwhelmed by the resulting decrease in the expected payment to expert 1.

**Proposition 2.** Let \( c \in C \) be such that the derivative \( c' \) is convex on \([0, 0.5]\) and \( c'(0.5) < +\infty \). Then there is a continuous and strictly increasing function \( f : [0, 0.5] \rightarrow [0, 0.5] \) with \( f(0) = 0 \) and \( f(0.5) < 0.5 \) such that, for any \( v \in V \), if \((e_1,e_2)\) is a maximizer of \( \pi_{c,v} \) then \( e_1 \geq f(e_2) \) and \( e_2 \geq f(e_1) \).

Moreover, in the special case where \( v \in V \) is the maximum of two linear functions that cross at the prior \( q = 0.5 \) (as in Examples 1 and 2 above), if \((e_1,e_2) > 0\) is a maximizer of \( \pi_{c,v} \) then either \( e_1 > e_2 \) and \( e_2 = f(e_1) \) or \( e_2 > e_1 \) and \( e_1 = f(e_2) \).

**Proof.** We start with a lemma.

**Lemma 2.** Under the condition of the proposition, for any fixed \( e_1 > 0 \) the cost \( \psi(e_1,\cdot) \) is strictly convex in \( e_2 \), strictly decreasing when \( e_2 \) is sufficiently close to 0, and strictly increasing when \( e_2 \) is sufficiently close to 0.5. The analogous result for the function \( \psi(\cdot, e_2) \) of \( e_1 \) holds as well.

It follows from Lemma 2 that for every \( e_1 > 0 \) there is \( f(e_1) \in (0,0.5) \) such that \( \psi(e_1,\cdot) \) is strictly decreasing for \( e_2 \in [0,f(e_1)] \) and strictly increasing for \( e_2 \in [f(e_1),0.5] \). The properties of \( f \) (continuous, increasing, \( f(0) = 0 \), \( f(0.5) < 0.5 \)) follow immediately from the formula in the proof of Lemma 2 that characterizes \( f(e_1) \). By symmetry, the same \( f \) applies when \( e_2 \) is held fixed.

Suppose that \( e = (e_1,e_2) \) satisfy \( e_2 < f(e_1) \). Define \( e' = (e_1,f(e_1)) \). Then \( \psi(e) > \psi(e') \) by the definition of \( f(e_1) \) and \( V_v(e) \leq V_v(e') \) for any \( v \in V \) since \( b(e) \) is a garbling of \( b(e') \). It follows that \( \pi_{c,v}(e) < \pi_{c,v}(e') \). A similar argument applies when \( e_1 < f(e_2) \).

This proves the first part of the proposition.

As for the second part, we need the following.

**Lemma 3.** If \( v \in V \) is the maximum of two linear functions that cross at the prior \( q = 0.5 \) then there are \( a > 0 \), \( b \in \mathbb{R} \) such that \( V(e_1,e_2) = a \max\{e_1,e_2\} + b \), i.e., \( V \) is increasing and linear in the accuracy of the more accurate signal.

Suppose now that \((e_1,e_2) > 0\) is a maximizer of \( \pi_{c,v} \), and consider first the case \( e_1 > e_2 \). From the first part of the proposition we have \( e_2 \geq f(e_1) \). We claim that this inequality must hold as equality. Indeed, suppose by contradiction that \( e_2 > f(e_1) \). Then \( \max\{e_1,e_2\} = \max\{e_1,f(e_1)\} = e_1 \), so by Lemma 3 \( V(e_1,e_2) = V(e_1,f(e_1)) \). Also, \( \psi(e_1,e_2) > \psi(e_1,f(e_1)) \) by the definition of \( f(e_1) \). Hence \( \pi_{c,v}(e_1,e_2) < \pi_{c,v}(e_1,f(e_1)) \),
contradicting the optimality of \((e_1, e_2)\). By a similar argument, if \(e_2 > e_1\) is a maximizer then \(e_1 = f(e_2)\) must hold.

Finally, we need to show that equal efforts can’t be optimal when \(v\) is the maximum of two linear functions that cross at \(q = 0.5\). This follows by the same argument as in the proof of Proposition 1 (for the function \(v'\)). □

It is interesting to contrast Proposition 2 with the leading example in Rahman [21]. The principal in that example is interested in implementing an action profile in which the first agent (the ‘expert’) works and the second agent (the ‘monitor’) rests, on the grounds that monitoring is not productive. While Rahman does not study the cost of implementation, in the contract he proposes to virtually implement this profile the expected payment to the expert is independent of the probability that the monitor monitors, which implies that the overall expected payment is increasing in this probability. Here, on the other hand, even if we ignore the value of information provided by expert 2 (so his signal is only used to monitor expert 1, as in the case where \(v\) is the maximum of two linear functions that cross at the prior), it is optimal to induce him to exert strictly positive effort, as this significantly reduces the expected payment to expert 1.

2.4.2. Many experts. Deriving properties of optimal contracts with a large number of experts is more challenging, as the value and cost of information become complicated objects. However, when the number of experts \(n\) is even, Proposition 1 generalizes as follows.

**Proposition 3.** Let \(n \geq 2\) be even and fix \(v \in \mathcal{V}\) and \(\epsilon > 0\). Then there is \(\tilde{v} \in \mathcal{V}\) such that

(i) \(|\tilde{v}(q) - v(q)| \leq \epsilon\) for all \(q \in [0, 1]\); and

(ii) For every \(c \in \mathcal{C}\), if \(e = (e_1, \ldots, e_n) > 0\) is a maximizer of \(\pi_{c,\tilde{v}}\) among all vectors of \(n\) experts then there are \(i \neq j\) such that \(e_i \neq e_j\).

**Sketch of proof:**
Define the function \(v'\) in the same way as in the proof of Proposition 1. Consider any \(e_1 = \ldots = e_n > 0\). Then it is not hard to check that the function \(V_{v'}(e_1 + \delta, e_2, \ldots, e_{n-1}, e_n - \delta)\) of \(\delta\) has a strict local minimum at \(\delta = 0\), and, moreover, the right derivative is strictly positive and left derivative is strictly negative at that point. Defining \(\tilde{v} = v + v'\) and repeating the argument in the proof of Proposition 1 completes the proof. Note that if \(n\) is odd then the optimal action for the DM in the decision problem \(v'\) is independent of \(\delta\) (for \(\delta\) close to 0), which implies that \(V_{v'}(e_1 + \delta, e_2, \ldots, e_{n-1}, e_n - \delta)\) is smooth at \(\delta = 0\), hence the failure of the argument. □
Proposition 4. Suppose that $c \in C$ satisfies $c'(0.5) < +\infty$ and $c'(0) > 0$. Then there is $\bar{n}$ such that for every $v \in V$ hiring more than $\bar{n}$ experts is not optimal. More precisely, hiring two experts with efforts $(0.5, 0.5)$ to fully learn the state is less costly than hiring any number $n \geq \bar{n}$ of experts with any positive effort levels $e = (e_1, \ldots, e_n) > 0$.

