Constrained-Efficient Profit Division in a Dynamic Partnership*

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Professional service partnerships that value collegiality often use the lock-step system to compensate their members. The lock-step system distributes profit based solely on seniority, hence it fails to reward and encourage high performance. When there are two members and each member has two productivity types, we propose a profit division mechanism that screens members and offers a bigger profit share to a member who has a higher type. The proposed mechanism satisfies constrained efficiency, periodic ex-post incentive compatibility, and periodically ex-post Pareto dominates the lock-step system. In addition, a high-type member collects all welfare gain from replacing the lock-step system with the constrained efficient mechanism. The corresponding profit division rule is implemented in Nash equilibrium by a voting mechanism, in which each member is given several menus of partnership arrangements and is asked to vote. We suggest that, in each period, each member receive a compensation package of non-equity income (fixed wage payment) and equity share (share of current net profit, which is current profit net of current wage payments). Since wage payments are drawn from profit and all resulting profit (or loss) is fully distributed, budget is always balanced. For an \( n \)-member static partnership, we propose a mechanism that satisfies constrained efficiency, Bayesian incentive compatibility, and Bayesian Pareto dominates the lock-step system. Our mechanisms also apply to partnerships outside professional service industries.

Keywords: dynamic partnership, profit sharing, interdependent values, adverse selection, moral hazard

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1 Introduction

Partnership is a business entity formed by two or more individuals. Partnership members contribute capital and effort in the knowledge that any future profit or loss will be divided according to the partnership agreement.\textsuperscript{1} Compared with other business structures (such as limited liability company and corporation), forming a partnership requires the least cost and paperwork. In addition, a partnership bears less tax burden than a corporation.\textsuperscript{2,3} Due to these advantages, the partnership structure is commonly chosen by small businesses. According to U.S. Census Bureau, in 2010, partnerships account for 50\% of non-employer firms\textsuperscript{4} owned by two or more individuals. In the Survey of Law Firm Economics (2014), 60 out of 162 participants are partnerships.\textsuperscript{5} In the Medical Group Compensation and Productivity Survey (2015), the corresponding figures are 17 and 66.\textsuperscript{6}

In many cases, partnership members benefit from complementary strengths. A functioning business typically requires a combination of technical skills, sales expertise, and social connections. Each member is assigned a role given his strength.\textsuperscript{7} On the other hand, a professional partnership (that provides professional services, such as litigation, accounting, consulting, and medical care) might be preferred over a solo practice for the following reasons.\textsuperscript{8} First, each member incurs smaller financial risks as his income depends on his own performance as well as his partners’. Second, a partnership offers a collegial environment for experience sharing, referrals, and cross-selling. As a result, each member receives access to richer information and a broader client base. Third, a larger firm usually has more capital and a stronger bargaining position in the market.\textsuperscript{9}

When a partnership is formed, a partnership agreement specifies an equity share allocation

\textsuperscript{1}In the absence of a partnership agreement, profit or loss is divided equally among members.

\textsuperscript{2}Most corporations are subject to double taxation: a corporate pays corporate tax, then its shareholders pay income tax on dividends. Partnerships and limited liability companies are pass-through entities: profit is taxed only once (through owners’ income tax).

\textsuperscript{3}For references on business structures, see https://www.sba.gov/ (U.S. Small Business Administration) and https://www.irs.gov/ (Internal Revenue Service).

\textsuperscript{4}The U.S. Census Bureau defines non-employer firm as “one that has no paid employees, has annual business receipts of $1, 000 or more ($1 or more in the construction industries), and is subject to federal income taxes”.

\textsuperscript{5}The Survey of Law Firm Economics is conducted annually by the National Law Journal and ALM Legal Intelligence.

\textsuperscript{6}The Medical Group Compensation and Productivity Survey is conducted annually by the American Medical Group Association.

\textsuperscript{7}See Wasserman (2012) for some discussion on complementary strengths in a partnership.

\textsuperscript{8}Professional partnerships benefit little from complementary strengths. In a single-specialty practice, all partners perform similar duties. In a multi-specialty practice, although partners might perform different tasks, gathering multiple specialties in one practice is unnecessary for its survival.

\textsuperscript{9}For some comparison between group practice and solo practice, see http://www.apapracticecentral.org (the APA Practice Organization for promoting and supporting practicing psychologists), https://www.acponline.org (the American College of Physicians), and https://www.healio.com/ophtalmology (Ocular Surgery News).
that dictates how future profit will be divided. For instance, a partner who has 30% of equity will receive 30% of future profit.\textsuperscript{10,11} According to Wasserman (2012), many startups divide equity equally among founders. In the lock-step system used by professional service firms, partners with the same level of seniority receive equal equity shares (Anderson 2013). The lock-step system is particularly popular among law firms in the UK, Europe, and Australia (Wesemann and Jarrett-Kerr 2015).\textsuperscript{12} Such equal division rule encourages collaboration (via experience sharing, referrals, and cross-selling) as each partner’s income depends on the group performance. However, it fails to punish shirkers and reward hard-working members. As a result, conflicts might arise, pushing the most talented partners to join other companies that reward them better. When only the least productive members remain, the partnership would struggle to compete and survive.\textsuperscript{13}

Under some suitable assumptions, this paper proposes a profit division mechanism that Pareto improves upon the equal division mechanism via rewarding hard work. We study a dynamic partnership in which each member has a privately known type that affects her marginal cost of working. Types are independently distributed across members and across time. A member has a high type if she has high ability, enjoys working, or incurs low marginal opportunity cost. We assume that members gain experience through working and share work experience among themselves. Hence, in each period, all members have the same experience level, which depends on past experience and past efforts.\textsuperscript{14} Our objective is to incentivize members to reveal their types and give a bigger equity share to a member who has a higher type, thereby maximizing members’ aggregate payoff.

The proposed mechanism suggests that, in each period, each member receive a compensation package of non-equity income (a fixed wage payment) and equity share (a share of current net profit, which is current profit net of current wage payments). Since wage payments are

\textsuperscript{10}In a limited liability partnership, the equity share allocation might not dictate how future loss will be divided (see https://www.sba.gov/). In this paper, we assume that a partnership never incurs a loss.

\textsuperscript{11}For a professional partnership, since individual performances are usually recorded, there are several available profit sharing rules. Instead of distributing profit based on equity shares, a partnership can rely on individual performances and some subjective criteria. See Anderson (2013) for a description of some available profit sharing rules in a professional partnership.

\textsuperscript{12}According to Wesemann and Jarrett-Kerr (2015), in Europe, 23% of law firms use the lock-step system and another 56% of law firms use some variations of the lock-step system. These figures are 18% and 54% in the UK. See Bartling and von Siemens (2010) for additional statistics on the popularity of the equal division rule. They provide a rationale for this popularity: if each member suffers some utility loss when his partners receive different rents, then the equal division rule is the unique profit division rule that maximizes work incentives.

\textsuperscript{13}For some discussion on the weaknesses of the equal division rule/ the lock-step system, see Wasserman (2012), Anderson (2013), and Henderson (2006).

\textsuperscript{14}The assumption of experience sharing is suitable for professional single-specialty partnerships. For professional multi-specialty partnerships, we assume that each member works and contributes to broadening the client base, which benefits everyone via referrals and cross-selling. For other types of partnerships, we assume that a firm with a bigger profit can afford more advanced technologies that reduce marginal costs of labor. In other words, in any partnership, an agent’s current effort lowers everyone’s future marginal cost.
drawn from the partnership profit and any resulting net profit (or loss) is fully distributed, budget is always balanced. We note that an equity share and a wage payment play different roles. A member’s effort contribution depends on his equity share, as a bigger equity share implies higher marginal benefit of effort. On the other hand, wage payments are independent of effort contributions and are designed to incentivize members to reveal their types. In particular, giving a member some fixed wage payment can deter her from attempting to acquire a bigger equity share by lying. Hence, for each member, two different compensation packages create different incentives even if they are equivalent to the same income.

The aforementioned compensation packages are part of a partnership arrangement that enters the partnership agreement in each period. A partnership arrangement specifies (enforceable) equity shares, (enforceable) wage payments, and (unenforceable) effort recommendations for all members. Our mechanism design ensures that members voluntarily follow effort recommendations given the specified profit division.\textsuperscript{15} For a two-member static partnership in which each member has two types, a welfare-maximizing arrangement is the outcome of a voting game that proceeds as follows. Each member is given several menus of arrangements and is asked to select one arrangement from each menu. The game outcome is an arrangement (across all menus) that receives the highest number of votes. In this voting game, members reveal their types through their voting decisions.

Here we highlight some features of the model. First, an equal division mechanism (such as the lock-step system) favors lower-type members and might push higher-type members to leave the partnership. Second, equity shares and wage payments are independent of individual efforts. In many cases, individual efforts are difficult to measure. Even when individual performances can be perfectly recorded, many partnerships choose to distribute the total output based on equity shares rather than rely on individual performances so as to avoid intra-firm competition (Anderson 2013). Third, a member’s valuation of his equity share depends on the partnership profit, which is determined by members’ effort contributions. Effort contributions, in turn, are endogenously determined (partly) by members’ private types and equity shares. Hence, our model exhibits adverse selection, moral hazard, and interdependent valuations.

Although an optimal partnership arrangement maximizes members’ aggregate payoff, it is constrained-efficient rather than efficient. Holmstrom (1982) studies a complete information static game in which members jointly contribute efforts to a team whose profit is distributed according to a pre-determined division rule.\textsuperscript{16} He shows that if members are risk neutral, then no budget-balanced profit division rule can induce the first-best efforts in a

\textsuperscript{15}Including effort recommendations simplifies our theoretical analysis. From a practical perspective, these recommended efforts might assist members in their decision making. A partnership agreement also usually has a section that specifies duties among members (see https://www.sba.gov/).

\textsuperscript{16}Profit shares depend on the team output rather than individual efforts (moral hazard).
pure strategy Nash equilibrium. This impossibility result is due to the free-riding problem: since each rational member does not internalize the positive externality of his effort on his partners, he fails to supply the first-best effort. Since an efficient arrangement (that recommends first-best efforts) typically\textsuperscript{17} cannot be enforced, our proposed mechanism suggests a constrained-efficient arrangement that maximizes members’ aggregate payoff subject to the constraint that members voluntarily follow the recommended efforts given the assigned compensation packages.

We note that for a self-managed partnership, a profit division mechanism that involves surplus destruction cannot be credible. Whenever the mechanism suggests that the partnership forgo some profit, its members would be better off re-negotiating and distributing that amount among themselves. According to Holmstrom (1982), if a team is managed by an outsider, then inducing the first-best efforts is possible: the manager threatens to confiscate all surplus if the first-best output is not achieved. This result implies that an efficient arrangement can be enforced by a mechanism that involves surplus destruction. However, as discussed earlier, we restrict our work to budget-balanced mechanisms for the case of self-managed partnerships.

The voting game described above is derived from a direct revelation mechanism that satisfies constrained efficiency, periodic ex-post incentive compatibility, and periodically ex-post Pareto dominates the equal division mechanism. In each period $t$, members report their types and receive an arrangement that is constrained-efficient if these reports are truthful.\textsuperscript{18} In each period $t$, even after all current private types are revealed, each member optimizes by reporting his type truthfully and following effort recommendation from period $t$ onwards (assuming that all other members report truthfully and follow effort recommendations from period $t$ onwards).\textsuperscript{19} In each period $t$, even after all current private types are revealed, all members weakly prefer to have the proposed mechanism implemented from period $t$ onwards rather than the equal division mechanism; in addition, the preference is strict for some member and some current type profile. Hence, as long as members prefer the equal division mechanism to quitting the partnership, they would be willing to participate in our mechanism.

Here we show how wage payments are designed to incentivize truth-telling. For illustration, we assume that, in the last period, a low-type member prefers to over-report her type if her partner has a high type and no wage payments are made.\textsuperscript{20} For preventing such over-

\textsuperscript{17}As discussed in the Literature Review below, achieving efficiency might require that members have unlimited endowments.

\textsuperscript{18}We assume that, from period $t + 1$ onwards, members report their types truthfully and constrained efficient arrangements are assigned.

\textsuperscript{19}Under periodic ex-post incentive compatibility, it is possible that after a member learns some new information in period $t + 1$, he regrets reporting truthfully and following effort recommendation in period $t$.

\textsuperscript{20}Here we make the following additional assumptions. First, the high-type member reports truthfully
reporting, a member receives some wage payment $x$ if she reports low type and her partner reports high type. A member receives no wage payment if she reports high type and her partner reports low type. When members report the same types, no wage payments are made. The wage payment $x$ ensures that a low-type member is indifferent between telling truth and lying when her partner has a high type. Due to some \textit{monotonicity} property, a low-type member prefers to tell truth if her partner has a low type. Due to some \textit{increasing differences} property, the wage payment $x$ is sufficiently small so as not to affect a high-type member’s reporting decision (she prefers to tell truth regardless of her partner’s type and regardless of whether $x$ is paid). Wage payments for other periods are constructed by backward recursion.

We can verify that all members weakly prefer to have the proposed mechanism implemented in the last period rather than the equal division mechanism. When members have the same types, the two mechanisms give the same outcome: members receive equal equity shares and no wage payments. When members have different types, if the low-type member lies and the high-type member tells truth, then the proposed mechanism assigns the equal division outcome. The fact that the low-type member is indifferent between lying and telling truth (shown above) implies that she receives the same payoff under either mechanism. Budget balance then implies that the high-type member collects all welfare gain from replacing the equal division mechanism by the proposed \textit{constrained efficient} mechanism. As a result, the high-type member strictly prefers the proposed mechanism.

For an $n$-member static partnership, we design a direct revelation mechanism that satisfies \textit{constrained efficiency}, \textit{Bayesian incentive compatibility}, and \textit{Bayesian Pareto dominates} the equal division mechanism. For each member, given her belief about her partners’ private types, (a) it is optimal to report truthfully and follow effort recommendation (assuming that all other members report truthfully and follow effort recommendations), (b) the proposed mechanism is at least as good as the equal division mechanism, and (c) the proposed mechanism is better than the equal division mechanism if she has a high type. Wage payments are sufficiently large to induce low-type members to reveal their types. They are also sufficiently small so as not to affect high-type members’ reporting decisions. Such wage payments exist due to the \textit{increasing differences} property.

The paper is organized as follows. Section 2 presents the model. Section 3 defines a \textit{constrained-efficient} arrangement rule, which assigns a \textit{constrained-efficient} arrangement to each type profile. Section 4 defines a \textit{direct revelation mechanism}, \textit{periodic ex-post incentive compatibility}, and \textit{periodic ex-post Pareto dominance over the equal division mechanism}. Section 5 presents some preliminary results. Sections 6 and 7 describe the proposed direct
revelation mechanisms for two-member partnerships and multiple-member partnerships respectively. Section 8 describes the voting mechanism. Section 9 concludes.

Literature Review.

The seminal paper on profit division in a team is Holmstrom (1982), who shows that a credible threat of surplus destruction can induce the first-best efforts in Nash equilibrium (as discussed above). The impracticality of surplus destruction for self-managed partnerships has led to a literature that provides sufficient conditions under which a budget-balanced profit division rule can induce the first-best outcome. Williams and Radner (1995) study the case of stochastic output and obtain a positive result under some conditions on the production function. However, these conditions appear strong and difficult to interpret. Rasmusen (1987) assumes that team members are risk averse and output is deterministic. He proposes some budget-balanced profit division rules that are feasible only if each member has a sufficiently large endowment. For instance, under the scapegoat contract, if the first-best output is not achieved, then a member (chosen at random) donates his endowment to the partnership instead of receiving any profit share. Legros and Matthews (1993) assume that team members are risk neutral and output is deterministic. They show that if each member has unlimited endowment, then the first-best outcome can be approximated by a mixed strategy equilibrium, in which one member deviates from his first-best effort with a small probability. In particular, when the output takes on certain values, the mixing member collects arbitrarily large payments from other team members. Legros and Matsushima (1991) assume that team members are risk neutral, output is stochastic, and the sets of effort levels are finite. One of their main findings is a budget-balanced profit division rule that induces the first-best efforts when the average of members’ endowments exceeds some finite threshold (as some member might be required to pay a large amount to his partners). Radner (1986) studies an infinitely repeated partnership and assumes that members do not discount future payoffs. In each period, each member receives some information about the state of the partnership, but cannot observe his partners’ efforts and information (imperfect monitoring). He shows that under certain conditions, the first-best outcome can be achieved in a Nash equilibrium of the supergame (Folk Theorem). Matsushima (1989) provides a Folk Theorem for an infinitely repeated partnership (with imperfect monitoring) in which members are very patient.

These papers suggest that efficiency might be unattainable if members do not have sufficiently large endowments or if the partnership is short-lived. According to Anderson (2013) and the U.S. Small Business Administration, popular profit division rules ensure that each member receives some income as long as the partnership is profitable. These compensation practices might indicate that members do not have sufficient endowments for the penalty rules described above to work. They might also indicate that the idea of penalizing some
random members (especially when the partnership is making a profit) is unappealing. Our paper makes no assumption on endowments, discount factor, or the length of a partnership. Our mechanism design, which offers combinations of equity incomes and wage payments, closely resembles popular compensation practices. Since the total profit is divided based on equity shares, the free-riding problem is always present. Hence, our mechanism is constrained efficient rather than efficient. Whether efficiency can be achieved under incomplete information is an open question. To the best of our knowledge, our paper is the first to study both moral hazard and adverse selection in a partnership.

Here we discuss several related papers in the mechanism design literature. Jehiel and Moldovanu (2001) study a static environment in which each agent’s payoff is a linear combination of all agents’ signals (interdependent values). When signals are one dimensional and some increasing differences property is met, they design a direct revelation mechanism that is efficient and ex-post incentive compatible. For our partnership problem, we prove that a similar increasing differences property holds. Our wage rule shares some features with the money transfer rule constructed by Jehiel and Moldovanu (2001). Our notion of periodic ex-post incentive compatibility follows Bergemann and Välimäki (2010), who generalize the pivot mechanism (Green and Laffont 1977) to dynamic settings with private values. Liu (2017) studies a dynamic environment with interdependent values under the assumption that an agent’s current signal is correlated with the other agents’ future signals. Our paper assumes that signals are independent across agents and across time.

2 Model

There are \( n \) agents seeking to form a partnership. The partnership lasts for \( T \) periods \((T < \infty)\). In each period \( t \in \{1, 2, \ldots, T\} \), each agent \( i \in N \equiv \{1, 2, \ldots, n\} \) has a privately known type \( \theta_{i,t} \) that is distributed according to a cumulative distribution function \( F_{i,t} \) with support \( \Theta_{i,t} \) (types are independently distributed across agents and across periods). We assume that \( \Theta_{i,t} \subset \mathbb{R}_{++} \) and \( \Theta_{i,t} \) is finite. Agent \( i \)'s marginal cost of working for the partnership in period \( t \) is strictly decreasing in \( \theta_{i,t} \).

Each period \( t \) consists of two stages.

**Stage 1.** A direct revelation mechanism (described in Section 4) or a voting mechanism (described in Section 8)

(a) chooses an equity share allocation \( r_t \equiv (r_{1,t}, \ldots, r_{n,t}) \in \Delta^{n-1} \equiv \{(r'_{1,t}, \ldots, r'_{n,t}) \in \mathbb{R}^n_+ \mid \sum_{i \in N} r'_{i,t} = 1\} \) and wages \( m_t \equiv (m_{1,t}, \ldots, m_{n,t}) \in \mathbb{R}^n_+ \) (the equity share allocation and wages are enforceable),
mechanism designer. For each period and each state, the mechanism designer chooses an
In this section, we assume that types are publicly observed and describe the objective of the
3 Constrained-Efficient Arrangement Rule
Stage 2. Agents simultaneously invest efforts $e_t = (e_{1,t}, \ldots, e_{n,t}) \in \mathbb{R}_+^n$ in the partnership. The partnership profit is $\pi(e_t)$. The cost of effort for each agent $i$ is $c_{i,t}(e_{i,t}, \theta_{0,t}, \theta_{i,t})$, where $\theta_{0,t} \in \mathbb{R}_{++}$ is a publicly observed state. At the end of period $t$, each agent $i$ receives his wage $m_{i,t}$ and net profit $[\pi(e_t) - \sum_{i \in N} m_{i,t}]$ is divided among partners according to the equity share allocation $r_t$. We note that the partnership treats wages $m_t$ as expenses.

We interpret the publicly observed state $\theta_{0,t}$ as the partnership’s level of experience at the beginning of period $t$. Period-1 experience $\theta_{0,1}$ takes some given positive real value. Agents gain experience through working and share work experience among themselves. Hence, for each period $t > 1$, the current experience level $\theta_{0,t}$ depends on past experience $\theta_{0,t-1}$ and effort contributions $e_{t-1}$ as follows: $\theta_{0,t} = \theta_{0,t-1} + \gamma \sum_{i \in N} e_{i,t-1}$, where $\gamma > 0$. We can write $\theta_{0,t} = \theta_{0,1} + \gamma \sum_{t=1}^{T} \sum_{i \in N} e_{i,t}$. Each agent $i$’s marginal cost of effort $\partial c_{i,t}(e_{i,t}, \theta_{0,t}, \theta_{i,t}) / \partial e_{i,t}$ is decreasing in the experience level $\theta_{0,t}$.

We refer to $\theta_t = (\theta_{0,t}, \theta_{1,t}, \ldots, \theta_{n,t})$ as the state of the partnership in period $t$, which consists of a publicly observed state $\theta_{0,t}$ and an unobservable state $\theta_{-0,t} = (\theta_{1,t}, \ldots, \theta_{n,t})$. Let $\Theta_t = \mathbb{R}_{++} \times \prod_{i=1}^{n} \Theta_{i,t}$ be the period-$t$ state space. In addition, let $\Theta_{-0,t} = \prod_{i=1}^{n} \Theta_{i,t}$ and $\Theta_{-i,t} = \mathbb{R}_{++} \times \prod_{j \neq i} \Theta_{j,t}$ for each $i \in N$.

Period-$t$ payoff of agent $i$ is

$u_{i,t}(r_t, m_t, e_t, \theta_t) \equiv r_{i,t} [\pi(e_t) - \sum_{j \in N} m_{j,t}] + m_{i,t} - c_{i,t}(e_{i,t}, \theta_{0,t}, \theta_{i,t})$.

Period-$t$ aggregate payoff is $u_t(r_t, m_t, e_t, \theta_t) \equiv \sum_{i \in N} u_{i,t}(r_t, m_t, e_t, \theta_{0,t}, \theta_{i,t})$.

Agents have a common discount factor $\delta \in (0, 1)$. The total payoff of each agent $i$ is

$$\sum_{t=1}^{T} \delta^{t-1} u_{i,t}(r_t, m_t, e_t, \theta_{0,t}, \theta_{i,t}).$$

For notational convenience, we let $\tilde{u}_{i,t}(r_t, e_t, \theta_{0,t}, \theta_{i,t}) \equiv r_{i,t} \pi(e_t) - c_{i,t}(e_{i,t}, \theta_{0,t}, \theta_{i,t})$ be agent $i$’s period-$t$ payoff before wages are paid and $\tilde{u}_{t}(r_t, e_t, \theta_t) \equiv \sum_{i \in N} \tilde{u}_{i,t}(r_t, e_t, \theta_{0,t}, \theta_{i,t})$ be the total surplus created in period $t$.

3 Constrained-Efficient Arrangement Rule
In this section, we assume that types are publicly observed and describe the objective of the mechanism designer. For each period and each state, the mechanism designer chooses an
arrangement which consists of equity shares, wage payments, and effort recommendations. Due to the free-riding problem, an *efficient* arrangement (that maximizes the expected sum of discounted aggregate payoffs) generally cannot be enforced (Holmstrom 1982). Hence, we introduce a weaker welfare-maximizing criterion: *constrained efficiency*. We define *efficiency* and *constrained efficiency* for a static partnership, then extend these notions for a dynamic partnership.

### 3.1 Arrangement Rule

At the beginning of period 1, the mechanism designer chooses an arrangement rule that assigns an arrangement for each period \( t \geq 1 \) and each state \( \theta_t \in \Theta_t \). An **arrangement rule** is a triple \( \alpha \equiv (s, w, \sigma) \), where

- \( s \equiv \{ s_t : \Theta_t \to \Delta_1^{n-1} \}_{t=1}^T \) is an equity sharing rule,
- \( w \equiv \{ w_t : \Theta_t \to \mathbb{R}_n^+ \}_{t=1}^T \) is a wage rule, and
- \( \sigma \equiv \{ \sigma_t : \Theta_t \to \mathbb{R}_n^+ \}_{t=1}^T \) is an effort recommendation rule.

For each period \( t \in \{1, \ldots, T\} \), we refer to \( s_t, w_t, \sigma_t \) as period-\( t \) equity sharing rule, period-\( t \) wage rule, period-\( t \) effort recommendation rule respectively, and refer to \( \alpha_t \equiv (s_t, w_t, \sigma_t) \) as period-\( t \) arrangement rule. For each state \( \theta_t \in \Theta_t \), the period-\( t \) arrangement rule \( \alpha_t \) assigns equity shares \( s_t(\theta_t) \equiv [s_{1,t}(\theta_t), \ldots, s_{n,t}(\theta_t)] \in \Delta_1^{n-1} \), wages \( w_t(\theta_t) \equiv [w_{1,t}(\theta_t), \ldots, w_{n,t}(\theta_t)] \in \mathbb{R}_n^+ \), and recommended efforts \( \sigma_t(\theta_t) \equiv [\sigma_{1,t}(\theta_t), \ldots, \sigma_{n,t}(\theta_t)] \in \mathbb{R}_n^+ \). We refer to \( \alpha_t(\theta_t) \equiv [s_t(\theta_t), w_t(\theta_t), \sigma_t(\theta_t)] \) as a period-\( t \) arrangement.

### 3.2 Static Partnership

Here we define *efficiency* and *constrained efficiency* for a static partnership. An arrangement rule \( \alpha \) is *efficient* if it selects an arrangement that maximizes the aggregate payoff (assuming that recommended efforts are enforceable): for each state \( \theta \in \Theta \), we have

\[
\alpha(\theta) \in \arg \max_{(r,m,e) \in \Delta_1^{n-1} \times \mathbb{R}_n^+} u(r, m, e, \theta).
\]

Fix a state \( \theta \in \Theta \) and let \( \alpha(\theta) \equiv (r^*, m^*, e^*) \) be the arrangement selected by an *efficient* arrangement rule \( \alpha \). Let \( G(\theta, r^*, m^*) \equiv [\mathbb{R}_+, u_i(r^*, m^*, \cdot, \theta, \theta_i)]_{i \in N} \) be a strategic-form game in which

- each agent \( i \) chooses an effort \( e_i \in \mathbb{R}_+ \),
- if agents make efforts \( e \in \mathbb{R}_n^+ \), then each agent \( i \) earns
\[ u_i(r^*, m^*, e, \theta_0, \theta_i) = r_i^* \left[ \pi(e) - \sum_{j \in N} m_j^* \right] + m_i^* - c_i(e_i, \theta_0, \theta_i). \]

We note that the recommended effort vector \( e^* \) solves \( \max_{e \in \mathbb{R}^n} \sum_{i \in N} u_i(r^*, m^*, e, \theta_0, \theta_i) \).

Due to the free-riding problem, there is no pure strategy Nash equilibrium of \( G(\theta, r^*, m^*) \) in which agents choose the recommended efforts \( e^* \) (Holmstrom 1982). To explain this impossibility result, we fix an agent \( i \) and assume that other agents choose \( e_{i-}^* \). An agent’s effort creates some positive externality on his partners who hold some equity shares. Since agent \( i \) does not internalize this positive externality, he chooses an effort level \( e_i' \) that equates his marginal gain and his marginal cost, which is below the efficient effort level \( e_i^* \) that equates the team’s marginal gain and his marginal cost. In particular, we have

\[ r_i' \frac{\partial \pi(e_i', e_{i-}^*)}{\partial e_i} = \frac{\partial c_i(e_i', \theta_0, \theta_i)}{\partial e_i} \quad \text{and} \quad \frac{\partial \pi(e^*)}{\partial e_i} = \frac{\partial c_i(e_i^*, \theta_0, \theta_i)}{\partial e_i}. \]

Fix \( \theta \in \Theta \).

\[ \Delta^{n-1} \times \mathbb{R}^{2n} \]
\[ \{(r, m, e) \mid e_i \in \arg \max_{e_i' \in \mathbb{R}^+} u_i(r, m, e_i', e_{i-}, \theta_0, \theta_i) \quad \forall i\} \]

Figure 1: The bigger ellipse represents the set of all feasible arrangements \( \Delta^{n-1} \times \mathbb{R}^{2n} \). The arrangement \((r^*, m^*, e^*)\) is selected by an efficient arrangement rule: it gives the highest aggregate payoff among \( \Delta^{n-1} \times \mathbb{R}^{2n} \) (assuming that recommended efforts are enforceable). However, due to the free-riding problem, \( e^* \) cannot be a Nash equilibrium of \( G(\theta, r^*, m^*) \), which implies that agents do not voluntarily choose \( e^* \) given \((r^*, m^*)\). Hence, \((r^*, m^*, e^*)\) does not belong to \( \mathcal{R}(\theta) \). A constrained-efficient arrangement rule selects an arrangement that gives the highest aggregate payoff among \( \mathcal{R}(\theta) \).

Since agents do not voluntarily choose the efforts recommended by an efficient arrangement rule, we weakens our welfare-maximizing criterion as follows. For each \( \theta \in \Theta \), let \( \mathcal{R}(\theta) \) be the set of arrangements such that for each arrangement \((r, m, e) \in \mathcal{R}(\theta)\), the recommended effort vector \( e \) is a Nash equilibrium of \( G(\theta, r, m) \). Formally,

\[ \mathcal{R}(\theta) \equiv \{(r, m, e) \in \Delta^{n-1} \times \mathbb{R}^{2n} \mid e_i \in \arg \max_{e_i' \in \mathbb{R}^+} u_i(r, m, e_i', e_{i-}, \theta_0, \theta_i) \quad \text{for each} \ i \in N\}. \]
Informally, agents are willing to follow effort recommendations if and only if they are offered an arrangement in $\mathcal{R}(\theta)$. An arrangement rule $\alpha$ is constrained-efficient if it searches within the set $\mathcal{R}(\theta)$ and selects an arrangement that gives the highest aggregate payoff: for each $\theta \in \Theta$, we have

$$\alpha(\theta) \in \arg \max_{(r, m, e) \in \mathcal{R}(\theta)} u(r, m, e, \theta).$$  \hfill (3.1)

**Remark 3.1** Since efforts do not depend on wages and budget is balanced, if $(r, m, e)$ solves the maximization problem in (3.1), then so does $(r, m', e)$ for each $m' \in \mathbb{R}_+^n$. As a result, for a static partnership, if an arrangement rule $(s, w, \sigma)$ is constrained-efficient, then $(s, w', \sigma)$ is also constrained-efficient for each wage rule $w'$. This statement is not true for a dynamic partnership (see Remark 3.3). □

### 3.3 Efficiency

For each period $t$ and each state $\theta_t \in \Theta_t$, the mechanism designer chooses a sequence of decisions $\{\alpha_t\}_{\tau \geq t}$ that assigns an arrangement for each period $\tau \geq t$ and each state $\theta_{\tau} \in \Theta_{\tau}$. An arrangement rule $\alpha \equiv (s, w, \sigma)$ is efficient if for each $t \geq 1$ and each $\theta_t \in \Theta_t$,

$$\{\alpha_{\tau}\}_{\tau \geq t} \in \arg \max_{\{\alpha'_{\tau}\}_{\tau \geq t}} \mathbb{E}(\theta_{-0,\tau})_{\tau \geq t} \left[ \sum_{\tau \geq t} \delta^{t-\tau} u_r[\alpha'_t(\theta_{\tau}), \theta_{\tau}] \mid \theta_t \right],$$

where for each $(\theta_{-0,\tau})_{\tau \geq t} \in \prod_{\tau \geq t} \Theta_{-\tau}$ and each $\tau \geq t + 1$,

$$\theta_{0,\tau} = \theta_{0,\tau-1} + \gamma \sum_{i \in N} \sigma'_{i,\tau-1}(\theta_{\tau-1}).$$  \hfill (3.2)

Informally, if recommended efforts are enforceable, then $\{\alpha_{\tau}\}_{\tau \geq t}$ maximizes the expected sum of discounted aggregate payoffs starting from period $t$. Although all past states $(\theta_1, \ldots, \theta_{t-1})$ and the current state $\theta_t$ are publicly observed, only the current state $\theta_t$ is relevant to the mechanism designer’s maximization problem (since past states do not affect current and future payoffs). For each sequence of decisions $\{\alpha'_{\tau}\}_{\tau \geq t} \equiv \{(s'_t, w'_{\tau}, \sigma'_t)\}_{\tau \geq t}$, the mechanism designer computes the expected sum of discounted aggregate payoffs as follows. Based on the cumulative distribution functions $(F_1, \tau, \ldots, F_n, \tau)_{\tau \geq t}$, he assigns some probability to each realization of future types $(\theta_{-0,\tau})_{\tau \geq t}$. For each such realization,

(a.1) the current experience level $\theta_{0,\tau}$ and the recommended efforts $\sigma'_t(\theta_t)$ (which are enforceable) determine the experience level $\theta_{0,\tau+1}$,

(a.2) recursively, for each period $\tau \geq t + 2$, the experience level $\theta_{0,\tau-1}$ and the recommended efforts $\sigma'_{\tau-1}(\theta_{\tau-1})$ determine the experience level $\theta_{0,\tau}$.

After deriving all future states $(\theta_{\tau})_{\tau \geq t}$, the mechanism designer can compute the sum of discounted aggregate payoffs for the realization $(\theta_{-0,\tau})_{\tau \geq t}$. Then he can compute the expected
sum of discounted aggregate payoffs given the probability distribution on $\prod_{\tau > t} \Theta_{-\tau \tau}$. An arrangement rule $\alpha$ is efficient if \( \{\alpha_{\tau}\}_{\tau \geq t} \) gives the highest expected sum of discounted aggregate payoffs among all feasible sequences of decisions.

We define efficiency in recursive form below. For each $t \geq 1$ and each $\theta_{t} \in \Theta_{t}$, let

$$U_{t}(\theta_{t}) = \max_{(r_{t}, m_{t}, e_{t}) \in \Delta^{n-1} \times \mathbb{R}^{2n}_{+}} u_{t}(r_{t}, m_{t}, e_{t}, \theta_{t}) + \delta \mathbb{E}[U_{t+1}(\theta_{t+1}) | \theta_{t}, e_{t}],$$

where $\mathbb{E}[U_{t+1}(\theta_{t+1}) | \theta_{t}, e_{t}] \equiv 0$ if $t = T < \infty$. For each $\theta_{t} \in \Theta_{t}$, the value function $U_{t}$ gives the maximal expected sum of discounted aggregate payoffs starting from period $t$ obtainable when the current state is $\theta_{t}$. We note that

(b.1) future types $(\theta_{i,t+1})_{i \in N}$ are distributed according to the cumulative distribution functions $(F_{i,t+1})_{i \in N}$, which are independent of the current state $\theta_{t}$ and current arrangement $(r_{t}, m_{t}, e_{t})$,

(b.2) the future experience level $\theta_{0,t+1}$ depends on the current experience level $\theta_{0,t}$ and current effort contributions $e_{t}$ as specified in (3.2).

**Definition 3.1** An arrangement rule $\alpha \equiv (s, w, \sigma)$ is efficient if for each $t \geq 1$ and each $\theta_{t} \in \Theta_{t}$,

$$\alpha_{t}(\theta_{t}) \in \arg \max_{(r_{t}, m_{t}, e_{t}) \in \Delta^{n-1} \times \mathbb{R}^{2n}_{+}} u_{t}(r_{t}, m_{t}, e_{t}, \theta_{t}) + \delta \mathbb{E}[U_{t+1}(\theta_{t+1}) | \theta_{t}, e_{t}]. \quad (3.3)$$

We say that $\alpha_{t}(\theta_{t})$ is an efficient arrangement and interpret condition (3.3) as follows. Suppose that optimal arrangements are assigned from period $t + 1$ onwards so that, for each $\theta_{t+1} \in \Theta_{t+1}$, the maximal expected sum of discounted aggregate payoffs $U_{t+1}(\theta_{t+1})$ is obtained. Then the arrangement $\alpha_{t}(\theta_{t})$ maximizes the expected sum of discounted aggregate payoffs starting from period $t$. We note that a current arrangement $(r_{t}, m_{t}, e_{t})$ affects not only the current aggregate payoff $u_{t}(r_{t}, m_{t}, e_{t}, \theta_{t})$ but also the experience level $\theta_{0,t+1}$ via the recommended efforts $e_{t}$. By the Principle of Optimality, our two definitions of efficiency are equivalent.