Proof. First, note that the ratio 
\[
\frac{e(N) + \bar{e}(N)}{e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})}
\]
is bounded below by $\frac{1}{2}$, which implies that for every $e = (e_1, \ldots, e_n) > 0$ 
\[
\psi(e) = [e(N) + \bar{e}(N)] \sum_{i=1}^{n} \frac{c'(e_i)}{e(N \setminus \{i\}) - \bar{e}(N \setminus \{i\})} \geq \frac{1}{2} \sum_{i=1}^{n} c'(e_i).
\]
Convexity of $c$ implies that $c'(e_i) \geq c'(0)$ for all $i$, which gives $\psi(e) \geq \frac{1}{2} c'(0)$. On the other hand, the cost of learning the state by hiring two experts is $\psi(0.5, 0.5) = 2c'(0.5)$. Thus, $\bar{n} = \frac{4c'(0.5)}{c'(0)}$ satisfies the claim of the proposition. \qed

3. A General Implementability Result

We now leave the binary-binary model of the previous section and consider a more general framework. Our goal in this section is to characterize the information structures that the DM can achieve as equilibrium outcomes of some contract.

The set of possible states of nature is $\Omega = \{\omega_1, \ldots, \omega_K\}$, and the prior belief over $\Omega$ is $\gamma = (\gamma_1, \ldots, \gamma_K)$. For expositional reasons we assume that $\gamma_k > 0$ for all $k = 1, \ldots, K$. The DM may obtain information regarding the state of nature from a group of $n \geq 2$ experts, where $N = \{1, \ldots, n\}$ denotes the set of experts. The finite set of signals that expert $i$ may (privately) observe is $S_i$, with a typical element denoted by $s_i$.

In the model of the previous section experts choose effort levels $e_i$ and reporting strategies $r_i$. Here instead we assume that experts directly choose information structures.\\footnote{In Appendix B we construct an alternative model similar to the one of the previous section (where experts choose efforts and reports), and show that the two models are equivalent in terms of the information that the DM can implement.} More precisely, each expert $i$ chooses a mapping $m_i : \Omega \to \Delta(S_i)$ from a set $M_i$ of such mappings. As is standard, we view each $m_i \in M_i$ as a stochastic matrix with $K$ rows and $|S_i|$ columns, where $m_i(k, s_i)$ is the probability that signal $s_i \in S_i$ realizes conditional on the state being $\omega_k \in \Omega$. The set $M_i$ is viewed as a subset of $\mathbb{R}^{K|S_i|}$ endowed with the standard Euclidean norm $\| \cdot \|$.
To capture the idea that experts can freely use any reporting strategy, as well as to randomize their actions, we make the following assumptions on the sets $M_i$, $i = 1, \ldots, n$:

(A1) $M_i$ is convex.

(A2) $M_i$ is closed under garblings: If $m_i \in M_i$ and $m'_i$ is a garbling of $m_i$ then $m'_i \in M_i$.

(A3) $M_i$ is non-empty and closed.

Convexity (A1) allows experts to play mixed strategies: If $m_i$ and $m'_i$ are both possible choices for $i$ then by randomizing between the two $i$ can induce any matrix between them.

(A2) captures the ability of an expert to misreport his signal. Indeed, if $i$ chooses $m_i$ and then reports to the DM using the reporting strategy $r_i : S_i \rightarrow \Delta(S_i)$ then from the point of view of the DM this is as if $i$ has chosen a garbling of $m_i$. By including all such garblings in $M_i$ we allow $i$ to use any reporting strategy. Finally, closedness (A3) is assumed for technical reasons.

For each expert $i$, the cost of choosing $m_i \in M_i$ is described by a function $C_i : M_i \rightarrow \mathbb{R}_+$. We make the following assumptions:

(A4) $C_i$ is convex on $M_i$.

(A5) $C_i$ is increasing with respect to informativeness: If $m'_i$ is a garbling of $m_i$ then $C_i(m_i) \geq C_i(m'_i)$.

(A6) If $m_i$ is uninformative (i.e., constant) then $C_i(m_i) = 0$.

(A7) $C_i$ is Lipschitz continuous on $M_i$: There is a constant $B > 0$ such that $|C_i(m_i) - C_i(m'_i)| \leq B\|m_i - m'_i\|$ for every $m_i, m'_i \in M_i$.

To understand why these are reasonable assumptions, consider first (A4). If $m_i = \lambda m'_i + (1 - \lambda)m''_i$ for some $\lambda \in (0,1)$ then one possible way for $i$ to induce $m_i$ is by randomizing between $m'_i$ and $m''_i$ with the corresponding probabilities $\lambda$ and $1 - \lambda$. Thus, the cost $C_i(m_i)$ cannot exceed $\lambda C_i(m'_i) + (1 - \lambda)C_i(m''_i)$. For (A5), recall that $i$ can achieve any garbling of $m_i$ by randomizing his report in different ways after privately observing his signal. This implies that a garbling of $m_i$ cannot be more costly than $m_i$ itself. (A6) is a normalization that captures the idea that pure noise is costless. (A7) is a strong continuity assumption, which implies that the cost can’t increase too steeply when informativeness increases.\footnote{It is worth pointing out that convexity of $C_i$ implies that it is Lipschitz on any closed subset of the relative interior of $M_i$ (Rockafellar [23, Theorem 10.4]). Therefore, (A7) only has a bite on the relative boundary of $M_i$.}

Let $S = S_1 \times \ldots \times S_n$ with $s = (s_1, \ldots, s_n) \in S$ denoting a vector of signal realizations, and let $M = M_1 \times \ldots \times M_n$ with a typical element $m = (m_1, \ldots, m_n)$. Any $m \in M$
induces a distribution over $S$ that we denote by $\mathbb{P}_m$:

\[
\mathbb{P}_m(s) = \sum_{k=1}^{K} \gamma_k \prod_{i=1}^{n} m_i(k, s_i).
\]

Note that this assumes signals of different experts are independent conditional on the state of nature.

The DM can incentivize the experts to gather information by offering them payments contingent on the vector of reported signals. A contract is a list $x = (x_1, \ldots, x_n)$, where $x_i : S \rightarrow \mathbb{R}_+$ is the payment to $i$ given $s \in S$. Each contract induces a game between the experts. The set of strategies for expert $i$ is $M_i$ and his payoff given strategy profile $m$ is

\[
U_i(m; x_i) := \mathbb{E}_{s \sim \mathbb{P}_m}[x_i(s)] - C_i(m_i).
\]

**Definition 1.** Say that $m^* \in M$ is implementable if there exists a contract $x$ such that $m^*$ is an equilibrium of the game induced by $x$.

To characterize implementable information structures we resort to an idea of Rahman [21, Theorem 1]: Fix $m^*$ and an expert $i$. Suppose that all other experts except $i$ follow their prescribed choices $m^*_{-i}$, and the DM tries to design a contract $x_i$ with expert $i$ to induce him to choose $m^*_i$. Note first that if there is $m_i \neq m^*_i$ that induces the same distribution over $S$ (i.e., $\mathbb{P}_{m^*} = \mathbb{P}_{(m_i, m^*_{-i})}$) and is less costly (i.e., $C_i(m_i) < C_i(m^*_i)$), then there is no way the DM can implement $m^*$. Suppose now that this is not the case, i.e., that every $m_i \neq m^*_i$ is either (i) at least as costly as $m^*_i$, or (ii) induces a different distribution over $S$ than $m^*_i$ (or both). Then a deviation from $m^*_i$ to some $m_i$ can be discouraged either by choosing a constant contract (in case (i)) or by choosing $x_i(s)$ sufficiently large for $s$ that has higher probability under $m^*$ than under $(m_i, m^*_{-i})$ (in case (ii)). Of course, the difficulty is to find one contract that simultaneously discourages all possible deviations from $m^*_i$. The existence of such contract is a consequence of the minmax theorem.

**Theorem 3.** Information $m^*$ is implementable if and only if for every $i \in N$ and every $m_i \in M_i$ at least one of the following is true:

(i) $\mathbb{P}_{m^*} \neq \mathbb{P}_{(m_i, m^*_{-i})}$, or

(ii) $C_i(m_i) \geq C_i(m^*_i)$.

**Proof.** The ‘only if’ part is obvious, so we only prove the ‘if’ part. Fix $m^*$ and $i$. Let $D > 0$ be a large constant to be determined later. Define the function $f : M_i \times [0, D]^S \rightarrow$
Thus, $f(m_i, x_i)$ is the payoff gain (or loss) for expert $i$ when he chooses $m_i$ rather than the prescribed $m_i^*$, all other experts choose according to $m_{-i}^*$, and given contract $x_i$. The following lemma gives some basic properties of $f$ and its domain.

**Lemma 4.** The set $M_i$ is compact and convex. The function $f$ defined in (8) is concave and continuous in $m_i$ for each fixed $x_i$, and affine (and continuous) in $x_i$ for each fixed $m_i$.

By Lemma 4 the conditions of the minmax theorem (see Rockafellar [23, Corollary 37.6.2]) are satisfied and therefore

$$
\max_{m_i \in M_i} \min_{x_i \in [0,D]^S} f(m_i, x_i) = \min_{x_i \in [0,D]^S} \max_{m_i \in M_i} f(m_i, x_i). \tag{9}
$$

We now show that the left-hand side of (9) equals zero. Note first that $f$ vanishes whenever $m_i = m_i^*$, and hence that

$$
\max_{m_i \in M_i} \min_{x_i \in [0,D]^S} f(m_i, x_i) \geq 0. \tag{10}
$$

To show the other inequality, define the mapping $T : M_i \to \mathbb{R}^S$ by $T(m_i)(s) = \mathbb{P}_{m_i, m_{-i}^*}(s) - \mathbb{P}_{m_i^*}(s)$ for each $s \in S$. In words, $T$ sends each $m_i$ to the difference between the distribution over $S$ that it induces (together with $m_{-i}^*$) and the distribution over $S$ that the desired $m_i^*$ induces (together with $m_{-i}^*$). Now, given some $m_i \in M_i$, define the contract $x_i$ by

$$
x_i(s) = \begin{cases} 
D & \text{if } T(m_i)(s) < 0 \\
0 & \text{if } T(m_i)(s) \geq 0.
\end{cases}
$$

Thus, according to $x_i$ the expert gets paid $D$ at signal realizations that are more likely to occur under $m^*$ than under $(m_i, m_{-i}^*)$, and gets nothing at the other realizations. The value of $f$ with these $m_i, x_i$ is

$$
f(m_i, x_i) = D \left[ \sum_{s : T(m_i)(s) < 0} \left( \mathbb{P}_{m_i, m_{-i}^*}(s) - \mathbb{P}_{m_i^*}(s) \right) \right] + (C_i(m_i^*) - C_i(m_i))
$$

$$
= -\frac{D}{2} \sum_{s \in S} \left| \mathbb{P}_{m_i, m_{-i}^*}(s) - \mathbb{P}_{m_i^*}(s) \right| + (C_i(m_i^*) - C_i(m_i))
$$

$$
\leq -\frac{D \sqrt{|S|}}{2} \|T(m_i)\| + (C_i(m_i^*) - C_i(m_i)),
$$
where the second equality follows from the fact that both \( \mathbb{P}_{(m_i, m^*_i)} \) and \( \mathbb{P}_{m^*} \) are distributions (sum-up to 1), and the inequality by a standard relation between the \( l_1 \) and \( l_2 \) norms.\(^{13}\)

The mapping \( T \) is affine. Denote by \( \tilde{m}_i \) the projection of \( m_i \) to the (compact, convex, non-empty) set \( M_i \cap \text{Ker}(T) \), where \( \text{Ker}(T) \) is the kernel of \( T \). Then the assumption of the theorem implies that \( C_i(\tilde{m}_i) \geq C_i(m^*_i) \). Hence,

\[
f(m_i, x_i) \leq -\frac{D\sqrt{|S|}}{2} \|T(m_i)\| + (C_i(\tilde{m}_i) - C_i(m_i)).
\]

To complete the proof we need the following.