### 3.4 Constrained Efficiency

As explained in Section 3.2, due to the free-riding problem, agents do not voluntarily choose the efforts recommended by an efficient arrangement. In this section, we define constrained efficiency as a weaker welfare-maximizing criterion.

For each $i \in N$, each $t \geq 1$, each $\theta_{t} \in \Theta_{t}$, and each arrangement rule $\alpha \equiv (s, w, \sigma)$, let
where $E[W_{i,t+1}(\theta_{t+1}, \alpha) | \theta_t, \sigma_t(\theta_t)] = 0$ if $t = T < \infty$. Suppose agents always follow effort recommendations from period $t$ onwards. Then, for each current state $\theta_t \in \Theta_t$, the function $W_{i,t}$ gives agent $i$’s expected sum of discounted payoffs starting from period $t$, given that the arrangement rule $\alpha$ is chosen by the mechanism designer. We note that $W_{i,t}(\theta_t, \alpha)$ is independent of $\{\alpha_{\tau}\}_{\tau < t}$, which is included only for notational convenience.

For each $t \geq 1$, each $\theta_t \in \Theta_t$, and each arrangement rule $\alpha \equiv (s, w, \sigma)$, let $\hat{G}_t(\theta_t, \alpha) \equiv [\mathbb{R}_+, \hat{u}_{i,t}(\alpha, \cdot, \theta_t)]_{i \in N}$ be a strategic-form game in which

- each agent $i$ chooses an effort $e_{i,t} \in \mathbb{R}_+$,
- if agents make efforts $e_t \in \mathbb{R}_+^n$, then each agent $i$ earns

$$\hat{u}_{i,t}(\alpha, e_t, \theta_t) \equiv u_{i,t}[s_t(\theta_t), w_t(\theta_t), e_t, \theta_{0,t}, \theta_{i,t}] + \delta E[W_{i,t+1}(\theta_{t+1}, \alpha) | \theta_t, e_t].$$

We interpret $\hat{G}_t(\theta_t, \alpha)$ as the period-$t$ effort-choosing game under the assumption that agents follow effort recommendations from period $t + 1$ onwards.

For each $t \geq 1$, each $\theta_t \in \Theta_t$, and each arrangement rule $\alpha$, let $\hat{R}_t(\theta_t, \alpha)$ be the set of period-$t$ arrangements such that for each arrangement $(r_t, m_t, e_t) \in \hat{R}_t(\theta_t, \alpha)$, the recommended effort vector $e_t$ is a Nash equilibrium of $\hat{G}_t(\theta_t, \alpha)$. Formally,

$$\hat{R}_t(\theta_t, \alpha) \equiv \left\{ (r_t, m_t, e_t) \in \Delta^{n-1} \times \mathbb{R}_+^{2n} \mid \text{for each } i \in N, \right.$$  
$$e_{i,t} \in \arg \max_{e_{i,t}' \in \mathbb{R}_+} u_{i,t}(r_t, m_t, e_{i,t}', e_{-i,t}, \theta_{0,t}, \theta_{i,t}) + \delta E[W_{i,t+1}(\theta_{t+1}, \alpha) | \theta_t, e_{i,t}']. \left\}.$$  

We note that $\hat{R}_t(\theta_t, \alpha)$ is independent of $\{\alpha_{\tau}\}_{\tau \leq t}$. Fix an arrangement rule $\alpha$ and some $t \geq 1$. If $\alpha_{\tau}(\theta_{\tau}) \in \hat{R}_t(\theta_{\tau}, \alpha)$ for each $\tau \geq t$ and each $\theta_{\tau} \in \Theta_{\tau}$, then agents voluntarily follow effort recommendations from period $t$ onwards.

For each $t \geq 1$, each $\theta_t \in \Theta_t$, and each arrangement rule $\alpha$, let

$$W_t(\theta_t, \alpha) = \max_{(r_t, m_t, e_t) \in \hat{R}_t(\theta_t, \alpha)} u_t(r_t, m_t, e_t, \theta_t) + \delta E[W_{t+1}(\theta_{t+1}, \alpha) | \theta_t, e_t],$$

where $E[W_{t+1}(\theta_{t+1}, \alpha) | \theta_t, e_t] = 0$ if $t = T < \infty$. For each current state $\theta_t \in \Theta_t$, the value function $W_t$ gives the maximal expected sum of discounted aggregate payoffs (starting from period $t$) obtainable subject to the following constraint: for each $\tau \geq t$ and each $\theta_{\tau} \in \Theta_{\tau}$, if the mechanism designer uses $\{\alpha_{\tau}^*\}_{\tau > t}$ from period $\tau + 1$ onwards, then he can only assign some arrangement $(r_{\tau}, m_{\tau}, e_{\tau}) \in \hat{R}_{\tau}(\theta_{\tau}, \{\alpha_{\tau}\}_{\tau < \tau}, \{\alpha_{\tau}^*\}_{\tau > \tau})$, where $\{\alpha_{\tau}\}_{\tau < \tau}$ is arbitrary. This constraint ensures that agents voluntarily follow effort recommendations from period $t$ onwards. We note that $W_t(\theta_t, \alpha)$ is independent of $\{\alpha_{\tau}\}_{\tau < t}$.
**Definition 3.2** An arrangement rule \( \alpha \) is **constrained-efficient** if for each \( t \geq 1 \) and each \( \theta_t \in \Theta_t \),

\[
\alpha_t(\theta_t) \in \arg \max_{(r_t,m_t,e_t) \in \mathcal{R}_t(\theta_t,\alpha)} u_t(r_t, m_t, e_t, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha) \mid \theta_t, e_t]. \tag{3.4}
\]

We say that \( \alpha_t(\theta_t) \) is a **constrained-efficient** arrangement and interpret condition (3.4) as follows. Suppose that optimal arrangements are assigned from period \( t+1 \) onwards subject to the aforementioned constraint. Then \( \alpha_t(\theta_t) \) gives the highest expected sum of discounted aggregate payoffs (starting from period \( t \)) among all arrangements in \( \mathcal{R}_t(\theta_t, \alpha) \).

For further analysis, we define the following. For each \( t \geq 1 \), each \( \theta_t \in \Theta_t \), and each arrangement rule \( \alpha \), let

\[
\tilde{\mathcal{R}}_t(\theta_t, \alpha) \equiv \left\{ (r_t, e_t) \in \Delta^{n-1} \times \mathbb{R}^n_+ \mid \text{for each } i \in N, \right. \\
\left. e_{i,t} \in \arg \max_{e_{i,t} \in \mathbb{R}_+} \tilde{u}_{i,t}(r_t, e_{i,t}, e_{-i,t}, \theta_{0,t}, \theta_{t}) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha) \mid \theta_t, e_{i,t}, e_{-i,t}] \right\}. 
\]

Since current efforts do not depend on current wages, if agents voluntarily follow effort recommendations \( e_t \) given compensation packages \( (r_t, m_t) \), then they will continue to follow \( e_t \) voluntarily if wage payments \( m_t \) are removed. Hence, \( (r_t, m_t, e_t) \in \mathcal{R}_t(\theta_t, \alpha) \) if and only if \( (r_t, e_t) \in \tilde{\mathcal{R}}_t(\theta_t, \alpha) \). As a result, \( (r^*_t, m^*_t, e^*_t) \) solves the maximization problem in (3.4) if and only if

\[
(r^*_t, e^*_t) \in \arg \max_{(r_t, e_t) \in \tilde{\mathcal{R}}_t(\theta_t, \alpha)} \tilde{u}_t(r_t, e_t, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha) \mid \theta_t, e_t]
\]

(noting that \( u_t(r_t, m_t, e_t, \theta_t) = \tilde{u}_t(r_t, e_t, \theta_t) \) by budget balance).

**Remark 3.2** Fix \( t \geq 1 \) and an arrangement rule \( \alpha \equiv (s, w, \sigma) \). Suppose that \( \alpha_{r}(\theta_{r}) \) satisfies (3.4) for each \( r \geq t \) and each \( \theta_r \in \Theta_r \); equivalently,

\[
[ s_r(\theta_r), \sigma_r(\theta_r) ] \in \arg \max_{(r_r, e_r) \in \tilde{\mathcal{R}}_r(\theta_r, \alpha)} \tilde{u}_r(r_r, e_r, \theta_r) + \delta \mathbb{E}[W_{r+1}(\theta_{r+1}, \alpha) \mid \theta_r, e_r].
\]

Then we say that \( \{ \alpha_r \}_{r \geq t} \) is constrained-efficient. \( \Box \)

**Remark 3.3** Unlike a static partnership, for a dynamic partnership, that \( \alpha \equiv (s, w, \sigma) \) is constrained-efficient does not imply \( \alpha' \equiv (s, w', \sigma) \) is constrained-efficient for \( w' \neq w \). To see why, fix some \( t < T \) and \( \theta_t \in \Theta_t \). If \( \tilde{\mathcal{R}}_t(\theta_t, \alpha) \neq \tilde{\mathcal{R}}_t(\theta_t, \alpha') \), then it might be the case that \( [s_t(\theta_t), \sigma_t(\theta_t)] \) \( \in \tilde{\mathcal{R}}_t(\theta_t, \alpha) \) but \( [s_t(\theta_t), \sigma_t(\theta_t)] \) \( \not\in \tilde{\mathcal{R}}_t(\theta_t, \alpha') \), which implies that \( \alpha' \) is not constrained-efficient. Informally, agents voluntarily follow effort recommendations \( \sigma_t(\theta_t) \) given equity share allocation \( s_t(\theta_t) \) if \( \{ \alpha_r \}_{r \geq t} \) applies from period \( t+1 \) onwards, but they are not willing to do so if \( \{ \alpha'_r \}_{r \geq t} \) applies from period \( t+1 \) onwards. \( \Box \)
4 Direct Revelation Mechanism

As discussed in Section 3, the objective of the mechanism designer is to assign a constrained-efficient arrangement whenever agents’ types are revealed. In this section, we provide sufficient conditions (on endogenous variables) under which there exists a direct revelation mechanism that incentivizes agents to reveal their types. We define desirable properties of a direct revelation mechanism: constrained efficiency, periodic Bayesian/ex-post incentive compatibility, and periodic Bayesian/ex-post Pareto dominance over the equal division mechanism. In Section 5, we provide sufficient conditions on the primitives so that the aforementioned sufficient conditions on endogenous variables are met.

A direct revelation mechanism is a tuple $\mathcal{D} \equiv \{\Theta_{-0,t}, \alpha_t\}_{t \geq 1}$, where for each period $t$,

(a.1) $\Theta_{i,t}$ is the message space for agent $i$, and

(a.2) $\alpha_t : \Theta_t \to \Delta^{n-1} \times \mathbb{R}^{2n}_+$ is a period-$t$ arrangement rule.

Definition 4.1 A direct revelation mechanism $\mathcal{D} \equiv \{\Theta_{-0,t}, \alpha_t\}_{t \geq 1}$ is constrained-efficient if the arrangement rule $\alpha \equiv \{\alpha_t\}_{t \geq 1}$ is constrained-efficient.

4.1 Incentive Compatibility

Periodic Bayesian incentive compatibility requires that each agent $i$ optimize by reporting her type truthfully and following effort recommendation in each period $t$ (given her belief about her partners’ current types) under the following assumptions:

(b.1) agent $i$ reports her type truthfully and chooses her effort optimally from period $t + 1$ onwards, and

(b.2) each agent $j \neq i$ reports her type truthfully and follows effort recommendation from period $t$ onwards.

Definition 4.2 A direct revelation mechanism $\mathcal{D} \equiv \{\Theta_{-0,t}, \alpha_t\}_{t \geq 1}$, where $\alpha \equiv (s, w, \sigma)$, is periodic Bayesian incentive compatible if for each $i \in N$, each $t \geq 1$, each $\theta_{0,t} \in \mathbb{R}^n_+$, and each $\theta_{i,t} \in \Theta_{i,t}$, the following hold:

(i) $\theta_{i,t}$ solves

$$\max_{\theta_{i,t}} \max_{\theta_{-i,t}} \mathbb{E}_{(\theta_{j,t})_{j \neq i}} \left\{ u_{i,t}(s_t(\hat{\theta}_{i,t}, \theta_{-i,t}), w_t(\hat{\theta}_{i,t}, \theta_{-i,t}), e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_{0,t}, \theta_{i,t}) \right. \\
+ \delta \mathbb{E}\left[ W_{i,t+1}^{(\theta_{i,t})_{j \neq i}}(\theta_{i,t}, \alpha) \mid \theta_{t}, e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) \right] \}.$$

(ii) for each $(\theta_{j,t})_{j \neq i} \in \prod_{j \neq i} \Theta_{j,t}$.
σ_{i,t}(θ_t) \in \arg \max_{e_{i,t} \in \mathbb{R}_+} u_{i,t}[s_t(θ_t), w_t(θ_t), e_{i,t}, \sigma_{-i,t}(θ_t), θ_{0,t}, θ_{i,t}]
+ \delta \mathbb{E}[W_{i,t+1}(θ_{t+1}, α) \mid θ_t, e_{i,t}, \sigma_{-i,t}(θ_t)].

Under assumptions (b.1) and (b.2), for each θ_{t+1} ∈ Θ_{t+1}, agent i’s expected sum of discounted payoffs (starting from period t + 1) is W_{i,t+1}(θ_{t+1}, α). If the mechanism designer uses a periodic Bayesian incentive compatible direct revelation mechanism, then there is a perfect Bayesian equilibrium in which agents report truthfully and follow effort recommendations in each period. At the beginning of each period t, each agent i forms a belief about her partners’ current types based on the cumulative distribution functions (F_j,t)_{j \neq i}. Given this belief, it is optimal for agent i to report her type truthfully (assuming that she chooses an optimal effort conditional on each reported type ˆθ_{i,t}). Once agent i reveals her type and other agents report their types as (θ_j,t)_{j \neq i}, by Bayes’ rule, she assigns probability one to the event that her partners’ true types are (θ_j,t)_{j \neq i}. Given this revised belief, it is optimal for her to follow the effort recommendation σ_{i,t}(θ_t). In Section 8, we define a perfect Bayesian equilibrium in which agents vote rather than report their types. Except for the message space, Section 8 formalizes the Bayesian revision and sequential rationality discussed here.

When there are only two agents, our mechanism design satisfies a stronger property: periodic ex-post incentive compatibility.

**Definition 4.3** A direct revelation mechanism \( D \equiv \{Θ_{-0,t}, α_t\}_{t \geq 1} \), where \( α \equiv (s, w, σ) \), is periodic ex-post incentive compatible if for each \( i \in N \), each \( t \geq 1 \), and each \( θ_t \in Θ_t \),
\[
[θ_{i,t}, σ_{i,t}(θ_t)] \in \arg \max_{(θ_{i,t}, e_{i,t})} u_{i,t}[s_t(θ_{i,t}, θ_{-i,t}), w_t(θ_{i,t}, θ_{-i,t}), e_{i,t}, σ_{-i,t}(θ_{i,t}, θ_{-i,t}), θ_{0,t}, θ_{i,t}]
+ \delta \mathbb{E}[W_{i,t+1}(θ_{t+1}, α) \mid θ_t, e_{i,t}, σ_{-i,t}(θ_{i,t}, θ_{-i,t})].
\]

In each period \( t \), under assumptions (b.1) and (b.2) specified above, each agent optimizes by reporting her type truthfully and following effort recommendation regardless of her partners’ current types. In other words, each agent does not regret reporting truthfully and following effort recommendation after she learns about her partners’ current types. However, under periodic ex-post incentive compatibility, an agent might regret reporting truthfully in period \( t \) after she acquires some new information in period \( t + 1 \).

### 4.2 Pareto Dominance

Suppose a direct revelation mechanism \( D \) periodically Bayesian Pareto dominates the equal division mechanism. Then, in each period \( t \), each agent weakly prefers to have \( D \) implemented from period \( t \) onwards rather than the equal division mechanism (given her belief about her partners’ current types). In addition, this preference is strict for some agent \( i \).
and some current type \( \theta_{i,t} \). We formalize this property below.

An equal division mechanism is a direct revelation mechanism \( D^e \equiv \{ \Theta_{-0,t}, \alpha^e_t \}_{t \geq 1} \equiv \{ \Theta_{-0,t}, s^e_t, w^e_t, \sigma^e_t \}_{t \geq 1} \), where for each \( t \geq 1 \) and each \( \theta_t \in \Theta_t \),

\[
\begin{align*}
s^e_t(\theta_t) &= (1/n, \ldots, 1/n) \\
w^e_t(\theta_t) &= (0, \ldots, 0) \\
\alpha^e_t(\theta_t) &\in R_t(\theta_t, \alpha^e).
\end{align*}
\]

That is, the equal division mechanism \( D^e \) always assigns equal equity shares and no wage payments regardless of reports. In addition, if agents report truthfully, then the recommended efforts \( \sigma^e_t(\theta_t) \) is a Nash equilibrium of \( \hat{G}_t(\theta_t, \alpha^e) \); equivalently, agents voluntarily follow \( \sigma^e_t(\theta_t) \) (assuming that they voluntarily follow effort recommendations by \( \sigma^e \) from period \( t+1 \) onwards). We refer to \( \alpha^e \) as the equal division rule.