**Lemma 5.** There is \( B' > 0 \) such that \( \|T(m_i)\| \geq B'\|m_i - \tilde{m}_i\| \) for every \( m_i \), where \( \tilde{m}_i \) is the projection of \( m_i \) to the set \( M_i \cap \text{Ker}(T) \).

Combining Lemma 5 and assumption (A7) we get that

\[
f(m_i, x_i) \leq -\frac{D\sqrt{|S|}}{2} B'\|m_i - \tilde{m}_i\| + B\|m_i - \tilde{m}_i\| = \left(-\frac{D\sqrt{|S|}}{2} B' + B\right)\|m_i - \tilde{m}_i\|
\]

for some constants \( B, B' > 0 \) independent of \( m_i \). Therefore, for \( D = \frac{2B}{B'\sqrt{|S|}} \) we have

\[
\max_{m_i \in M_i} \min_{x_i \in [0,D]^S} f(m_i, x_i) \leq 0,
\]

which together with (10) imply that \( \max_{m_i \in M_i} \min_{x_i \in [0,D]^S} f(m_i, x_i) = 0 \).

To conclude the proof, it follows from the minmax equality (9) that for \( D \) large enough \( \min_{x_i \in [0,D]^S} \max_{m_i \in M_i} f(m_i, x_i) = 0 \). That is, there exists \( x_i \in [0,D]^S \) such that \( f(m_i, x_i) \leq 0 \) for every \( m_i \in M_i \). Repeating the same argument for each expert \( i \) we get a contract \( x \) that implements \( m^* \).

**Example 4.** Suppose \( \Omega = \{\omega_1, \omega_2, \omega_3\} \), \( \gamma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \), \( n = 2 \), and \( S_1 = S_2 = \{0,1\} \). Let \( m^*_1, m^*_2, m_2 \) be given by

\[
\begin{array}{c|c|c}
\omega_1 & \omega_2 & \omega_3 \\
\hline
m^*_1 & 3/4 & 1/4 \\
\hline
m^*_2 & 1/2 & 1/2 \\
\hline
m_2 & 1/4 & 3/4
\end{array}
\]

It is straightforward to check that \( \mathbb{P}_{(m^*_1, m^*_2)} = \mathbb{P}_{(m^*_1, m_2)} \). Hence, if \( C_2(m^*_2) > C_2(m_2) \), then by Theorem 3 \( (m^*_1, m^*_2) \) is not implementable.

\(^{13}\)Note that we use \( \|\cdot\| \) also for the \( l_2 \) norm on the space \( \mathbb{R}^S \).
When is it the case that a given profile \( m^* \) can always be implemented, regardless of the cost functions? In the above Example 4 \( m^* \) may not be implementable since \( m_2^* \) and \( m_2 \) induce the same distribution over \( S \) when combined with \( m_1^* \). But if every two information structures of each expert \( i \) induce different distributions over \( S \) (when combined with \( m_{-i}^* \)) then by Theorem 3 implementation is always possible. To formalize this, denote by \( \mathbb{P}_{m_{-i}^*|\omega_k} \) the distribution over \( S_{-i} \) induced by \( m_{-i}^* \) conditional on state \( \omega_k \), that is, \( \mathbb{P}_{m_{-i}^*|\omega_k}(s_{-i}) = \prod_{j \neq i} m_j^*(k, s_j) \). In the next proposition we view each \( \mathbb{P}_{m_{-i}^*|\omega_k} \) as a vector in \( \mathbb{R}^{S_{-i}} \).

**Proposition 5.** Given \( m^* \in M \), if for every \( i \in N \) the \( K \) vectors \( \left\{ \mathbb{P}_{m_{-i}^*|\omega_k} \right\}_{k=1}^K \) are linearly independent then \( m^* \) is implementable.

**Proof.** We will show that if the vectors \( \left\{ \mathbb{P}_{m_{-i}^*|\omega_k} \right\}_{k=1}^K \) are linearly independent then \( \mathbb{P}_{m^*} \neq \mathbb{P}_{(m_i, m_{-i}^*)} \) for every \( m_i \neq m_i^* \). Note that, by Theorem 3, this is sufficient to complete the proof.

Suppose that for some \( m_i \) we have \( \mathbb{P}_{m^*} = \mathbb{P}_{(m_i, m_{-i}^*)} \). Fix some \( s_i \in S_i \). Then for every \( s_{-i} \in S_{-i} \) the probability of \((s_i, s_{-i})\) is the same under \( m_i^* \) and \( m_i \), that is

\[
\sum_{k=1}^K \gamma_k \left( \prod_{j \neq i} m_j^*(k, s_j) \right) m_i^*(k, s_i) = \sum_{k=1}^K \gamma_k \left( \prod_{j \neq i} m_j^*(k, s_j) \right) m_i(k, s_i).
\]

Denoting \( y_k = m_i^*(k, s_i) - m_i(k, s_i) \) \((1 \leq k \leq K)\) we get that for every \( s_{-i} \)

\[
\sum_{k=1}^K \gamma_k y_k \mathbb{P}_{m_{-i}^*|\omega_k}(s_{-i}) = 0,
\]

but by linear independence this implies that \( y_k = 0 \) for all \( k \). It follows that \( m_i(k, s_i) = m_i^*(k, s_i) \) for every \( k \), and since \( s_i \) was arbitrary we get \( m_i^* = m_i \). \( \square \)

We conclude this section with the following two corollaries of Proposition 5. The first one shows that increasing informativeness cannot hinder implementability. The second shows that in the special case of \( K = 2 \) states implementation of any (informative) structure is possible.