**Definition 4.4** Let \( D \equiv \{ \Theta_{-0,t}, \alpha_t \}_{t \geq 1} \) be a periodic Bayesian incentive compatible direct revelation mechanism. We say that \( D \) periodically Bayesian Pareto dominates the equal division mechanism \( D^e \equiv \{ \Theta_{-0,t}, \alpha^e_t \}_{t \geq 1} \) if

(i) for each \( i \in N \), each \( t \geq 1 \), each \( \theta_{0,t} \in R_{++} \), and each \( \theta_{i,t} \in \Theta_{i,t} \),

\[
E_{(\theta_{i,t}, j \neq i)}[W_{i,t}[\theta_{0,t}, \theta_{i,t}, (\theta_{j,t})_{j \neq i}, \alpha]] > E_{(\theta_{i,t}, j \neq i)}[W_{i,t}[\theta_{0,t}, \theta_{i,t}, (\theta_{j,t})_{j \neq i}, \alpha^e]],
\]

(ii) for each \( t \geq 1 \) and each \( \theta_{0,t} \in R_{++} \), there exist \( i \in N \) and \( \theta_{i,t} \in \Theta_{i,t} \) such that

\[
E_{(\theta_{i,t}, j \neq i)}[W_{i,t}[\theta_{0,t}, \theta_{i,t}, (\theta_{j,t})_{j \neq i}, \alpha]] > E_{(\theta_{i,t}, j \neq i)}[W_{i,t}[\theta_{0,t}, \theta_{i,t}, (\theta_{j,t})_{j \neq i}, \alpha^e]].
\]

Suppose the mechanism designer uses the direct revelation mechanism \( D \), which is periodic Bayesian incentive compatible. Since agents optimize by reporting truthfully and following effort recommendations in each period, for each current state \( \theta_t \in \Theta_t \), each agent \( i \)’s expected sum of discounted payoff is \( W_{i,t}(\theta_t, \alpha) \) (as defined in Section 3.4). At the beginning of each period \( t \), each agent \( i \) forms a belief about her partners’ current types based on the cumulative distribution functions \( (F_j)_{j \neq i} \). Given this belief, her expected payoff is \( E_{(\theta_{i,t}, j \neq i)}[W_{i,t}[\theta_{0,t}, \theta_{i,t}, (\theta_{j,t})_{j \neq i}, \alpha]] \). We can also verify that the equal division mechanism \( D^e \) is periodic Bayesian incentive compatible. Hence, if the mechanism designer uses \( D^e \), then agent \( i \)’s expected payoff is \( E_{(\theta_{i,t}, j \neq i)}[W_{i,t}[\theta_{0,t}, \theta_{i,t}, (\theta_{j,t})_{j \neq i}, \alpha^e]] \). Condition (i) ensures that each agent weakly prefers to have \( D \) implemented from period \( t \) onwards rather than \( D^e \). Condition (ii) ensures that this preference is strict for some agent \( i \) and current type \( \theta_{i,t} \).

When there are only two agents, our mechanism design satisfies a stronger property: periodic ex-post Pareto dominance over the equal division mechanism.

**Definition 4.5** Let \( D \equiv \{ \Theta_{-0,t}, \alpha_t \}_{t \geq 1} \) be a periodic Bayesian incentive compatible direct revelation mechanism. We say that \( D \) **periodically ex-post Pareto dominates** the equal division mechanism \( D^e \equiv \{ \Theta_{-0,t}, \alpha^e_t \}_{t \geq 1} \) if
(i') for each $i \in N$, each $t \geq 1$, and each $\theta_t \in \Theta_t$, we have $W_{i,t}(\theta_t, \alpha) \geq W_{i,t}(\theta_t, \alpha^e)$,
(ii') for each $t \geq 1$, there exist $i \in N$ and $\theta_t \in \Theta_t$ such that $W_{i,t}(\theta_t, \alpha) > W_{i,t}(\theta_t, \alpha^e)$.

That is, even after agent $i$ learns about her partners’ true types, she still weakly prefers to have $D$ implemented from period $t$ onwards rather than $D^e$; in addition, this preference is strict for some agent $i$ and some current state $\theta_t$.

5 Preliminary Results

Assumption 5.1 For each $i \in N$, each $t \geq 1$, each $e_t \in \mathbb{R}^n_+$, and each $\theta_t \in \Theta_t$,

\[
\pi(e_t) = \sum_{j \in N} e_{j,t} \quad \text{and} \quad c_{i,t}(e_{i,t}, \theta_{0,t}, \theta_{i,t}) = e_{i,t}^2/(2\theta_{0,t}\theta_{i,t}).
\]

Definition 5.1 For each $t \geq 1$ and each arrangement rule $\alpha$, the sequence $\{\alpha_\tau\}_{\tau \geq t}$ periodically allows individual payoffs to be weakly increasing in experience level at constant rates if for each $i \in N$, each $\tau \geq t$ and each $\theta_\tau \in \Theta_\tau$,

(a) $0 \leq \partial W_{i,\tau}(\theta_\tau, \alpha)/\partial \theta_{0,\tau} < \infty$, and
(b) $\partial W_{i,\tau}(\theta_\tau, \alpha)/\partial \theta_{0,\tau}$ is independent of $\theta_{0,\tau}$.

5.1 Efforts

Lemma 5.1 Fix $t \geq 1$ and an arrangement rule $\alpha$. Suppose that $\{\alpha_\tau\}_{\tau \geq t}$ periodically allows individual payoffs to be weakly increasing in experience level at constant rates. Then for each $i \in N$, each $\tau \geq t$ and each $r_t \in \Delta^{n-1}$, there is a unique effort profile $e_t \equiv \sigma_t(\theta_t, r_t, \alpha) \in \mathbb{R}^n_+$ such that $(r_t, e_t) \in \tilde{R}_t(\theta_t, \alpha)$. In addition, for each $i \in N$,

(a) $\frac{\partial \sigma_{i,t}(\theta_t, r_t, \alpha)}{\partial r_{i,t}} > 0$ and $\frac{\partial^2 \sigma_{i,t}(\theta_t, r_t, \alpha)}{\partial^2 r_{i,t}} = 0$,

(b) $\sigma_{i,t}(\theta_t, r_t, \alpha) > \sigma_{i,t}(\theta_{i,t}', \theta_{-i,t}, \alpha) \text{ for } \theta_{i,t} > \theta_{i,t}'$, and

(c) $\frac{\partial \sigma_{i,t}(\theta_t, r_t, \alpha)}{\partial r_{i,t}} > \frac{\partial \sigma_{i,t}(\theta_{i,t}', \theta_{-i,t}, r_t, \alpha)}{\partial r_{i,t}} \text{ for } \theta_{i,t} > \theta_{i,t}'$.

5.2 Equity Shares

Proposition 5.1 Fix $t \geq 1$ and an arrangement rule $\alpha$. Suppose that $\{\alpha_\tau\}_{\tau \geq t}$ is constrained-efficient and periodically allows individual payoffs to be weakly increasing in experience level at constant rates. Then there is a unique pair $(s^*_t, \sigma^*_t)$ such that for each $\theta_t \in \Theta_t$,
\[
[s_t^*(\theta_t), \sigma_t^*(\theta_t)] \in \arg\max_{(r_t, e_t) \in R_t(\theta_t, \alpha)} \tilde{u}_t(r_t, e_t, \theta_t) + \delta \mathbb{E}\left[W_{t+1}(\theta_{t+1}, \alpha) \mid \theta_t, e_t\right].
\]

In addition, for each \(i \in N\) and each \(\theta'_{i,t} > \theta_{i,t}\), we have \(s_{i,t}^*(\theta'_{i,t}, \theta_{-i,t}) \geq s_{i,t}^*(\theta_{i,t})\).

### 5.3 Increasing Differences

For each \(i \in N\), each \(t \geq 1\), each \(\theta_t \in \Theta_t\), each \(\hat{\theta}_{i,t} \in \Theta_{i,t}\), and each arrangement rule \(\alpha \equiv (s, w, \sigma)\), define

\[
\bar{W}_{i,t}(\theta_t; \hat{\theta}_{i,t}; \alpha) \equiv \max_{e_{i,t} \in \mathbb{R}_+} \tilde{u}_{i,t}[s_t(\hat{\theta}_{i,t}, \theta_{-i,t}), e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_0,t, \theta_{i,t}]
\]

\[
+ \delta \mathbb{E}\left[W_{i,t+1}(\theta_{t+1}, \alpha) \mid \theta_t, e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t})\right].
\]

**Lemma 5.2** Fix \(t \geq 1\) and an arrangement rule \(\alpha \equiv (s, w, \sigma)\). Suppose that \(\{\alpha_t\}_{t \geq 1}\) periodically allows individual payoffs to be weakly increasing in experience level at constant rates. Then for each \(i \in N\), each \(\theta_t \in \Theta_t\), and each \(\hat{\theta}_{i,t} \in \Theta_{i,t}\), the effort level \(\hat{\sigma}_{i,t}[\theta_t, s_t(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]\) is the unique solution to the following maximization problem:

\[
\max_{e_{i,t} \in \mathbb{R}_+} \tilde{u}_{i,t}[s_t(\hat{\theta}_{i,t}, \theta_{-i,t}), e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_0,t, \theta_{i,t}]
\]

\[
+ \delta \mathbb{E}\left[W_{i,t+1}(\theta_{t+1}, \alpha) \mid \theta_t, e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t})\right].
\]

**Definition 5.2** For each \(i \in N\), each \(t \geq 1\), each \(\theta_{-i,t} \in \Theta_{-i,t}\), and each arrangement rule \(\alpha\), the payoff function \(\bar{W}_{i,t}(:, \theta_{-i,t}; \alpha)\) satisfies increasing differences if

\[
\bar{W}_{i,t}(\theta'_{i,t}; \theta_{-i,t}; \hat{\theta}_{i,t}^'; \alpha) - \bar{W}_{i,t}(\theta'_{i,t}; \theta_{-i,t}; \hat{\theta}_{i,t}^'; \alpha) \geq \bar{W}_{i,t}(\theta_{i,t}; \theta_{-i,t}; \hat{\theta}_{i,t}'; \alpha) - \bar{W}_{i,t}(\theta_{i,t}; \theta_{-i,t}; \hat{\theta}_{i,t}; \alpha)
\]

for \(\theta'_{i,t} \geq \theta_{i,t}\) and \(\hat{\theta}_{i,t}^' \geq \hat{\theta}_{i,t}\).

**Proposition 5.2** Fix \(t \geq 1\) and an arrangement rule \(\alpha \equiv (s, w, \sigma)\). Suppose that

(a) \(\{\alpha_t\}_{t \geq 1}\) is constrained-efficient and periodically allows individual payoffs to be weakly increasing in experience level at constant rates, and

(b) for each \(\theta_t \in \Theta_t\),

\[
[s_t(\theta_t), \sigma_t(\theta_t)] \in \arg\max_{(r_t, e_t) \in R_t(\theta_t, \alpha)} \tilde{u}_t(r_t, e_t, \theta_t) + \delta \mathbb{E}\left[W_{t+1}(\theta_{t+1}, \alpha) \mid \theta_t, e_t\right].
\]

Then for each \(i \in N\) and each \(\theta_{-i,t} \in \Theta_{-i,t}\), the payoff function \(\bar{W}_{i,t}(:, \theta_{-i,t}; \alpha)\) satisfies increasing differences.
6 Two-Member Partnership

Lemma 6.1 Fix \( t \geq 1 \) and an arrangement rule \( \alpha \equiv (s, w, \sigma) \). Suppose that
(a) \( \{\alpha_T\}_{T>t} \) is constrained-efficient and periodically allows individual payoffs to be weakly increasing in experience level at constant rates, and
(b) for each \( \theta_t \in \Theta_t \),
\[
[s_t(\theta_t), \sigma_t(\theta_t)] \in \arg \max_{(r_t, e_t) \in \mathcal{R}_t(\theta_t, \alpha)} \tilde{u}_t(r_t, e_t, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha) | \theta_t, e_t].
\]
Then for each \( i \in N \) and each \( \theta_{0,t} \in \Theta_{0,t} \),
\[
\tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_H; \alpha) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha) \\
\geq \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_L; \alpha).
\]

6.1 Finite Horizon

Step 1. Fix an arbitrary arrangement rule \( \alpha \equiv (s, w, \sigma) \). By Proposition 5.1, there is a unique pair \((s_T^*, \sigma_T^*)\) such that for each \( \theta_T \in \Theta_T \),
\[
[s_T^*(\theta_T), \sigma_T^*(\theta_T)] \in \arg \max_{(r_T, e_T) \in \mathcal{R}_T(\theta_T, \alpha)} \tilde{u}_T(r_T, e_T, \theta_T).
\]
Let \( \alpha^{(T)} \) be an arrangement rule such that \( \alpha_T^{(T)} = (s_T^*, \sigma_T^*) \) and \( \alpha_T^{(T)} = \alpha_T \) for each \( T \neq T \).

We construct \( w_T^* \) as follows. For each \( i \in N \) and each \( \theta_{0,T} \in \mathbb{R}_{++} \), let \( w_{i,T}(\theta_{0,T}, \theta_L, \theta_L) = w_{i,T}(\theta_{0,T}, \theta_H, \theta_H) = 0 \).

For each \( i \in N \) and each \( \theta_{0,T} \in \mathbb{R}_{++} \), define
\[
\Delta \tilde{W}_{i,T}(\theta_{0,T}, \theta_L, \theta_H; \theta_H; \alpha^{(T)}) \equiv \tilde{W}_{i,T}(\theta_{0,T}, \theta_L, \theta_H; \theta_H; \alpha^{(T)}) - \tilde{W}_{i,T}(\theta_{0,T}, \theta_L, \theta_H; \theta_H; \alpha^{(T)}).
\]
If \( \Delta \tilde{W}_{i,T}(\theta_{0,T}, \theta_L, \theta_H; \alpha^{(T)}) \geq 0 \), let \( w_{i,T}^*(\theta_{0,T}, \theta_L, \theta_H) = 0 \) and
\[
w_{i,T}^*(\theta_{0,T}, \theta_L, \theta_H) = \frac{\Delta \tilde{W}_{i,T}(\theta_{0,T}, \theta_L, \theta_H; \alpha^{(T)})}{s_{i,T}^*(\theta_{0,T}, \theta_L, \theta_H)}.
\]
If \( \Delta \tilde{W}_{i,T}(\theta_{0,T}, \theta_L, \theta_H; \alpha^{(T)}) < 0 \), let \( w_{i,T}^*(\theta_{0,T}, \theta_L, \theta_H) = 0 \) and
\[
w_{i,T}^*(\theta_{0,T}, \theta_L, \theta_H) = -\frac{\Delta \tilde{W}_{i,T}(\theta_{0,T}, \theta_L, \theta_H; \alpha^{(T)})}{s_{i,T}^*(\theta_{0,T}, \theta_L, \theta_H)}.
\]

Lemma 6.2 The period-\( T \) arrangement rule \((s_T^*, w_T^*, \sigma_T^*)\) is constrained-efficient and periodically allows individual payoffs to be weakly increasing in experience level at constant rates.
Step $t$. Let $\alpha^{(t+1)}$ be an arrangement rule such that $\alpha^{(t+1)} = (s^*, w^*, \sigma^*)$ for each $t \geq t + 1$ and $\alpha^{(t+1)} = \alpha_t$ for each $\tau < t + 1$. By Proposition 5.1, there is a unique pair $(s^*_t, \sigma^*_t)$ such that for each $\theta_t$,

$$[s^*_t(\theta_t), \sigma^*_t(\theta_t)] \in \arg \max_{(r_t, e_t) \in R_t(\theta_t, \alpha^{(t+1)})} \tilde{u}_t(r_t, e_t, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha^{(t+1)}) | \theta_t, e_t].$$

Let $\alpha^{(t)}$ be an arrangement rule such that $\alpha^{(t)} = (s^*_t, w^*_t, \sigma^*_t)$ and $\alpha^{(t)} = \alpha^{(t+1)}$ for each $\tau \neq t$. We construct $w^*_t$ as follows. For each $i \in N$ and each $\theta_{0,t} \in \mathbb{R}^+$, let $w_{i,t}(\theta_{0,t}, \theta_L, \theta_H) = w_{i,t}(\theta_{0,t}, \theta_H, \theta_H) = 0$.

For each $i \in N$ and each $\theta_{0,t} \in \mathbb{R}^+$, define

$$\Delta W_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^{(t)}) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_H; \alpha^{(t)}) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_H; \alpha^{(t)}).$$

If $\Delta W_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^{(t)}) \geq 0$, let $w^*_{i,t}(\theta_{0,t}, \theta_L, \theta_H) = 0$ and

$$w^*_{i,t}(\theta_{0,t}, \theta_L, \theta_H) = \frac{\Delta \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^{(t)})}{s^*_i(\theta_{0,t}, \theta_L, \theta_H)}.$$

If $\Delta W_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^{(t)}) < 0$, let $w^*_{i,t}(\theta_{0,t}, \theta_L, \theta_H) = 0$ and

$$w^*_{i,t}(\theta_{0,t}, \theta_L, \theta_H) = \frac{-\Delta \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^{(t)})}{s^*_i(\theta_{0,t}, \theta_L, \theta_H)}.$$

Lemma 6.3 The sequence $\{s^*_t, w^*_t, \sigma^*_t\}_{t \geq 1}$ is constrained-efficient and periodically allows individual payoffs to be weakly increasing in experience level at constant rates.

Theorem 6.1 The direct revelation mechanism $\mathcal{D}^* = \{\Theta_{-i,t}, s^*_t, w^*_t, \sigma^*_t\}_{t=1}^T$ satisfies constrained efficiency, periodic ex-post incentive compatibility, and periodically ex-post Pareto dominates the equal division mechanism $\mathcal{D}^e$.

6.2 Infinite Horizon

7 Multiple-Member Partnership

For each $i \in N$ and each $\theta_{-i} \in \Theta_{-i}$, let $\eta(\theta_{-i})$ be the number of agents in the set $N \setminus \{i\}$ who have type $\theta_H$. That is, $\eta(\theta_{-i}) \equiv |\{j \in N \setminus \{i\} | \theta_j = \theta_H\}|$. For each $n' \in \{0, \ldots, n-1\}$, let $\Psi_{-i}(n') \equiv \{\theta_{-i} \in \Theta_{-i} | \eta(\theta_{-i}) = n'\}$ and let $\xi_{-i}(n')$ be an element of $\Psi_{-i}(n')$. That is, $\xi_{-i}(n')$ is a type profile of agent $i$’s partners, in which $n'$ agents have type $\theta_H$. Let $\rho_{-i}(n')$ be the cardinality of $\Psi_{-i}(n')$. 
Fix an agent $i \in N$ and a type profile $\theta \in \Theta$. Let $w^*_i(\theta) = 0$ if either $\theta_i = \theta_H$ or $\theta = (\theta_L, \ldots, \theta_L)$. Suppose $\theta_i = \theta_L$ and $\theta_j = \theta_H$ for some $j \neq i$. Let $\theta' \in \Theta$ such that $\theta'_j = \theta_L$ and $\theta'_k = \theta_k$ for each $k \in N \setminus \{j\}$. Let $n^* \equiv \eta(\theta_{-i})$ and
\[
\left[\rho(n^* - 1)(n - n^*)\mu(\theta_L) + \rho(n^*)n^*\mu(\theta_H)\right] s^*_i(\theta) w^*_i(\theta) =
\mu(\theta_L) \max \left\{0, \quad \rho(n^*)[W^*_i(\theta) - W^*_i(\theta)]\right\}
+ \mu(\theta_H) \min \left\{\rho(n^*)[W^*_i(\theta_H, \theta_{-i}) - \tilde{W}_i(\theta_H, \theta_{-i}; \theta_L)], \quad \rho(n^*)[W^*_i(\theta_H, \theta_{-i}) - W^*_i(\theta_H, \theta_{-i})]\right\},
\]
where $W^*_i(\theta) \equiv u_i[s^*(\theta), w^*(\theta), \sigma^c(\theta), \theta_0, \theta_1]$ for each $\theta \in \Theta$.