**Corollary 2.** Suppose that \( m^* \) satisfies the assumption of Proposition 5, so it is implementable regardless of the cost functions. Let \( m^{**} \) be such that, for each \( i \in N \), \( m_{-i}^{**} \) is at least as informative as \( m_{-i}^* \) (i.e., \( m_{-i}^{**} \) is a garbling of \( m_{-i}^* \) in the sense of Blackwell). Then \( m^{**} \) also satisfies the the assumption of Proposition 5 and is therefore implementable regardless of the cost functions.
Proof. Let \( m^* \) and \( m^{**} \) be as in the corollary and fix \( i \). Then there is a stochastic matrix \( r \) with dimensions \( |S_{-i}| \times |S_{-i}| \) such that \( m_{-i}^{**} r = m_{-i}^* \). Suppose that \( \{ y_k \}_{k=1}^K \) are real numbers such that for all \( s_{-i} \)

\[
\sum_{k=1}^K y_k \mathbb{P}_{m_{-i}^{**}|\omega_k}(s_{-i}) = 0.
\]

Then for every \( s_{-i} \)

\[
\sum_{k=1}^K y_k \mathbb{P}_{m_{-i}^{**}|\omega_k}(s_{-i}) = \sum_{k=1}^K y_k \sum_{s_{-i}'} \mathbb{P}_{m_{-i}^{**}|\omega_k}(s_{-i}') r(s_{-i}', s_{-i}) = \sum_{s_{-i}'} \sum_{k=1}^K y_k \mathbb{P}_{m_{-i}^{**}|\omega_k}(s_{-i}') r(s_{-i}', s_{-i}) = 0,
\]

which by assumption implies that \( \{ y_k \}_{k=1}^K \) are all zero. Thus, \( \{ \mathbb{P}_{m_{-i}^{**}|\omega_k} \}_{k=1}^K \) are linearly independent.

Corollary 3. Suppose \( K = 2 \) and let \( m^* \) be such that, for all \( i \in N \), \( m^*_i \) is not completely uninformative (constant). Then \( m^* \) is implementable.

Proof. Fix \( i \). Since for each \( j \neq i \) the distribution over \( S_j \) conditional on \( \omega_1 \) is different than the distribution over \( S_j \) conditional on \( \omega_2 \) under \( m_j^* \), it follows that \( \mathbb{P}_{m_{-i}^*|\omega_1} \neq \mathbb{P}_{m_{-i}^*|\omega_2} \). Since both are distributions they must be linearly independent.

Note that implementability of any positive efforts’ vector in the binary-binary model of Section 2 can be viewed as a special case of Corollary 3.

4. The cost of implementation

To be added...

5. Concluding remarks

To be added...

References


Appendix A. Proofs of auxiliary lemmata

Proof of Lemma 1:
Fix $e_1 \geq e_2$. The set of possible signals in the information structures $b(e)$ can be identified with $\{\emptyset, 1, 2, 12\}$, corresponding to the coalition of experts who got signal $h$. For each signal $A$ in this set denote by $p_e(A) = \frac{1}{2} [e(A)e(A^c) + \bar{e}(A)e(A^c)]$ the probability that signal $A$ is observed, and by $q_e(A) = \frac{\frac{1}{2}e(A)e(A^c)}{p_e(A)}$ the posterior probability that the state is $H$ after signal $A$ is observed (assuming a uniform prior). We view the posterior of state $H$ as a $[0, 1]$-valued random variable which takes the values $\{q_e(A)\}$ with corresponding probabilities $\{p_e(A)\}$. The cumulative distribution function (cdf) of this variable is

$$F_e(t) = \sum_{\{A: q_e(A)\leq t\}} p_e(A).$$

Let $e'_1 \geq e'_2$ be obtained from $(e_1, e_2)$ by a PD transfer, i.e., $e_1 + e_2 = e'_1 + e'_2$ and $e_1 \geq e'_1$. The probabilities $p_{e'}(A)$ and $q_{e'}(A)$, and the cdf $F_{e'}(t)$ are defined in an analogous way to the above definitions. By Blackwell and Girshick [4, Theorem 12.4.1 on page 332], $b(e)$ is more informative than $b(e')$ if and only if

$$\int_0^x F_e(t)dt \geq \int_0^x F_{e'}(t)dt$$

holds for every $x \in [0, 1]$. To complete the proof we now show that (12) holds at the four atoms of $F_e$, i.e. at the points $x = q_e(\emptyset), q_e(1), q_e(2),$ and $q_e(12)$. Since $F_e$ and $F_{e'}$ are non-decreasing step-functions this would imply that (12) holds for every $x \in [0, 1]$. Indeed, if $\int_0^x F_e(t)dt < \int_0^x F_{e'}(t)dt$ at some $x \in [0, 1]$, then the same must be true at one of the jumps of $F_e$ adjacent to $x$.

We will need the following simple observations, whose proofs can be found at the end of this proof:

(a) $q_e(\emptyset) \leq q_e(2) \leq \frac{1}{2} \leq q_e(1) \leq q_e(12)$.
(b) $q_{e'}(\emptyset) \leq q_{e'}(2) \leq \frac{1}{2} \leq q_{e'}(1) \leq q_{e'}(12)$.
(c) $q_e(\emptyset) \leq q_{e'}(\emptyset), q_e(2) \leq q_{e'}(2), q_e(1) \leq q_{e'}(1),$ and $q_e(12) \leq q_{e'}(12)$.
(d) $F_e(t) = 1 - F_e(1-t)$ and $F_{e'}(t) = 1 - F_{e'}(1-t)$ for every $t \in [0, 1]$.

1. $x = q_e(\emptyset)$:

From observations (b) and (c) it immediately follows that $q_e(\emptyset)$ is smaller than the four possible posteriors under $e'$. Thus, $F_{e'}(t) = 0$ for every $t \in [0, q_e(\emptyset)]$, which implies $\int_0^{q_e(\emptyset)} F_{e'}(t)dt = 0$. Inequality (12) at $x = q_e(\emptyset)$ follows.

2. $x = q_e(2)$:

From observation (a) we have that $\int_0^{q_e(2)} F_e(t)dt = [q_e(2) - q_e(\emptyset)]p_e(\emptyset)$, and from observations (b) and (c) we have that either $\int_0^{q_e(2)} F_{e'}(t)dt = [q_e(2) - q_{e'}(\emptyset)]p_{e'}(\emptyset)$ or
\[ \int_0^{q_e(2)} F_e(t) \, dt = 0. \] In the latter case there is nothing to prove, so suppose the former is true. We therefore need to show that

\[ [q_e(2) - q_e(\emptyset)]p_e(\emptyset) \geq [q_e(2) - q_e'(\emptyset)]p_e'(\emptyset), \]

or equivalently that

\[ q_e(2)[p_e(\emptyset) - p_e'(\emptyset)] \geq q_e(\emptyset)p_e(\emptyset) - q_e'(\emptyset)p_e'(\emptyset). \]  

Using the equality \( e_1 + e_2 = e_1' + e_2' \), simple algebra gives that the right-hand side of (13) is equal to \( \frac{1}{2}(e_1e_2 - e_1'e_2') \). Also, it is easy to verify that \( p_e(\emptyset) - p_e'(\emptyset) = e_1e_2 - e_1'e_2' \), so (13) becomes

\[ q_e(2)(e_1e_2 - e_1'e_2') \geq \frac{1}{2}(e_1e_2 - e_1'e_2'). \]

Since the area of a rectangle with a given perimeter decreases in the difference between its length and its width, we have that \( e_1e_2 - e_1'e_2' \leq 0 \), and by observation (a) we have that \( q_e(2) \leq \frac{1}{2} \). This proves (13).