**Theorem 7.1** The direct revelation mechanism $D^*$ satisfies constrained efficiency, Bayesian incentive compatibility, and Bayesian Pareto dominates the equal division mechanism.

When agents have the same type, the mechanism $D^*$ and the equal division mechanism give the same outcome: agents receive the same equity share and no wage payment. When agents have different types, the high-type agents receive no wage and the low-type agents receive some positive wage. This wage payment depends on the fraction of low-type agents. We ensure that these wage payments are sufficiently large for low-type agents to be truthful and better off compared with the equal division mechanism. They must also be sufficiently small for high-type agents to be truthful and better off compared with the equal division mechanism (we note that a high-type’s income is strictly decreasing in the wage payment offered to a low-type). These wage payments exist due to the increasing differences property of the payoff structure. The wage rule $w^*$ is demonstrated in the following examples.

**Example 7.1** Suppose there are two partners.

For each $i \in N$, let $w^*_i(\theta_L, \theta_L) = w^*_i(\theta_H, \theta_H) = w^*_i(\theta_L, \theta_H) = 0$ and $s^*_i(\theta_L, \theta_H) w^*_i(\theta_H, \theta_L) = \mu(\theta_L) \max \left\{0, \quad \max_{\theta_{-i}} \left(W^*_i(\theta_L, \theta_{-i}; \theta_H) - \tilde{W}_i(\theta_L, \theta_{-i}; \theta_L)\right)\right\}$
\[
+ \mu(\theta_H) \min_{\theta_{-i}} \left[\tilde{W}_i(\theta_H, \theta_{-i}; \theta_H) - W^*_i(\theta_L, \theta_{-i}; \theta_L)\right],
\]

**Example 7.2** Suppose there are three partners. For each $i \in N$, let $w^*_i(\theta_L, \theta_L, \theta_L) = w^*_i(\theta_H, \theta_H, \theta_H) = w^*_i(\theta_H, \theta_{-i}) = 0$ for each $\theta_{-i}$
\[
2s^*_i(\theta_H, \theta_L, \theta_L) w^*_i(\theta_L, \theta_H, \theta_H) =
+ \mu(\theta_L) \max \left\{0, \quad 2[W^*_i(\theta_L, \theta_H, \theta_H) - \tilde{W}_i(\theta_L, \theta_H, \theta_H)], \quad 2[\tilde{W}_i(\theta_L, \theta_H, \theta_H) - W^*_i(\theta_L, \theta_H, \theta_H)]\right\}
+ \mu(\theta_H) \min \left\{[\tilde{W}_i(\theta_H, \theta_L, \theta_H) - W^*_i(\theta_H, \theta_L, \theta_H), \quad 2[\tilde{W}_i(\theta_H, \theta_L, \theta_H) - W^*_i(\theta_H, \theta_L, \theta_H)]\right\},
\]
\[
2s^*_i(\theta_L, \theta_L, \theta_L) w^*_i(\theta_L, \theta_H, \theta_L) =
+ \mu(\theta_L) \max \left\{0, \quad 2[W^*_i(\theta_L, \theta_L, \theta_H) - \tilde{W}_i(\theta_L, \theta_L, \theta_H)], \quad \tilde{W}_i(\theta_L, \theta_L, \theta_H) - W^*_i(\theta_L, \theta_L, \theta_H)]\right\}
+ \mu(\theta_H) \min \left\{[\tilde{W}_i(\theta_H, \theta_L, \theta_H) - \tilde{W}_i(\theta_L, \theta_L, \theta_H), \quad \tilde{W}_i(\theta_H, \theta_L, \theta_H) - W^*_i(\theta_H, \theta_L, \theta_H)]\right\},
\]

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When there are two agents, the property stated in Lemma 5.2 allows us to construct a mechanism $D' \equiv \{(\Theta_1, \Theta_2), (s', w', \sigma')\}$ that is \textit{ex post incentive compatible} and Pareto dominates the equal division mechanism. That is, each agent always prefers mechanism $D'$ regardless of his partner’s type; and a high-type agent strictly prefers $D'$ when his partner has low type. Let $(s', \sigma')$ be a constrained-efficient equity-effort rule for two agents (characterized in Section 3). We construct the wage rule $w'$ as follows. For each $i \in N$, let $w'_i(\theta_L, \theta_L) = w'_i(\theta_H, \theta_H) = w'_i(\theta_H, \theta_L) = 0$, and

$$s'_i(\theta_H, \theta_L)w'_i(\theta_L, \theta_H) = \begin{cases} 0, & \tilde{W}_i(\theta_L, \theta_H) - \tilde{W}_i(\theta_L, \theta_H) \\ \end{cases}.$$ 

8 Voting Mechanism

In this section, we design a voting mechanism that implements the arrangement rule $(s^*, w^*, \sigma^*)$ in Nash equilibrium.

8.1 Description

In a voting mechanism, each agent is given a few menus of some suggested arrangements and is asked to vote for one arrangement from each menu. These menus are designed so that if agents have type $\theta \in \Theta$, then $[s^*(\theta), w^*(\theta), \sigma^*(\theta)]$ is the unique arrangement (across all menus) that receives $n$ votes (which is the highest number of votes that an arrangement can receive). We define a voting game formally below.

A \textbf{voting mechanism} is a tuple $V \equiv \{(A_i, V_i)_{i \in N}, \xi\}$, where $A_i$ is a set of menus, $V_i$ is a set of voting strategies, and $\xi$ is an outcome function.

**Menus.**

In stage 1, each agent $i$ receives a set of $L$ menus, each of which consists of $K$ arrangements ($1 \leq L, K < \infty$). Each menu is represented by a column in the following matrix:

$$A_i \equiv \begin{bmatrix} a^i_{11} & a^i_{12} & \cdots & a^i_{1L} \\ a^i_{21} & a^i_{22} & \cdots & a^i_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ a^i_{K1} & a^i_{K2} & \cdots & a^i_{KL} \end{bmatrix}$$

where each entry $a^i_{kl}$ is an arrangement in menu $l$. Let $\tilde{A}_i$ be the set of all entries in matrix $A_i$. We require that
(a) all entries in $A_i$ are distinct (each agent can vote for/against an arrangement only once), and 
(b) $\tilde{A}_i = \tilde{A}_j$ for $i \neq j$ (all agents can vote for/against any suggested arrangement).

Let $\tilde{A} \equiv \tilde{A}_i$ be the set of suggested arrangements.

**Voting strategies.**

Each agent is asked to vote for a unique arrangement in each menu. Formally, a voting strategy for agent $i$ is a mapping $\nu_i : \tilde{A} \to \{0, 1\}$ such that $\sum_{k=1}^{K} \nu_i(a_{kl}) = 1$ for each $l \in \{1, 2, ..., L\}$. For instance, $\nu_i(a_{kl}) = 1$ implies that arrangement $a_{kl}$ receives one vote and all other arrangements in menu $l$ receive no vote from agent $i$ ($\nu_i(a_{k'l}) = 0$ for each $k' \neq k$).

Let $V_i$ be the set of voting strategies for agent $i$ and $V \equiv V_1 \times \cdots \times V_n$ be the set of voting strategy profiles.

**Outcome function.**

An outcome function maps from each voting strategy profile to an arrangement. For each $\nu \in V$, 

$$\xi(\nu) \in \arg \max_{a \in \tilde{A}} \sum_{i \in N} \nu_i(a).$$

Let $\xi^\pi(\nu) \in \Delta^{n-1} \times \mathbb{R}^n_+$ be the selected compensation plan and $\xi^e(\nu) \in \mathbb{R}^n_+$ be the selected effort recommendation.

**Remark 8.1** If there are multiple arrangements that receive the same number of votes, then the outcome function $\xi$ selects an arbitrary such arrangement. However, in equilibrium (defined in Section 8.2), there is a unique arrangement that receives the highest number of votes. □

### 8.2 Implementation

A voting mechanism implements the arrangement rule $\alpha^* = (s^*, w^*, \sigma^*)$ in Bayesian Nash equilibrium if the corresponding voting game (defined below) has a Bayesian Nash equilibrium that induces the outcome $[(s^*(\theta), w^*(\theta), \sigma^*(\theta))]$ for each type profile $\theta \in \Theta$. We formalize the concept of implementation as follow.

Fix a voting mechanism $\mathcal{V} \equiv \{(A_i, V_i)_{i \in N}, \xi\}$. A **voting game** is a tuple 

$$\mathcal{G}^\mathcal{V} \equiv \{N, \Theta, p, H, Z, (\varphi_i)_{i \in N}\},$$
where

- $H \equiv \{h^1\} \cup V$ is the set of non-terminal histories ($h^1$ represents the root of the game tree and each $\nu \in V$ describes the voting strategies chosen in stage 1),
- $Z \equiv V \times \mathbb{R}_+^n$ is the set of terminal histories (each $(\nu, e) \in V \times \mathbb{R}_+^n$ describes the chosen voting strategies and effort levels),
- $\varphi_i : Z \times \Theta \rightarrow \mathbb{R}$ is the payoff for agent $i$: if agent $i$ has true cost parameter $\theta_i \in \Theta_i$ and the terminal history is $(\nu, e) \in Z$, then the selected compensation plan is $\xi^\pi(\nu)$ and the payoff for agent $i$ is $\varphi_i(\nu, e, \theta_i) = v_i[\xi^\pi(\nu), e, \theta_i]$.

**Strategies.**

An agent’s strategy is a mapping from each information state to an action available at that information state. For each agent $i$, a stage-one information state is his private information $\theta_i \in \Theta_i$ and a stage-two information state is a pair $(\theta_i, \nu) \in \Theta_i \times V$, which consists of his private information $\theta_i$ and the voting strategies $\nu$ observed in stage 1. A strategy for agent $i$ is some

$$\lambda_i : \Theta_i \cup \Theta_i \times V \rightarrow V_i \cup \mathbb{R}_+$$

such that

(a) $\lambda_i(\theta_i) \in V_i$ for each $\theta_i \in \Theta_i$ (agent $i$ chooses a voting strategy in stage 1), and
(b) $\lambda_i(\theta_i, \nu) \in \mathbb{R}_+$ for each $(\theta_i, \nu) \in \Theta_i \times V$ (agent $i$ chooses an effort level in stage 2).

Let $\Lambda_i$ be the set of strategies for agent $i$ and $\Lambda \equiv \Lambda_1 \times \cdots \times \Lambda_n$ be the set of strategy profiles. Suppose agents have true type $\theta \in \Theta$ and choose strategies $\lambda \in \Lambda$. Let $\zeta^\nu(\lambda, \theta)$ be the on-path voting strategies and $\zeta^e(\lambda, \theta)$ be the on-path efforts. Let $\zeta(\lambda, \theta) \equiv [\zeta^\nu(\lambda, \theta), \zeta^e(\lambda, \theta)]$ be the terminal history. Then the ex-post payoff for agent $i$ is $\phi_i(\lambda, \theta) = \varphi_i[\zeta(\lambda, \theta), \theta_i]$.

**Nash equilibrium.**

A Nash equilibrium of voting mechanism $\mathcal{V}$ is a strategy profile $\lambda \in \Lambda$ such that for each agent $i \in N$, each $\theta_i \in \Theta_i$, and each strategy $\lambda'_i \in \Lambda_i$, we have

$$\phi_i(\lambda, \theta) \geq \phi_i(\lambda'_i, \lambda_{-i}, \theta).$$

**Definition 8.1** The voting mechanism $\mathcal{V}' \equiv \{(A_i, \mathcal{V}_i)_{i \in N}, \xi\}$ implements the arrangement rule $\alpha^\ast$ in Nash equilibrium if there is a Nash equilibrium $\lambda$ of $\mathcal{V}'$ such that $\xi[\zeta^\nu(\lambda, \theta)] = \alpha^\ast(\theta)$ for each $\theta \in \Theta^\ast$. 26
8.3 Design

Here we design a voting mechanism $V^* \equiv \{(A_i^*, V_i^*)_{i \in N}, \xi^*\}$ that implements the arrangement rule $\alpha^* = (s^*, w^*, \sigma^*)$ in Bayesian Nash equilibrium. For each $i \in N$ and each $\theta_{-i} \in \Theta^{n-1}$, let $\mu_i^*(\theta_{-i}) \equiv (\alpha^*(\theta_i, \theta_{-i}))_{\theta_i \in \Theta}$ be a menu of $|\Theta|$ arrangements ($\mu_i^*(\theta_{-i})$ is a column vector). Let $A_i^* = (\mu_i^*(\theta_{-i}))_{\theta_{-i} \in \Theta^{n-1}}$ be a set of $|\Theta^{n-1}|$ menus. By construction, $A_i^*$ is a $|\Theta| \times |\Theta^{n-1}|$ matrix, whose each column is a menu of $|\Theta|$ arrangements. It is easy to verify that for each $i \in N$, all entries of $A_i^*$ are distinct and $\tilde{A}_i^* = \tilde{A}_j^*$ for each $j \neq i$. The sets of menus $(A_i^*)_{i \in N}$ induce sets of voting strategies $(V_i^*)_{i \in N}$ and an outcome function $\xi^*$ as defined in Section 8.1.

Theorem 8.1 The voting mechanism $V^* \equiv \{(A_i^*, V_i^*)_{i \in N}, \xi^*\}$ implements the arrangement rule $\alpha^*$ in Nash equilibrium.

Hence, if agents have types $\theta \in \Theta$, the game outcome is $\alpha^*(\theta)$ in some Nash equilibrium. We recall that $\alpha^*(\theta)$ is a constrained-efficient arrangement that maximizes agents’ aggregate welfare subject to the constraint that agents choose their efforts rationally conditional on their equity shares.

9 Conclusion
References


A Preliminary Results

A.1 Efforts

Proof for Lemma 5.1. An effort profile \( e_t \in \mathbb{R}^n_+ \) satisfies \((r_t, e_t) \in \mathcal{R}_t(\theta_t, \alpha)\) if and only if for each \( i \in N \),

\[
    r_{i,t} + \frac{\partial \mathbb{E}[W_{i,t+1}(\theta_{t+1}, \alpha)]}{\partial e_{i,t}} |_{\theta_t, e_t} = r_{i,t} + \frac{\partial \mathbb{E}[W_{i,t+1}(\theta_{t+1}, \alpha) \delta \theta_{t+1}]}{\partial e_{i,t}} |_{\theta_t, e_t} = \frac{e_{i,t}}{\theta_{0,t+1}}. \tag{A.1}
\]

Note that \( \theta_{0,t+1} = \theta_{0,t} + \gamma \sum_{j \in N} e_{j,t} \). Hence, (A.1) holds if and only if

\[
e_{i,t} = \theta_{0,t} \lambda_{i,t} \left( r_{i,t} + \delta \gamma \mathbb{E}\left[ \partial W_{i,t+1}(\theta_{t+1}, \alpha)/\partial \theta_{0,t+1} \right] \right). \tag{A.2}
\]

For each \( i \in N \), let \( \tilde{\sigma}_{i,t}(\theta_t, r_t, \alpha) = \theta_{0,t} \lambda_{i,t} \left( r_{i,t} + \delta \gamma \mathbb{E}\left[ \partial W_{i,t+1}(\theta_{t+1}, \alpha)/\partial \theta_{0,t+1} \right] \right) \). Showing that \( \tilde{\sigma}_{i,t}(\theta_t, r_t, \alpha) \) satisfies (a)–(c) is straightforward.

A.2 Equity Shares

Fix \( t \geq 1 \), an arrangement rule \( \alpha \), and a pair \((s^*_t, \sigma^*_t)\). By Lemma 5.1, for each \( \theta_t \in \Theta_t \),

\[
    [s^*_t(\theta_t), \sigma^*_t(\theta_t)] = \arg \max_{(r_t, e_t) \in \mathcal{R}_t(\theta_t, \alpha)} \tilde{u}_t(r_t, e_t, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha) | \theta_t, e_t]
\]

if and only if \( \sigma^*_t(\theta_t) = \tilde{\sigma}_t[\theta_t, s^*_t(\theta_t), \alpha] \) and \( s^*_t(\theta_t) \) solves

\[
    \max_{r_t} \tilde{u}_t[r_t, \tilde{\sigma}_t(\theta_t, r_t, \alpha), \theta_t] + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha) | \theta_t, \tilde{\sigma}_t(\theta_t, r_t, \alpha)]
\]

subject to \( \sum_{i \in N} r_{i,t} = 1 \) and \( 0 \leq r_{i,t} \leq 1 \) for each \( i \in N \).