3. \( x = q_e(1) \):

This inequality is the “mirror image” of the inequality of the previous case. Indeed, using the symmetry of \( F_e \) around 0.5 (observation (d)) and a simple change of variables we get that

\[ \int_0^{q_e(1)} F_e(t) \, dt = q_e(1) - \int_0^1 F_e(t) \, dt + \int_0^{1-q_e(1)} F_e(t) \, dt, \]

and similarly that

\[ \int_0^{q_e(1)} F_e'(t) \, dt = q_e(1) - \int_0^1 F_e'(t) \, dt + \int_0^{1-q_e(1)} F_e'(t) \, dt. \]

Now, since the expected posterior is equal to the prior, we have that \( \int_0^1 F_e(t) \, dt = \int_0^1 F_e'(t) \, dt \). Thus, inequality (12) at \( x = q_e(1) \) is equivalent to \( \int_0^{1-q_e(1)} F_e(t) \, dt \geq \int_0^{1-q_e(1)} F_e'(t) \, dt \). But notice that \( 1 - q_e(1) = q_e(2) \), so the last inequality is the same as the one proved for \( x = q_e(2) \).

4. \( x = q_e(12) \):

As in the previous case, it is simple to show that inequality (12) at \( x = q_e(12) \) is equivalent to the inequality at \( x = q_e(\emptyset) \) proven above. We omit the details.

Proofs of observations (a)-(d):

(a): The posterior probability of state \( H \) is clearly nondecreasing (with respect to set inclusion) in the coalition of experts who obtained signal \( h \). Thus, to prove observation (a) we only need to check that \( q_e(2) \leq \frac{1}{2} \leq q_e(1) \). The latter inequality immediately
follows from $e_1 \geq e_2$, since

$$q_e(2) = \frac{1}{1 + \frac{(0.5+e_1)(0.5+e_2)}{(0.5-e_1)(0.5-e_2)}} \quad \text{and} \quad q_e(1) = \frac{1}{1 + \frac{(0.5+e_1)(0.5-e_1)}{(0.5+e_2)(0.5-e_2)}}.$$  

(b): The proof is identical to that of observation (a) (recall that $e'_1 \geq e'_2$).

(c): We have

$$q_e(\emptyset) = 1 + \frac{1}{(0.5+e_1)(0.5+e_2)} \quad \text{and} \quad q_e'(\emptyset) = 1 + \frac{1}{(0.5+e'_1)(0.5+e'_2)},$$

so we need to show that $(0.5+e_1)(0.5+e_2) \geq (0.5+e'_1)(0.5+e'_2)$. The latter is equivalent to $(0.5+e_1)(0.5+e_2)(e'_1e'_2 - e_1e_2) \geq (0.5-e_1)(0.5-e_2)(e'_1e'_2 - e_1e_2)$, which follows from $e'_1e'_2 \geq e_1e_2$.

Next,

$$q_e(2) = 1 + \frac{1}{(0.5+e_1)(0.5+e_2)} \quad \text{and} \quad q_e'(2) = 1 + \frac{1}{(0.5+e'_1)(0.5+e'_2)},$$

so $q_e(2) \leq q_e'(2)$ is equivalent to $(0.5+e_1)(0.5+e_2) \geq (0.5+e'_1)(0.5+e'_2)$, which follows from $e_1 \geq e'_1$ and $e_2 \leq e'_2$. The rest of the inequalities are proved in a similar fashion, the details are omitted.

(d): $F_e(t)$ is the probability that the posterior of state $H$ is less or equal to $t$, while $1 - F_e(1-t)$ is the probability that the posterior of state $L$ is less or equal to $t$. Since the prior and the information structure are symmetric between the two states, these two probabilities must be equal. The same argument holds for $F'_e$. \qed

Proof of Lemma 2:

From Corollary 1 we have

$$\frac{\partial \psi_2(e_1, e_2)}{\partial e_2} = c'(e_2) + \frac{0.5 + 2e_1e_2}{2e_1}c''(e_2)$$

and

$$\frac{\partial \psi_1(e_1, e_2)}{\partial e_2} = \frac{-c'(e_1)}{4e_2^2}.$$

The sum of these two terms is the partial derivative $\frac{\partial \psi}{\partial e_2}$. Fixing $0 < e_1 < 0.5$, $\frac{\partial \psi}{\partial e_2}$ is clearly negative when $e_2$ is sufficiently close to zero and positive when $e_2$ is sufficiently close to 0.5. As for coordinate-wise convexity, we have

$$\frac{\partial^2 \psi(e_1, e_2)}{\partial e_2^2} = 2c''(e_2) + \left(e_2 + \frac{1}{4e_1}\right)c'''(e_2) + \frac{c'(e_1)}{2e_2^3}.$$
Since $c'$ is assumed to be convex the third derivative $c'''$ is positive, which implies that \( \frac{\partial^2 \psi}{\partial e^2} \) is strictly positive for any \( e_1, e_2 > 0 \). □

Proof of Lemma 3:

Suppose that \( e_1 \geq e_2 \). Then the posterior probability of state \( H \) is greater or equal to \( q = 0.5 \) when expert 1 sends signal \( h \) and less or equal to \( q = 0.5 \) when he sends signal \( l \). Thus, the optimal choice for the DM is independent of expert 2's signal, which implies that \( V(e_1, e_2) = V(e_1) \). Finally, the distribution over posteriors induced by \( b_1(e_1) \) has a mass of 0.5 at the posterior \( q = 0.5 + e_1 \) and a mass of 0.5 at \( q = 0.5 - e_1 \). Therefore,

\[
V(e_1) = 0.5v(0.5 + e_1) + 0.5v(0.5 - e_1).
\]

Since \( v \) is the maximum of two linear functions that cross at \( q = 0.5 \) there are numbers \( a_1, b_1, a_2, b_2 \), with \( a_1 > a_2 \) such that

\[
v(0.5 + e_1) = a_1(0.5 + e_1) + b_1
\]

and

\[
v(0.5 - e_1) = a_2(0.5 - e_1) + b_2.
\]

Thus,

\[
V(e_1) = 0.5(a_1(0.5 + e_1) + b_1) + 0.5(a_2(0.5 - e_1) + b_2) = \frac{a_1 - a_2}{2} e_1 + \frac{0.5a_1 + b_1 + 0.5a_2 + b_2}{2}.
\]

The case \( e_2 > e_1 \) is similar. □

Proof of Lemma 4:

The set \( M_i \) is closed by assumption (A3) and bounded since it contains only stochastic matrices, which proves compactness. It is convex by (A1). The mapping \( m_i \mapsto \mathbb{P}_{(m_i, m^*_i)} \) is affine, and therefore \( \mathbb{E}_{x \sim \mathbb{P}_{(m_i, m^*_i)}}[x_i(s)] \) is affine in \( m_i \) for any fixed \( x_i \). Since \( C_i \) is convex (assumption (A4)) it follows that \( f(m_i, x_i) \) is concave in its first argument, and since \( C_i \) is continuous (assumption (A7)) it follows that \( f(m_i, x_i) \) is continuous in its first argument. Finally, \( f \) is clearly linear and continuous in \( x_i \) for any given \( m_i \). □

Proof of Lemma 5:

Let \( g : \mathbb{R}^p \to \mathbb{R}^q \) be a non-constant linear function and define the set \( B = Ker(g) \cap S \), where \( Ker(g) \) is the orthogonal complement of \( Ker(g) \) (the kernel of \( g \)) and \( S \) is the unit sphere of \( \mathbb{R}^p \). Let \( D_2 = \min_{z \in B} \| g(z) \| \), where the minimum is attained due to compactness and continuity. Moreover, \( D_2 > 0 \) by construction. Fix some \( y \in \mathbb{R}^p \) and
let \( \bar{y} \) be the projection of \( y \) to \( \text{Ker}(g) \). We claim that \( \|g(y)\| \geq D_2\|y - \bar{y}\| \). Indeed, the inequality is trivially satisfied when \( y \in \text{Ker}(g) \), and if \( y \notin \text{Ker}(g) \) then

\[
\|g(y)\| = \|g(y) - g(\bar{y})\| = \|y - \bar{y}\| \left\| g\left(\frac{y - \bar{y}}{\|y - \bar{y}\|}\right) \right\| \geq D_2\|y - \bar{y}\|,
\]

where the first equality follows from \( \bar{y} \in \text{Ker}(g) \), the second and third follow from the linearity of \( g \), and the inequality is by the definition of \( D_2 \).

To prove the lemma, note first that the above conclusion remains valid if \( g \) is affine rather than linear, so we may apply this to the function \( T \) of the lemma. Second, by convexity of \( M_i \), the projection \( \bar{m}_i \) of \( m_i \) to the set \( M_i \cap \text{Ker}(T) \) is the same as the projection of \( m_i \) to \( \text{Ker}(T) \). It follows that there is \( B' > 0 \) such that \( \|T(m_i)\| \geq B'\|m_i - \bar{m}_i\| \) for every \( m_i \in M_i \) as needed.

\[\square\]

**Appendix B. An alternative general framework**

The purpose of this appendix is to show how the model of Section 3 can be derived from a more primitive framework in which experts choose how much effort to exert and what to report to the DM (as in the binary-binary model of Section 2). In particular, we show that assumptions (A1)-(A7) of Section 3 are consequences of standard assumptions in this basic framework; and that the notion of implementability of information structures \( m^* \) in Definition 1 is equivalent to implementability of strategies in this basic framework that induce \( m^* \).

As in the model of Section 3, the set of possible states of nature is \( \Omega = \{\omega_1, \ldots, \omega_K\} \), and the common prior over \( \Omega \) is \( \gamma = (\gamma_1, \ldots, \gamma_K) \). The set of experts is \( N = \{1, \ldots, n\} \), and \( S_i \) is the set of signals that expert \( i \in N \) may observe. Each expert \( i \) chooses an effort level \( e_i \in [0, \bar{e}] \). Every \( e_i \) determines a mapping \( b_i(e_i) : \Omega \to \Delta(S_i) \), which we identify with a stochastic matrix with \( K \) rows and \( |S_i| \) columns. Here \( b_i(e_i)(k, s^i) \) is the probability that signal \( s^i \in S^i \) realizes conditional on the state being \( \omega_k \in \Omega \), given effort level \( e_i \).

We assume that higher effort increases informativeness: If \( e_i > e'_i \) then \( b_i(e_i) \) is strictly more informative than \( b_i(e'_i) \) in the sense of Blackwell [3]. Moreover, we assume that no effort results in no information, i.e., \( b_i(0) \) is constant. We will also need to assume that the technology \( b_i \) is continuous on \( [0, \bar{e}] \), and that its inverse \( b_i^{-1} \) is Lipschitz continuous on the image of \( b_i \)[14].

The cost of exerting effort \( e_i \) is \( c_i(e_i) \), where \( c_i : [0, \bar{e}] \to \mathbb{R}_+ \) is strictly increasing, Lipschitz continuous, and satisfies \( c_i(0) = 0 \).

[14] Since informativeness strictly increases with effort, it follows that \( b_i \) is one-to-one.
Let $S = S_1 \times \ldots \times S_n$. A contract is a list $x = (x_1, \ldots, x_n)$ with $x_i : S \to \mathbb{R}_+$ for each $i \in N$. A contract $x$ defines a game between the experts: A pure strategy for expert $i$ is a pair $(e_i, r_i)$, where $e_i$ is $i$'s effort level and $r_i : S_i \to S_i$ determines the report that $i$ sends to the DM as a function of the signal he observes. It is convenient to think of $r_i$ as a (stochastic) matrix of dimensions $|S_i| \times |S_i|$, with $r_i(s_i', s_i)$ equals 1 if $r_i(s_i') = s_i$ and equals zero otherwise. Notice that the information that the DM receives from $i$ under the strategy $(e_i, r_i)$ is a garbling of the information that $i$ privately observes, and that the stochastic matrix that describes this information structure is the product $b_i(e_i) r_i$.