For each \( r_t \in \Delta^{n-1} \), let

\[
    U_t(\theta_t, r_t, \alpha) \equiv \tilde{u}_t[r_t, \tilde{\sigma}_t(\theta_t, r_t, \alpha), \theta_t] + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha) | \theta_t, \tilde{\sigma}_t(\theta_t, r_t, \alpha)].
\]

The Lagrangean function

\[
    L(r_t) = U_t(\theta_t, r_t, \alpha) + \sum_{i \in N} \lambda_{i,t} r_{i,t} - \sum_{i \in N} \gamma_{i,t} (r_{i,t} - 1) - \mu_t \left( \sum_{i \in N} r_{i,t} - 1 \right).
\]

The Kuhn-Tucker conditions

\[
    \frac{\partial U_t(\theta_t, r_t, \alpha)}{\partial r_{i,t}} + \lambda_{i,t} - \gamma_{i,t} - \mu_t = 0 \text{ for each } i \in N,
\]

\[
    \lambda_{i,t} \geq 0, \ r_{i,t} \geq 0, \ \lambda_{i,t} r_{i,t} = 0 \text{ for each } i \in N,
\]

\[
    \lambda_{i,t} \geq 0, \ r_{i,t} \geq 0, \ \lambda_{i,t} r_{i,t} = 0 \text{ for each } i \in N,
\]
$\gamma_{i,t} \geq 0$, $r_{i,t} \leq 1$, $\gamma_{i,t}(r_{i,t} - 1) = 0$ for each $i \in N$,
\[ r_{1,t} + r_{2,t} + \ldots + r_{n,t} = 1. \]

**Lemma A.1** For each $t \geq 1$, each $\theta_t \in \Theta_t$, each $r_t \in \Delta^{n-1}$, and each arrangement rule $\alpha$ such that $\{\alpha_\tau\}_{\tau > t}$ is constrained-efficient,
\[
U_t(\theta_t, r_t, \alpha) = \sum_{i \in N} \left( \bar{u}_{i,t}[r_t, \bar{\sigma}_t(\theta_t, r_t, \alpha) \sigma_{i,t,0}] + \delta \mathbb{E}[W_{i,t+1}(\theta_{t+1}, \alpha) \mid \theta_t, \bar{\sigma}_t(\theta_t, r_t, \alpha)] \right).
\]

**Proof.** Since $\{\alpha_\tau\}_{\tau > t}$ is constrained-efficient, for each $\theta_{t+1} \in \Theta_{t+1}$,
\[
W_{t+1}(\theta_{t+1}, \alpha) = \mathbb{E}(\theta_{-0,\tau})_{\tau > t+1} \left[ \sum_{\tau \geq t+1} \delta^{\tau-t-1} u_{\tau}[\alpha(\theta_\tau), \theta_\tau] \mid \theta_{t+1} \right]
\]
\[ = \sum_{i \in N} \mathbb{E}(\theta_{-0,\tau})_{\tau > t+1} \left[ \sum_{\tau \geq t+1} \delta^{\tau-t-1} u_{i,\tau}[\alpha_\tau(\theta_\tau), \theta_{0,\tau}, \theta_{i,\tau}] \mid \theta_{t+1} \right]
\]
\[ = \sum_{i \in N} W_{i,t+1}(\theta_{t+1}, \alpha), \text{ which implies (A.3)}. \]

**Lemma A.2** For each $i \in N$, each $t \geq 1$, each $\theta_t \in \Theta_t$, each $r_t \in \Delta^{n-1}$, and each arrangement rule $\alpha$ such that $\{\alpha_\tau\}_{\tau > t}$ is constrained-efficient and periodically allows individual payoffs to be weakly increasing in experience level at constant rates,
\[
(a) \quad \frac{\partial U_t(\theta_t, r_t, \alpha)}{\partial r_{i,t}} = \left[ 1 - r_{i,t} + \delta \gamma \sum_{j \neq i} \frac{\partial \mathbb{E} W_{j,t+1}(\theta_{t+1}, \alpha)}{\partial \theta_{0,t+1}} \right] \theta_{0,t} \theta_{i,t},
\]
\[
(b) \quad \text{for each } r'_{i,t} \in \Delta^{n-1} \text{ and each } \theta'_t \in \Theta_t \text{ such that } r'_{i,t} > r_{i,t}, \theta'_{0,t} = \theta_{0,t}, \text{ and } \theta'_{i,t} \leq \theta_{i,t},
\]
\[ \frac{\partial U_t(\theta_t, r_t, \alpha)}{\partial r_{i,t}} > \frac{\partial U_t(\theta'_t, r'_{i,t}, \alpha)}{\partial r'_{i,t}}. \]

**Proof.**

(a) By Lemma A.1, $U_t(\theta_t, r_t, \alpha) = \sum_{j \in N} \bar{\sigma}_{j,t}(\theta_t, r_t, \alpha) + \delta \sum_{j \in N} \mathbb{E}[W_{j,t+1}(\theta_{t+1}, \alpha) \mid \theta_t, \bar{\sigma}_t(\theta_t, r_t, \alpha)] - \sum_{j \in N} c_{j,t}[\bar{\sigma}_{j,t}(\theta_t, r_t, \alpha), \theta_{0,t}, \theta_{j,t}].$

We have $\frac{\partial U_t(\theta_t, r_t, \alpha)}{\partial r_{i,t}} = \sum_{j \in N} \bar{\sigma}_{j,t}(\theta_t, r_t, \alpha) + \delta \sum_{j \in N} \mathbb{E}[W_{j,t+1}(\theta_{t+1}, \alpha) \mid \theta_t, \bar{\sigma}_t(\theta_t, r_t, \alpha)] - \sum_{j \in N} c_{j,t}[\bar{\sigma}_{j,t}(\theta_t, r_t, \alpha), \theta_{0,t}, \theta_{j,t}].$
\[
1 + \delta \sum_{j \in N} \frac{\partial \mathbb{E}[W_{j,t+1}(\theta_{t+1}, \alpha) \mid \theta_t, \hat{s}_t(\theta_t, \alpha)]}{\partial e_{i,t}} \cdot \frac{\partial c_{i,t}[\hat{s}_t(\theta_t, \alpha), \theta_0, \theta_t]}{\partial e_{i,t}} \cdot \frac{\partial \hat{s}_t(\theta_t, \alpha)}{\partial r_{i,t}}.
\]

By (A.1), we have
\[
\frac{\partial U_t(\theta_t, r_t, \alpha)}{\partial r_{i,t}} = \left[1 - r_{i,t} + \delta \gamma \sum_{j \neq i} \frac{\partial \mathbb{E}[W_{j,t+1}(\theta_{t+1}, \alpha)]}{\partial \theta_{0,t+1}}\right] \theta_0 \theta_t.
\]

Hence,
\[
\frac{\partial U_t(\theta_t, r_t, \alpha)}{\partial r_{i,t}} = 1 - r_{i,t} + \delta \gamma \sum_{j \neq i} \frac{\partial \mathbb{E}[W_{j,t+1}(\theta_{t+1}, \alpha)]}{\partial \theta_{0,t+1}}
\]

(b) From Step (a),
\[
\frac{\partial U_t(\theta_t, r_t, \alpha)}{\partial r_{i,t}} = \left[1 - r_{i,t} + \delta \gamma \sum_{j \neq i} \frac{\partial \mathbb{E}[W_{j,t+1}(\theta_{t+1}, \alpha)]}{\partial \theta_{0,t+1}}\right] \theta_0 \theta_t.
\]

Since \(1 - r_{i,t} > 1 - r_{i,t}'\) and \(\theta_{i,t} > \theta_{i,t}'\), we have
\[
\frac{\partial U_t(\theta_t, r_t, \alpha)}{\partial r_{i,t}} > \frac{\partial U_t(\theta_t, r_t, \alpha)}{\partial r_{i,t}'}.
\]

Proof for Proposition 5.1.

**Step 1.** We show that, for each \(\theta_t \in \Theta_t\), there is a unique pair \([s^*_t(\theta_t), \sigma^*_t(\theta_t)]\) that solves
\[
\max_{(r_t, \hat{e}_t) \in \hat{R}_t(\theta_t, \alpha)} \hat{u}_t(r_t, \hat{e}_t, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha) \mid \theta_t, \hat{c}].
\]

By Lemma 5.1 and Lemma A.1, it suffices to show that the following maximization problem has a unique solution:

\[
\max_{r_t \in \Delta_{t-1}} \sum_{i \in N} \left(\hat{u}_i(r_t, \hat{s}_i(\theta_t, \alpha), \theta_0, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}, \alpha) \mid \theta_t, \hat{s}_i(\theta_t, \alpha)]\right). \tag{A.4}
\]

By Weierstrass Theorem, the maximization problem (A.4) has a solution \(s^*_t(\theta_t)\). Suppose there is some \(r_t \neq s^*_t(\theta_t)\) that solves (A.4). Since \(r_t \neq s^*_t(\theta_t)\), there is a pair \([i, j] \subseteq N\) such that \(r_{i,t} < s^*_t(\theta_t)\) and \(r_{j,t} > s^*_t(\theta_t)\). By the Kuhn-Tucker conditions,

\[
\frac{\partial U_t(\theta_t, s^*_t(\theta_t), \alpha)}{\partial r_{i,t}} \geq \frac{\partial U_t(\theta_t, s^*_t(\theta_t), \alpha)}{\partial r_{j,t}} \text{ and } \frac{\partial U_t(\theta_t, s^*_t(\theta_t), \alpha)}{\partial r_{i,t}} \leq \frac{\partial U_t(\theta_t, r_{i,t}, \alpha)}{\partial r_{j,t}}. \tag{A.5}
\]

By Lemma A.2(b),
\[
\frac{\partial U_t(\theta_t, r_{i,t}, \alpha)}{\partial r_{i,t}} > \frac{\partial U_t(\theta_t, s^*_t(\theta_t), \alpha)}{\partial r_{i,t}} \text{ and } \frac{\partial U_t(\theta_t, s^*_t(\theta_t), \alpha)}{\partial r_{j,t}} > \frac{\partial U_t(\theta_t, r_{i,t}, \alpha)}{\partial r_{j,t}}.
\]

This contradicts (A.5). Hence, the maximization problem (A.4) has a unique solution \(s^*_t(\theta_t)\).

**Step 2.** For each \(i \in N\) and each \(\theta_{i,t} > \theta_{i,t}\), we show that \(s^*_t(\theta_{i,t}, \theta_{i,t}) \geq s^*_t(\theta_t)\).

Let \(\theta'_t \equiv (\theta_{i,t}', \theta_{i,t})\). If \(s^*_t(\theta_t) = 0\), then it is clear that \(s^*_t(\theta_t) \leq s^*_t(\theta_t')\). Suppose \(s^*_t(\theta_t) > 0\) and \(s^*_t(\theta_t') > s^*_t(\theta_t')\). Then there exists some \(j \neq i\) such that \(s^*_t(\theta_t) < s^*_t(\theta_t')\). By the
Kuhn-Tucker conditions,
\[
\frac{\partial U_t[\theta_t, s^*_t(\theta_t), \alpha]}{\partial r_{i,t}} \geq \frac{\partial U_t[\theta_t, s^*_t(\theta_t), \alpha]}{\partial r_{j,t}} \quad \text{and} \quad \frac{\partial U_t[\theta'_t, s^*_t(\theta'_t), \alpha]}{\partial r_{i,t}} \leq \frac{\partial U_t[\theta'_t, s^*_t(\theta'_t), \alpha]}{\partial r_{j,t}}.
\] (A.6)

By Lemma A.2(b),
\[
\frac{\partial U_t[\theta'_t, s^*_t(\theta'_t), \alpha]}{\partial r_{i,t}} > \frac{\partial U_t[\theta_t, s^*_t(\theta_t), \alpha]}{\partial r_{i,t}} \quad \text{and} \quad \frac{\partial U_t[\theta'_t, s^*_t(\theta'_t), \alpha]}{\partial r_{j,t}} > \frac{\partial U_t[\theta'_t, s^*_t(\theta'_t), \alpha]}{\partial r_{j,t}}.
\]

This contradicts (A.6). Hence, \(s^*_{i,t}(\theta_t) \leq s^*_{i,t}(\theta'_t)\).

**Lemma A.3** Fix \(t \geq 1\) and an arrangement rule \(\alpha\) such that \(\{\alpha_t\}_{t>1}\) is constrained-efficient and periodically allows individual payoffs to be weakly increasing in experience level at constant rates. For each \(\theta_t \in \Theta_t\), let \(s^*_t(\theta_t)\) be the solution to (A.4). We have \(s^*_t(\theta_t) = s^*_t(\theta'_{0,t}, \theta_{-0,t})\) for each \(\theta'_{0,t} \in \mathbb{R}^{+} \).

**Proof.** Let \(\theta'_t = (\theta'_{0,t}, \theta_{-0,t})\). Suppose \(s^*_t(\theta_t) \neq s^*_t(\theta'_t)\). Then there is a pair \(\{i, j\} \subseteq N\) such that \(s^*_{i,t}(\theta_t) < s^*_{i,t}(\theta'_t)\) and \(s^*_{j,t}(\theta_t) > s^*_{j,t}(\theta'_t)\). We note that \(s^*_{j,t}(\theta_t) > 0\).

For each \(k \in N\), let \(\xi_{k,t+1} \equiv \frac{\partial \mathbb{E} W_{k,t+1}(\theta_{t+1}, \alpha)}{\partial \theta_{0,t+1}}\).

By the Kuhn-Tucker conditions and Lemma A.2(a),
\[
\left[1 - s^*_{i,t}(\theta_t) + \delta \gamma \sum_{k \neq i} \xi_{k,t+1}\right] \theta_{0,t} \theta_{i,t} \leq \left[1 - s^*_{j,t}(\theta_t) + \delta \gamma \sum_{k \neq j} \xi_{k,t+1}\right] \theta_{0,t} \theta_{j,t},
\]
which implies
\[
\left[1 - s^*_{i,t}(\theta'_t) + \delta \gamma \sum_{k \neq i} \xi_{k,t+1}\right] \theta'_{0,t} \theta'_{i,t} < \left[1 - s^*_{j,t}(\theta'_t) + \delta \gamma \sum_{k \neq j} \xi_{k,t+1}\right] \theta'_{0,t} \theta'_{j,t}.
\]
This is a contradiction.

**A.3 Increasing Differences**

**Proof for Lemma 5.2.** An effort level \(e_{i,t}\) is a solution to the maximization problem (5.1) if and only if
\[
s_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta \frac{\partial \mathbb{E} W_{i,t+1}(\theta_{t+1}, \alpha) | \theta_{t}, e_{i,t}, \sigma^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})}{\partial e_{i,t}} = \frac{e_{i,t} \delta}{\theta_{0,t} \theta_{i,t}},
\]
which holds if and only if

\[
s_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta\mathbb{E}\left[\frac{\partial W_{i,t+1}(\theta_{t+1}, \alpha)}{\partial \theta_{0,t+1}} \frac{\partial \theta_{0,t+1}}{\partial e_{i,t}}\right] = \frac{e_{i,t}}{\theta_{0,t} \hat{\theta}_{i,t}}.
\]

(A.7)

Note that \(\theta_{0,t+1} = \theta_{0,t} + \gamma \sum_{j \in N} e_{j,t}\). Hence, (A.7) holds if and only if

\[
e_{i,t} = \theta_{0,t} \hat{\theta}_{i,t} \left(s_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta\mathbb{E}[\partial W_{i,t+1}(\theta_{t+1}, \alpha)/\partial \theta_{0,t+1}]\right) = \tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha],
\]

which implies that \(\tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]\) is the unique solution to (A.7).

\[
\text{Proof for Proposition 5.2. Since } [s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \sigma_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}, \alpha)] \in \tilde{R}_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}, \alpha), \text{ we have}
\]

\[
\sigma_{j,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) = \tilde{\sigma}_{j,t}[\hat{\theta}_{i,t}, \theta_{-i,t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha] \text{ for each } j \neq i \text{ (by Lemma 5.1).}
\]

By (A.2), we have \(\tilde{\sigma}_{j,t}[\hat{\theta}_{i,t}, \theta_{-i,t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]\) is independent of \(\hat{\theta}_{i,t}\), which implies

\[
\sigma_{j,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) = \tilde{\sigma}_{j,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha] \text{ for each } j \neq i.
\]

Also, by Lemma 5.2, \(\tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]\) solves the maximization problem (5.1). Hence,

\[
\tilde{W}_{i,t}(\theta_{t}; \hat{\theta}_{i,t}; \alpha) = s_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) \sum_{j \in N} \tilde{\sigma}_{j,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha] + \delta\mathbb{E}\left[W_{i,t+1}(\theta_{t+1}, \alpha) \mid \theta_{t}, \tilde{\sigma}_{j,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha] - c_{i,t}(\tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha], \theta_{0,t}, \theta_{t})\right].
\]

Without loss of generality, assume \(\theta'_{i,t} - \theta_{i,t} = \epsilon\) is sufficiently small. Then

\[
\left[\tilde{W}_{i,t}(\theta'_{i,t}, \theta_{-i,t}; \hat{\theta}_{i,t}; \alpha) - \tilde{W}_{i,t}(\theta_{i,t}, \theta_{-i,t}; \hat{\theta}_{i,t}; \alpha)\right] \epsilon^{-1} = \frac{\partial W_{i,t}(\theta_{i,t}, \theta_{-i,t}; \hat{\theta}_{i,t}; \alpha)}{\partial \hat{\theta}_{i,t}}
\]

\[
= \left[s_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta\mathbb{E}\left(\frac{\partial W_{i,t+1}(\theta_{t+1}, \alpha)}{\partial \theta_{0,t+1}} \frac{\partial \theta_{0,t+1}}{\partial e_{i,t}}\right)\right] \frac{\partial \tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]}{\partial \theta'_{i,t}} - \frac{\tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]}{\theta_{0,t} \hat{\theta}_{i,t}} \frac{\partial \tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]}{\partial \theta'_{i,t}} \
- \frac{\tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]}{\theta_{0,t} \hat{\theta}_{i,t}} \frac{\partial \tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]}{\partial \theta_{i,t}} \frac{\partial c_{i,t}(\tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha], \theta_{0,t}, \theta_{t})}{\partial \theta_{i,t}}.
\]

By (A.7), we have

\[
s_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta\mathbb{E}\left[\frac{\partial W_{i,t+1}(\theta_{t+1}, \alpha)}{\partial \theta_{0,t+1}} \frac{\partial \theta_{0,t+1}}{\partial e_{i,t}}\right] = \frac{\tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha]}{\theta_{0,t} \hat{\theta}_{i,t}}.
\]

Hence,

\[
\tilde{W}_{i,t}(\theta'_{i,t}, \theta_{-i,t}; \hat{\theta}_{i,t}; \alpha) - \tilde{W}_{i,t}(\theta_{i,t}, \theta_{-i,t}; \hat{\theta}_{i,t}; \alpha) = - \frac{\partial c_{i,t}(\tilde{\sigma}_{i,t}[\theta_{t}, s_{t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha], \theta_{0,t}, \theta_{t})}{\partial \theta_{i,t}} \epsilon.
\]

Likewise,
rates periodically allows individual payoffs to be weakly increasing in experience level at constant rates.

Proof for Lemma 6.1. Let $\bar{\theta}_{i,t} = \min(\bar{\theta}_{i,t}, \bar{\theta}_{-i,t})$ and $s_L = s_i, s_{-i} \equiv s_i, s_{-i} \equiv s_i, s_{-i}$ periodicity allows individual payoffs to be weakly increasing in experience level at constant rates. For each $\bar{\theta}_{i,t} \in \Theta_i$ and each $e_t \in \mathbb{R}_+^n$, we have

$$0 \leq \frac{\partial \mathbb{E}[W_{i,t+1}(\theta_{i,t}, \alpha) | \theta_{i,t}, e_t]}{\partial \theta_{0,t+1}} < \infty \quad \text{and} \quad \frac{\partial^2 \mathbb{E}[W_{i,t+1}(\theta_{i,t}, \alpha) | \theta_{i,t}, e_t]}{\partial^2 \theta_{0,t+1}} = 0.$$

Hence, we can write $\mathbb{E}[W_{i,t+1}(\theta_{i,t}, \alpha) | \theta_{i,t}, e_t] = \xi_{i,t+1} + \psi_{i,t+1}$ for some $0 \leq \xi_{i,t+1} < \infty$.

Also, for each $\bar{\theta}_{i,t} \in \Theta_i$ and each $r_t \in \Delta^{n-1}$, let

$$\bar{\sigma}_{i,t}(\theta_{i,t}, r_t, \alpha) = \theta_{0,t} \mathbb{E}[\bar{\theta}_{i,t} r_t + \delta \gamma \mathbb{E}[\partial W_{i,t+1}(\theta_{i,t}, \alpha)/\partial \theta_{0,t+1}]] = \theta_{0,t} \mathbb{E}[\theta_{i,t} f_{i,t}(r_t, \bar{\theta}_{i,t})].$$

We have

$$\bar{W}_{i,t}(\bar{\theta}_{i,t}, \alpha) - \bar{W}_{i,t}(\bar{\theta}_{i,t}, \alpha) = \frac{1}{n} [f_{i,t}(1/n) \theta_{0,t} \theta_L + f_{-i,t}(1/n) \theta_{0,t} \theta_H] - \frac{1}{2} [f_{i,t}(1/n)]^2 \theta_{0,t} \theta_L$$

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Proof for Lemma 6.1. Let $s_H \equiv s_i(\theta_{0,t}, \theta_H, \theta_L)$ and $s_L \equiv s_i(\theta_{0,t}, \theta_L, \theta_H)$. Since $\{\alpha_r\}_{r>0}$ periodically allows individual payoffs to be weakly increasing in experience level at constant rates, for each $\bar{\theta}_{i,t} \in \Theta_i$ and each $e_t \in \mathbb{R}_+^n$, we have

$$0 \leq \frac{\partial \mathbb{E}[W_{i,t+1}(\theta_{i,t}, \alpha) | \theta_{i,t}, e_t]}{\partial \theta_{0,t+1}} < \infty \quad \text{and} \quad \frac{\partial^2 \mathbb{E}[W_{i,t+1}(\theta_{i,t}, \alpha) | \theta_{i,t}, e_t]}{\partial^2 \theta_{0,t+1}} = 0.$$

Hence, we can write $\mathbb{E}[W_{i,t+1}(\theta_{i,t}, \alpha) | \theta_{i,t}, e_t] = \xi_{i,t+1} + \psi_{i,t+1}$ for some $0 \leq \xi_{i,t+1} < \infty$.

Also, for each $\bar{\theta}_{i,t} \in \Theta_i$ and each $r_t \in \Delta^{n-1}$, let

$$\bar{\sigma}_{i,t}(\theta_{i,t}, r_t, \alpha) = \theta_{0,t} \mathbb{E}[\bar{\theta}_{i,t} r_t + \delta \gamma \mathbb{E}[\partial W_{i,t+1}(\theta_{i,t}, \alpha)/\partial \theta_{0,t+1}]] = \theta_{0,t} \mathbb{E}[\theta_{i,t} f_{i,t}(r_t, \bar{\theta}_{i,t})].$$

We have

$$\bar{W}_{i,t}(\bar{\theta}_{i,t}, \alpha) - \bar{W}_{i,t}(\bar{\theta}_{i,t}, \alpha) = \frac{1}{n} [f_{i,t}(1/n) \theta_{0,t} \theta_L + f_{-i,t}(1/n) \theta_{0,t} \theta_H] - \frac{1}{2} [f_{i,t}(1/n)]^2 \theta_{0,t} \theta_L$$

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\[
-s_L \left[ f_{i,t}(s_L)\theta_{0,t}\theta_L + f_{-i,t}(s_H)\theta_{0,t}\theta_H \right] + \frac{1}{2} \left[ f_{i,t}(s_L) \right]^2 \theta_{0,t}\theta_L \\
-\delta \left( \theta_{0,t} + \gamma f_{i,t}(s_L)\theta_{0,t}\theta_L + \gamma f_{-i,t}(s_H)\theta_{0,t}\theta_H \right) \xi_{i,t+1} - \delta \psi_{i,t+1}
\]
\[
= \frac{1}{n} \left[ f_{i,t}(1/n)\theta_{0,t}\theta_L + f_{-i,t}(1/n)\theta_{0,t}\theta_L \right] - \frac{1}{2} \left[ f_{i,t}(1/n) \right]^2 \theta_{0,t}\theta_L \\
+ \delta \left( \theta_{0,t} + \gamma f_{i,t}(1/n)\theta_{0,t}\theta_L + \gamma f_{-i,t}(1/n)\theta_{0,t}\theta_L \right) \xi_{i,t+1} + \delta \psi_{i,t+1}
\]
\[
-s_L \left[ f_{i,t}(s_L)\theta_{0,t}\theta_L + f_{-i,t}(s_H)\theta_{0,t}\theta_L \right] + \frac{1}{2} \left[ f_{i,t}(s_L) \right]^2 \theta_{0,t}\theta_L \\
-\delta \left( \theta_{0,t} + \gamma f_{i,t}(s_L)\theta_{0,t}\theta_L + \gamma f_{-i,t}(s_H)\theta_{0,t}\theta_L \right) \xi_{i,t+1} - \delta \psi_{i,t+1}
\]
\[
+ \left[ \frac{1}{n} + \delta \gamma \xi_{i,t+1} \right] f_{-i,t}(1/n)\theta_{0,t}(\theta_H - \theta_L) + s_L + \delta \gamma \xi_{i,t+1} \right] f_{-i,t}(s_H)\theta_{0,t}(\theta_H - \theta_L),
\]
and
\[
\bar{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha) - \bar{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha) \\
= s_H \left[ f_{i,t}(s_H)\theta_{0,t}\theta_L + f_{-i,t}(s_L)\theta_{0,t}\theta_L \right] - \frac{1}{2} \left[ f_{i,t}(s_H) \right]^2 \theta_{0,t}\theta_L \\
+ \delta \left( \theta_{0,t} + \gamma f_{i,t}(s_H)\theta_{0,t}\theta_L + \gamma f_{-i,t}(s_L)\theta_{0,t}\theta_L \right) \xi_{i,t+1} + \delta \psi_{i,t+1}
\]
\[
- \frac{1}{n} \left[ f_{i,t}(1/n)\theta_{0,t}\theta_L + f_{-i,t}(1/n)\theta_{0,t}\theta_L \right] + \frac{1}{2} \left[ f_{i,t}(1/n) \right]^2 \theta_{0,t}\theta_L \\
-\delta \left( \theta_{0,t} + \gamma f_{i,t}(1/n)\theta_{0,t}\theta_L + \gamma f_{-i,t}(1/n)\theta_{0,t}\theta_L \right) \xi_{i,t+1} - \delta \psi_{i,t+1}.
\]

Let \( \theta'_t = (\theta_{0,t}, \theta_L, \theta_L) \), \( r'_t \equiv (s_L, s_H) \), \( m'_t = (0, 0) \), and
\[
e'_t = [\theta_{0,t}\theta_L f_{i,t}(s_L), \theta_{0,t}\theta_L f_{-i,t}(s_H)] = \tilde{\sigma}_t(\theta'_t, r'_t, \alpha).
\]

Note that \( (r'_t, m'_t, e'_t) \in \mathcal{R}_t(\theta'_t, \alpha) \). We have
\[
\left[ \bar{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha) - \bar{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha) \right] \\
- \left[ \bar{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha) - \bar{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha) \right] \\
= W_t(\theta'_t, \alpha) - \left[ u_t(r'_t, m'_t, e'_t, \theta'_t) + \delta \mathbb{E}[W_{i,t+1}(\theta_{i,t+1}, \alpha) | \theta'_t, e'_t] \right] \\
+ \left[ \frac{1}{n} + \delta \gamma \xi_{i,t+1} \right] f_{-i,t}(1/n)\theta_{0,t}(\theta_H - \theta_L) + s_L + \delta \gamma \xi_{i,t+1} \right] f_{-i,t}(s_H)\theta_{0,t}(\theta_H - \theta_L) \\
\geq \left[ \frac{1}{n} + \delta \gamma \xi_{i,t+1} \right]^2 - \left( s_L + \delta \gamma \xi_{i,t+1} \right) \left( s_H + \delta \gamma \xi_{i,t+1} \right) \theta_{0,t}(\theta_H - \theta_L) \\
= \left( \frac{1}{2^2} - s_L s_H \right) \theta_{0,t}(\theta_H - \theta_L) \geq 0.
\]
Proof for Lemma 6.2 and Lemma 6.3. By Remark 3.2, the sequence \( \{ \tilde{\alpha}^{(t)} \}_{\tau \geq t} \equiv \{ s^*_r, w^*_r, \sigma^*_r \}_{\tau \geq t} \) is constrained-efficient. Here we show that \( \{ \tilde{\alpha}^{(t)} \}_{\tau \geq t} \) periodically allows individual payoffs to be weakly increasing in experience level at constant rates. The proof is by induction. For each \( i \in \mathbb{N} \), let \( W_{i,T+1}(\theta_{T+1}, \tilde{\alpha}^{(T+1)}) \equiv 0 \equiv W_{i,0,T+1}(\theta_{0,T+1}) \). Fix \( t \in \{ 1, \ldots, T \} \). Suppose that, for each \( i \in \mathbb{N} \), each \( \tau \geq t + 1 \) and each \( \theta \in \Theta_t \), we have \( W_{i,\tau}(\theta, \tilde{\alpha}^{(\tau)}) = \theta_{0,\tau + 1} \xi_{i,\tau}(\theta_{-0,\tau}) \), where \( 0 < \xi_{i,\tau}(\theta_{-0,\tau}) < \infty \), which implies \( \{ \tilde{\alpha}^{(t+1)} \}_{\tau \geq t+1} \) periodically allows individual payoffs to be weakly increasing in experience level at constant rates.\(^{21}\) Then for each \( i \in \mathbb{N} \) and each \( \theta \in \Theta_t \), we have \( W_{i,t}(\theta, \tilde{\alpha}^{(t)}) = \theta_{0,t} \tilde{\xi}_{i,t}(\theta_{-0,t}) \), where \( 0 < \tilde{\xi}_{i,t}(\theta_{-0,t}) < \infty \), which implies \( \{ \tilde{\alpha}^{(t)} \}_{\tau \geq t} \) periodically allows individual payoffs to be weakly increasing in experience level at constant rates.

For each \( i \in \mathbb{N} \) and each \( \theta \in \Theta_t \),

\[
W_{i,t}(\theta; \theta_{i,t}; \tilde{\alpha}^{(t)}) = \max_{e_{i,t} \in \mathbb{R}_+} \tilde{u}_{i,t}[s^*_i(\theta), e_{i,t}, \sigma^*_{-i,t}(\theta), \theta_{0,t}, \theta_{i,t}] + \delta \mathbb{E}[W_{i,t+1}(\theta_{t+1}, \tilde{\alpha}^{(t+1)}) | \theta_t, e_{i,t}, \sigma^*_{-i,t}(\theta_t)].^{22}
\]

Since \( [s^*_i(\theta), \sigma^*_i(\theta)] \in \tilde{R}_{i,t}(\theta, \tilde{\alpha}^{(t+1)}) \), we have \( \sigma^*_j(\theta) = \tilde{\sigma}_j[\theta_i, s^*_i(\theta), \tilde{\alpha}^{(t+1)}] \) for each \( j \in \mathbb{N} \) (by Lemma 5.1). By Lemma 5.2, \( \tilde{\sigma}_{j,t}[\theta_i, s^*_i(\theta), \tilde{\alpha}^{(t+1)}] \) solves

\[
\max_{e_{j,t} \in \mathbb{R}_+} \tilde{u}_{j,t}[s^*_j(\theta), e_{j,t}, \sigma^*_{-j,t}(\theta), \theta_{0,t}, \theta_{j,t}] + \delta \mathbb{E}[W_{j,t+1}(\theta_{t+1}, \tilde{\alpha}^{(t+1)}) | \theta_t, e_{j,t}, \sigma^*_{-j,t}(\theta_t)].
\]

Hence,

\[
W_{i,t}(\theta; \theta_{i,t}; \tilde{\alpha}^{(t)}) = s^*_i(\theta) \sum_{j \in \mathbb{N}} \sigma^*_j(\theta) - \frac{[\sigma^*_i(\theta)]^2}{2\theta_{0,t}} + \delta \mathbb{E}[W_{i,t+1}(\theta_{t+1}, \tilde{\alpha}^{(t+1)}) | \theta_t, \sigma^*_i(\theta)].
\]

For each \( j \in \mathbb{N} \), let \( \xi_{j,t+1} \equiv \mathbb{E}[\tilde{\xi}_{j,t+1}(\theta_{-0,t+1})] \). By (A.2), for each \( j \in \mathbb{N} \),

\[
\sigma^*_j(\theta) = \tilde{\sigma}_j[\theta_i, s^*_i(\theta), \tilde{\alpha}^{(t+1)}] = \theta_{0,t} \theta_{j,t} [s^*_j(\theta) + \delta \gamma \xi_{j,t+1}] \equiv \theta_{0,t} \theta_{j,t} f_j[s^*_j(\theta)].
\]

We have

\[
W_{j,t}(\theta; \theta_{j,t}; \tilde{\alpha}^{(t)}) = s^*_j(\theta) \theta_{0,t} \sum_{j \in \mathbb{N}} f_j[s^*_j(\theta)] \theta_{j,t} - \frac{1}{2} \left(f_{\theta,t} [s^*_j(\theta)] \right)^2 \theta_{0,t} \theta_{i,t} + \delta \left( \theta_{0,t} + \gamma \theta_{0,t} \sum_{j \in \mathbb{N}} f_j[s^*_j(\theta)] \theta_{j,t} \right) \xi_{j,t+1} \]

\[
= \theta_{0,t} \left[ [s^*_j(\theta) + \delta \gamma \xi_{j,t+1}] \sum_{j \in \mathbb{N}} f_j[s^*_j(\theta)] \theta_{j,t} - \frac{1}{2} \left(f_{\theta,t} [s^*_j(\theta)] \right)^2 \theta_{i,t} + \delta \xi_{j,t+1} \right].
\]

By Lemma A.3, \( s^*_i(\theta) \) is independent of \( \theta_{0,t} \). Hence, we can write

\(^{21}\)Note that \( W_{i,\tau}(\theta, \tilde{\alpha}^{(\tau)}) = W_{i,\tau}(\theta, \tilde{\alpha}^{(\tau+1)}) \) for each \( \tau \geq t + 1 \) and each \( \theta \in \Theta_t \).

\(^{22}\)Note that \( W_{i,t+1}(\theta_{t+1}, \tilde{\alpha}^{(t+1)}) = W_{i,t+1}(\theta_{t+1}, \tilde{\alpha}^{(t+1)}) \) for each \( \theta_{t+1} \in \Theta_{t+1} \).
\[ \tilde{W}_{i,t}(\theta_i; \theta_{-i}; \alpha(t)) = \theta_{0,i}\tilde{\xi}_{i,t}(\theta_{-0,t}), \text{ where } 0 \leq \tilde{\xi}_{i,t}(\theta_{-0,t}) < \infty. \]

Likewise,
\[
\tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_H; \alpha(t)) = \frac{1}{n}\theta_{0,t} \sum_{j \in N} f_{j,t}(1/n)\theta_{j,t} - \frac{1}{2}\left[ f_{i,t}(1/n) \right]^2 \theta_{0,t}\theta_{i,t} + \delta\left(\theta_{0,t} + \gamma\theta_{0,t} \sum_{j \in N} f_{j,t}(1/n)\theta_{j,t}\right)\xi_{t,t+1}
\]
\[
= \theta_{0,t}\left[(1/n + \delta\xi_{t,t+1}) \sum_{j \in N} f_{j,t}(1/n)\theta_{j,t} - \frac{1}{2}\left[ f_{i,t}(1/n) \right]^2 \theta_{i,t} + \delta\xi_{t,t+1}\right]
\]
\[\equiv \theta_{0,i}\tilde{\xi}_{i,t}(\theta_L, \theta_H), \text{ where } 0 \leq \tilde{\xi}_{i,t}(\theta_L, \theta_H) < \infty.\]

We have
\[
W_{i,t}(\theta_i, \tilde{\alpha}(t)) = s_{i,t}^*(\theta_i)\left[ \sum_{j \in N} \sigma_{j,t}^*(\theta_i) - \sum_{j \in N} w_{j,t}^*(\theta_i) \right] - c_{i,t}[\sigma_{i,t}^*(\theta_i), \theta_{0,t}; t_{i,t}] + w_{i,t}^*(\theta_i)
\]
\[\quad + \delta\mathbb{E}[W_{i,t+1}(\theta_{t+1}, \tilde{\alpha}(t+1)) | \theta_i, \sigma_i^*(\theta_i)]
\]
\[= \tilde{W}_{i,t}(\theta_i; \theta_{i,t}; \alpha(t)) - s_{i,t}^*(\theta_i) \sum_{j \in N} w_{j,t}^*(\theta_i) + w_{i,t}^*(\theta_i).\]

**Case 1.** Suppose \( \theta_{-0,t} = (\theta_L, \theta_L) \) or \( \theta_{-0,t} = (\theta_H, \theta_H) \). Then \( w_{i,t}^*(\theta_i) = w_{-i,t}^*(\theta_i) = 0 \), which implies \( W_{i,t}(\theta_i, \tilde{\alpha}(t)) = \tilde{W}_{i,t}(\theta_i; \theta_{i,t}; \alpha(t)) \equiv \theta_{0,i}\tilde{\xi}_{i,t}(\theta_{-0,t}). \)

**Case 2.** Suppose \( \theta_{-0,t} = (\theta_L, \theta_H) \).

If \( \Delta \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t)) \geq 0 \), then
\[
W_{i,t}(\theta_{0,t}, \theta_L, \theta_H, \tilde{\alpha}(t)) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta; \tilde{\alpha}(t)) + [1 - s_{i,t}^*(\theta_{0,t}, \theta_L, \theta_H)] w_{i,t}^*(\theta_{0,t}, \theta_L, \theta_H)
\]
\[= \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t)) + \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t)) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t))
\]
\[= \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t)).\]

If \( \Delta \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t)) < 0 \), then
\[
W_{i,t}(\theta_{0,t}, \theta_L, \theta_H, \tilde{\alpha}(t)) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta; \tilde{\alpha}(t)) - s_{i,t}^*(\theta_{0,t}, \theta_L, \theta_H) w_{-i,t}^*(\theta_{0,t}, \theta_L, \theta_H)
\]
\[= \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t)) - [\tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t)) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t))]
\]
\[= \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t)).\]

In either case, \( W_{i,t}(\theta_{0,t}, \theta_L, \theta_H, \tilde{\alpha}(t)) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha(t)) \equiv \theta_{0,i}\tilde{\xi}_{i,t}(\theta_L, \theta_H). \)

**Case 3.** Suppose \( \theta_{-0,t} = (\theta_H, \theta_L). \)
In the following, we show that $\theta$

By Lemma 5.2, the effort level $\tilde{D}$ implies that the direct revelation mechanism

If $\Delta \tilde{W}_{i,t}(\theta_{0,t}, \theta_H; \alpha(t)) < 0$, then

If $\Delta \tilde{W}_{i,t}(\theta_{0,t}, \theta_H; \alpha(t)) \geq 0$, then

In either case, we can write $W_{i,t}(\theta_{0,t}, \theta_H, \tilde{\alpha}(t)) \equiv \tilde{W}_{i,t}(\theta_H, \theta_L)$.