We denote a pure strategy profile by $(e, r) = (e_1, \ldots, e_n, r_1, \ldots, r_n)$. Each $(e, r)$ induces a distribution over the vector of signals $s \in S$ that the DM observes, denoted $P_{(e, r)}$:

$$P_{(e, r)}(s) = \sum_{k=1}^{K} \gamma_k \prod_{i=1}^{n} (b_i(e_i) r_i)(k, s_i).$$

Note that (14) assumes that different experts’ signals are independent conditional on the state. The payoff to expert $i$ given pure strategy profile $(e, r)$ is

$$E_{s \sim P_{(e, r)}}[x_i(s)] - c_i(e_i).$$

Experts can also use mixed strategies. To avoid unnecessary technical issues we only consider finite-support distributions over pure strategies. When $i$ plays the mixed strategy $\sigma_i$ that assigns probability $\lambda^l$ to the pure strategy $(e_i^l, r_i^l) (l = 1, \ldots, L, \sum_l \lambda^l = 1)$, the induced information structure is the convex combination $m_i(\sigma_i) := \sum_l \lambda^l b_i(e_i^l) r_i^l$. With abuse of notation we denote the cost of such a mixed strategy $\sigma_i$ by $c_i(\sigma_i) = \sum_l \lambda^l c_i(e_i^l)$. The payoff in (15) is extended to profiles of mixed strategies as usual.

Let $M_i$ be the set of all information structures (i.e., mappings from $\Omega$ to $\Delta(S_i)$) that can be induced by some (pure or mixed) strategy of $i$:

$$M_i = \{ m_i(\sigma_i) : \sigma_i is a strategy of expert i \}.$$

For each $m_i \in M_i$ let $C_i(m_i)$ be the cost of the least costly way for $i$ to induce $m_i$:

$$C_i(m_i) = \inf \{ c_i(\sigma_i) : \sigma_i induces m_i \}.$$

**Lemma 6.** The set $M_i$ defined in (16) satisfies assumptions (A1)-(A3) of Section 3. The cost function $C_i$ defined in (17) satisfies assumptions (A4)-(A7) of Section 3.

**Proof.** First, every $m_i \in M_i$ is a convex combination of garblings of information structures from the image of $b_i$. Since informativeness increases with effort, and since the set of garblings of a given information structure is convex, it follows that every $m_i \in M_i$ is
a garbling of \( b_i(\bar{e}) \). But since \( i \) can choose effort \( \bar{e} \) and then garble his report in any way, it follows that \( M_i \) is equal to the set of all garblings of \( b_i(\bar{e}) \). This implies that \( M_i \) is a closed and convex polyhedral set, and that it is closed under garblings. Thus, (A1)-(A3) are satisfied.

Next, for every \( e_i \in [0, \bar{e}] \) let \( M_i(e_i) \) be the set of all garblings of \( b_i(e_i) \); this is the set of information structures that \( i \) can induce with effort of at most \( e_i \). By the previous paragraph we have \( M_i(\bar{e}) = M_i \), and increasing informativeness implies that \( M_i(e'_i) \subseteq M_i(e_i) \) whenever \( e'_i \leq e_i \). Define the mapping \( E_i : M_i \rightarrow [0, \bar{e}] \) by \( E_i(m_i) = \min\{ e_i \in [0, \bar{e}] : m_i \in M_i(e_i) \} \). In words, \( E_i(m_i) \) is the minimal effort needed to be able to induce \( m_i \). The minimum is attained due to the continuity of the function \( b_i \) and the upper hemi-continuity of the correspondence that assigns to each information structure the set of all of its garblings.

The cost function \( C_i \) is the convexification of the composite function \( c_i \circ E_i \), that is, \( C_i \) is the largest convex function on \( M_i \) that is point-wise below \( c_i \circ E_i \). This implies the convexity assumption (A4). Monotonicity with respect to informativeness (A5) follows from the monotonicity of \( b_i \) and \( c_i \). Next, (A6) follows from the un informativeness of \( b_i(0) \) and from \( c_i(0) = 0 \).

Finally, to prove (A7), ...

Now, consider the game of Section 3 with \( M_i \) from (16) and \( C_i \) from (17). We have the following.

**Lemma 7.** Fix a contract \( x \). Then \( m^* \in M \) is an equilibrium of the game of Section 3 if and only if there is an equilibrium \( \sigma \) such that \(^{15} m(\sigma) = m^* \). In particular, \( m^* \) is implementable according to Definition 1 if and only if there is a contract \( x \) and an equilibrium \( \sigma \) of the game induced by \( x \) such that \( m(\sigma) = m^* \).

**Proof.** We start by showing that the infimum in (17) is attained, i.e., that for any \( \bar{m}_i \in M_i \) there is \( \bar{\sigma}_i \) such that \( m_i(\bar{\sigma}_i) = \bar{m}_i \) and \( c_i(\bar{\sigma}_i) = C_i(\bar{m}_i) \). Indeed, it follows from the upper hemi-continuity of the correspondence that assigns to each information structure the set of its garblings that the mapping \( E_i \) from the previous proof is lower semi-continuous. Since \( c_i \) is continuous, we have that \( c_i \circ E_i \) is lower semi-continuous, and since \( C_i \) is the convexification of \( c_i \circ E_i \) it follows from standard arguments that the infimum in (17) is attained.

\(^{15}\) The notation \( m(\sigma) = m^* \) means that \( m^*_i = m_i(\sigma_i) \) for every \( i \in N \).
Now, suppose that \( \sigma \) is an equilibrium of the game induced by \( x \) and let \( m^* = m(\sigma) \). Consider a deviation \( m'_i \in M_i \) for expert \( i \). Let \( \sigma'_i \) be a strategy for \( i \) such that \( m_i(\sigma'_i) = m'_i \) and \( C_i(m'_i) = c_i(\sigma'_i) \) (existence of such \( \sigma'_i \) follows from the first part of the proof). Then the payoff to \( i \) by choosing \( m'_i \) in the game of Section 3 is the same as his payoff for choosing \( \sigma'_i \) in the game of this appendix, and his payoff by choosing \( m^*_i \) in the game of Section 3 is at least his payoff for choosing \( \sigma_i \) in the game of this appendix. Since \( \sigma_i \) is a best response to \( \sigma_{-i} \), it follows that deviating to \( m'_i \) is not profitable, hence \( m^* \) is an equilibrium of the game of Section 3.

To prove the converse, start with an equilibrium \( m^* \) of the game of Section 3, and consider \( \sigma \) such that \( m_i(\sigma_i) = m^*_i \) and \( c_i(\sigma_i) = C_i(m_i) \) for every \( i \). Then it is immediate to check in a similar way to the previous paragraph that \( \sigma \) is an equilibrium of the game of this appendix. This completes the proof. \( \square \)