By Lemma 6.3, the arrangement rule $\alpha^* \equiv \{s_t^*, w_t^*, \sigma_t^*\}_{t=1}^T$ is constrained-efficient, which implies that the direct revelation mechanism $D^* = \{\Theta_{-0,t}, \alpha_t^*\}_{t=1}^T$ is constrained-efficient. Hence, in order to prove Theorem 6.1, it is sufficient to prove the following lemmas.

**Lemma B.1** The direct revelation mechanism $D^*$ is periodic ex-post incentive compatible.

**Proof.** Fix $i \in N$, $t \in \{1, \ldots, T\}$, and $\theta_t \in \Theta_t$. For each $\hat{\theta}_{i,t} \in \Theta_{i,t}$ and each $e_{i,t} \in \mathbb{R}_+$,

By Lemma 5.2, the effort level $\tilde{\sigma}_{i,t}[\theta_t, s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha^*]$ is the unique solution to

In the following, we show that $\theta_{i,t}$ solves

which implies

\[ [\theta_{i,t}, \sigma_{i,t}^*(\theta_t)] \in \arg \max_{(\theta_{i,t}, e_{i,t})} u_{i,t}[s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), w_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), e_{i,t}, \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_{0,t}, \theta_t] \]
as required by periodic ex-post incentive compatibility.\footnote{Since \( [s_t^i(\theta_t), \sigma_t^i(\theta_t)] \in \mathcal{R}_t(\theta_t, \alpha^*) \), we have \( \sigma_t^i(\theta_t) = \tilde{\sigma}_{t,i}[\theta_t, s_t^i(\theta_t), \alpha^*] \) (by Lemmas 5.1 and 6.3).}

For each \( \hat{\theta}_{i,t} \in \Theta_{i,t} \), define \( Z_{i,t}(\theta_t; \hat{\theta}_{i,t}; \alpha^*) \equiv \)
\[
u_{i,t}\left[ s_t^i(\hat{\theta}_{i,t}, \theta_{-i,t}), w_t^i(\hat{\theta}_{i,t}, \theta_{-i,t}), \tilde{\sigma}_{t,i}(\theta_t, s_t^i(\hat{\theta}_{i,t}, \theta_{-i,t}), \alpha^*) \right.
\]
\[
+ \delta \mathbb{E}\left[ W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid \theta_{i,t}, e_{i,t}, \sigma_{-i,t}^i(\hat{\theta}_{i,t}, \theta_{-i,t}) \right],
\]

Then showing (B.1) is equivalent to showing \( Z_{i,t}(\theta_t; \hat{\theta}_{i,t}; \alpha^*) \geq Z_{i,t}(\theta_t; \hat{\theta}_{i,t}; \alpha^*) \) for each \( \hat{\theta}_{i,t} \).

**Case 1.** Suppose \( \Delta \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) \geq 0 \) and \( \theta_{-0,t} = (\theta_L, \theta_L) \).

By construction of \( w_t^i \),
\[
Z_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) - Z_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) = \left[ \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) \right] - \left[ \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) \right] \leq 0 \text{ (by Lemma 6.1).}
\]

**Case 2.** Suppose \( \Delta \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) \geq 0 \) and \( \theta_{-0,t} = (\theta_L, \theta_H) \).

By construction of \( w_t^i \),
\[
Z_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_H; \alpha^*) - Z_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_H; \alpha^*) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) \left[ 1 - s_t^i(\theta_{0,t}, \theta_L, \theta_H) \right] = \left[ \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) \right] - \left[ \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) \right] \geq 0 \text{ (by Proposition 5.2).}
\]

**Case 3.** Suppose \( \Delta \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) \geq 0 \) and \( \theta_{-0,t} = (\theta_L, \theta_L) \).

By construction of \( w_t^i \),
\[
Z_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) - Z_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) \left[ 1 - s_t^i(\theta_{0,t}, \theta_L, \theta_H) \right] = \left[ \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) \right] - \left[ \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_H; \alpha^*) \right] = 0,
\]

and
By construction of \( w \), we have

\[
Z_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) - Z_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_L; \alpha^*)
= \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) - s^*_i(\theta_{0,t}, \theta_H, \theta_L)w^*_i(\theta_{0,t}, \theta_H, \theta_L) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_L; \alpha^*)
= \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_L; \alpha^*)
- [\tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)]
= [\tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) + \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)]
- [\tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_L; \theta_L; \alpha^*) + \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)].
\]

We have \( \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) + \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) + \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) \)

\[
= [\tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_L; \alpha^*)] + [\tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)]
= [\tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_L; \alpha^*)] + [\tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)].
\]

We have \( V_t(\theta_L, \theta_H; \alpha^*) = 0 \)

(since \( V_t(\theta_L, \theta_H; \alpha^*) \) is the aggregate payoff given by the constrained-efficient arrangement rule \( \alpha^* \) when agents have types \( (\theta_L, \theta_H) \) in period \( t \)).

**Case 4.** Suppose \( \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) < \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) \) and \( \theta_t = (\theta_L, \theta_L) \).

By construction of \( w^*_t \),

\[
Z_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - Z_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)
= \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)
- [\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_L; \alpha^*)]
= [\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)] + [\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_L; \alpha^*)]
\]

\[
> 0 \text{ (by Lemma 6.1).}
\]

**Case 5.** Suppose \( \tilde{W}_{i,t}(\theta_L, \theta_H; \alpha^*) < \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) \) and \( \theta_t = (\theta_H, \theta_H) \).

By construction of \( w^*_t \),

\[
Z_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - Z_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)
= \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) + s^*_i(\theta_L, \theta_H)w^*_i(\theta_L, \theta_H)
= [\tilde{W}_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)] + [\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_L; \alpha^*)]
= [\tilde{W}_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)] + [\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_L; \alpha^*)]
\]

\[
> 0 \text{ (by Lemma 5.2).}
\]

**Case 6.** Suppose \( \tilde{W}_{i,t}(\theta_L, \theta_H; \alpha^*) < \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) \) and \( \theta_t = (\theta_L, \theta_H) \).

By construction of \( w^*_t \),

\[
> 0 \text{ (by Lemma 5.3).}
\]
\[ Z_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - Z_1(t, \theta_L; \theta_L; \alpha^*) \]
\[ = \hat{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \bar{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) + s_{i,t}^e(\theta_L, \theta_H)W_{i,t}(\theta_H, \theta_L) \]
\[ = [\hat{W}_{i,t}^c(\theta_L, \theta_H; \alpha^*) - \bar{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)] + [\bar{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \bar{W}_{i,t}(\theta_L, \theta_H; \alpha^*)] \]
\[ = [\hat{W}_{i,t}^c(\theta_L, \theta_H; \alpha^*) - \bar{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)] - [\bar{W}_{i,t}(\theta_L, \theta_H; \alpha^*) - \bar{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)] = 0, \]
and
\[ Z_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*) - Z_1(t, \theta_L; \theta_L; \alpha^*) \]
\[ = \hat{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*) + [1 - s_{i,t}^e(\theta_H, \theta_L)]W_{i,t}(\theta_H, \theta_L) - \bar{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*) \]
\[ = \hat{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*) + [\bar{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*) - \bar{W}_{i,t}(\theta_H, \theta_L; \alpha^*)] - \bar{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*) \]
\[ = [\hat{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*)] + [\bar{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*) - \bar{W}_{i,t}(\theta_H, \theta_L; \alpha^*)] \]
\[ = [\hat{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*)] + [\bar{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*) + \bar{W}_{i,t}(\theta_H, \theta_L; \alpha^*)] \]
\[ = V^e(\theta_H, \theta_L; \theta_L; \alpha^*) - V^e(\theta_H, \theta_L; \alpha^*) = 0 \]
(since \( V^e(\theta_L, \theta_H; \alpha^*) \) is the aggregate payoff given by the constrained-efficient arrangement rule \( \alpha^* \) when agents have types \((\theta_L, \theta_H)\) in period \( t \)).

\section*{Lemma B.2}

The direct revelation mechanism \( D^* \) periodically ex-post Pareto dominates the equal division mechanism \( D^e \).

\section*{Proof.}

Suppose that the following holds for some \( t \in \{2, \ldots, T\} \): for each \( i \in N \), each \( \theta_t \in \Theta_t \), and each \((\theta_{0:t}, \tau > t)\), we have \( U_{i,t}[((\theta_{0:t}, \tau), \alpha^* | \theta_t] \geq U_{i,t}[(\theta_{0:t}, \tau)_{\tau > t}, \alpha^* | \theta_t] \).

\section*{Step 1.}

We show that for each \( i \in N \), each \( \theta_{t-1} \in \Theta_{t-1} \), and each \((\theta_{0:t-1}, \tau > t-1)\),
\[ U_{i,t-1}[(\theta_{0:t-1}, \tau > t-1, \alpha^* | \theta_{t-1}] \geq U_{i,t-1}[(\theta_{0:t-1}, \tau > t-1, \alpha^* | \theta_{t-1}] \]

\section*{Case 1.}

Suppose \( \theta_{t-1} = (\theta_L, \theta_H) \) and \( \hat{W}^e_{i,t-1}(\theta_{t-1}) \geq \hat{W}^e_{i,t-1}(\theta_{t-1}; \theta_L; \alpha^*) \). Let
\[ \theta^*_{0:t-1} = \gamma[\theta_{0:t-1} + \sum_{j \in N} \sigma^e_{i,t-1}^j(\theta_{t-1})] \text{ and } \theta^*_{0:t} = \gamma[\theta_{0:t-1} + \sum_{j \in N} \sigma^e_{i,t-1}^j(\theta_{t-1})].\]

Then
\[ U_{i,t-1}[(\theta_{0:t-1}, \tau > t-1, \alpha^* | \theta_{t-1}] - U_{i,t-1}[(\theta_{0:t-1}, \tau > t-1, \alpha^* | \theta_{t-1}] \]
\[ = u_{i,t-1}[\alpha^*_{t-1}(\theta_{t-1}), \theta_{0:t-1}, \theta_{i,t-1}] + \delta U_{i,t}[(\theta_{0:t-1}, \tau > t, \alpha^* | \theta_{0:t}, \theta_{0:t})] \]
\[ - u_{i,t-1}[\alpha^*_{t-1}(\theta_{t-1}), \theta_{0:t-1}, \theta_{i,t-1}] - \delta U_{i,t}[(\theta_{0:t-1}, \tau > t, \alpha^* | \theta_{0:t-1}, \theta_{0:t-1})] \]
\[ = u_{i,t-1}[\sigma^*_{t-1}(\theta_{t-1}), \sigma^*_{t-1}(\theta_{t-1}) \theta_{0:t-1}, \theta_{i,t-1}] + \delta U_{i,t}[(\theta_{0:t-1}, \tau > t, \alpha^* | \theta_{0:t-1}, \theta_{0:t-1})] \]
\[ - u_{i,t-1}[\sigma^*_{t-1}(\theta_{t-1}), \sigma^*_{t-1}(\theta_{t-1}) \theta_{0:t-1}, \theta_{i,t-1}] - \delta U_{i,t}[(\theta_{0:t-1}, \tau > t, \alpha^* | \theta_{0:t-1}, \theta_{0:t-1})] \]
\[ + \hat{W}^e_{i,t-1}(\theta_{t-1}) - \hat{W}^e_{i,t-1}(\theta_{t-1}; \theta_L; \alpha^*) \]
\[
\begin{align*}
&= u_{i,t-1}[s_{i,t-1}^*(\theta_{t-1}), \sigma_{i,t-1}^*(\theta_{t-1}), \theta_{0,t-1}, \theta_{i,t-1}] + \delta U_{i,t}[(\theta_{0,t}, \theta_{-0,t}, \alpha^*, \theta_{0,t}, \theta_{-0,t})] \\
&\quad + \delta t_{0,t} h_1(t) - \delta U_{i,t}[(\theta_{0,t}, \theta_{-0,t}, \alpha^*, \theta_{0,t}, \theta_{-0,t})] \\
&\quad + W_{i,t-1}(\theta_{t-1}) - W_{i,t-1}(\theta_{t-1}; \theta; \alpha^*)
\end{align*}
\]

C Two-Member Partnership

C.1 Finite Horizon

C.2 Infinite Horizon

An arrangement rule \( \alpha \equiv (s, w, \sigma) \) selects an arrangement \( \alpha_t(\theta_t) \in \mathcal{R}_t(\theta_t, \alpha) \) for each \( t \geq 1 \) and each \( \theta_t \in \Theta_t \), if and only if: for each \( i \in \mathcal{N} \), each \( t \geq 1 \), and each \( \theta_t \in \Theta_t \), the sequence \( \{\sigma_{i,\tau}\}_{\tau \geq t} \) solves

\[
\max_{\{\sigma_{i,\tau}\}_{\tau \geq t}} \mathbb{E}_{(\theta_{0,t}, \tau) > t} \left[ \sum_{\tau \geq t} \delta^{\tau-t} u_{i,\tau} [s_{\tau}(\theta_{\tau}), w_{\tau}(\theta_{\tau}), \sigma_{i,\tau}^*(\theta_{\tau}), \sigma_{-i,\tau}^*(\theta_{\tau}), \theta_{0,\tau}, \theta_{i,\tau}] \mid \theta_t \right]
\]

subject to \( \theta_{0,\tau+1} = \theta_{0,\tau} + \gamma \left[ \sigma_{i,\tau}^*(\theta_{\tau}) + \sum_{j \neq i} \sigma_{j,\tau}(\theta_{\tau}) \right] \) for each \( (\theta_{0,t}, \tau) > t \) and each \( \tau \geq t \)

\( \sigma_{i,\tau}^*(\theta_{\tau}) \geq 0 \) for each \( \tau \geq t \) and each \( \theta_{\tau} \in \Theta_t \).

The Lagrangean function

\[
\mathcal{L} = \mathbb{E}_{(\theta_{0,t}, \tau) > t} \left[ \sum_{\tau \geq t} \delta^{\tau-t} u_{i,\tau} [s_{\tau}(\theta_{\tau}), w_{\tau}(\theta_{\tau}), \sigma_{i,\tau}^*(\theta_{\tau}), \sigma_{-i,\tau}(\theta_{\tau}), \theta_{0,\tau}, \theta_{i,\tau}] \mid \theta_t \right]
\]

\[
+ \sum_{(\theta_{0,t}, \tau) > t} \sum_{\tau \geq t} \xi_{\tau+1} [(\theta_{0,t}, \tau) > t] \left( \theta_{0,\tau} + \gamma \left[ \sigma_{i,\tau}(\theta_{\tau}) + \sum_{j \neq i} \sigma_{j,\tau}(\theta_{\tau}) \right] - \theta_{0,\tau+1} \right)
\]

\[
+ \sum_{(\theta_{0,t}, \tau) > t} \sum_{\tau \geq t} \chi_{\tau+1} [(\theta_{0,t}, \tau) > t] \sigma_{i,\tau}(\theta_{\tau})
\]

Recall that

\[
u_{i,\tau} [s_{\tau}(\theta_{\tau}), w_{\tau}(\theta_{\tau}), \sigma_{i,\tau}(\theta_{\tau}), \sigma_{-i,\tau}(\theta_{\tau}), \theta_{0,\tau}, \theta_{i,\tau}] = s_{i,\tau}(\theta_{\tau}) \left[ \sigma_{i,\tau}(\theta_{\tau}) + \sum_{j \neq i} \sigma_{j,\tau}(\theta_{\tau}) - \sum_{j \neq i} w_{j,\tau}(\theta_{\tau}) \right] + [1 - s_{i,\tau}(\theta_{\tau})] w_{i,\tau}(\theta_{\tau}) - \frac{[\sigma_{i,\tau}(\theta_{\tau})]^2}{2 \theta_{0,\tau} \theta_{i,\tau}}.
\]

The Kuhn-Tucker conditions: for each \( \tau \geq t \) and each \( (\theta_{0,t}, \tau) > t \),

\[
p[(\theta_{0,t}, \tau) > t] \delta^{\tau-t} \left[ s_{i,\tau}(\theta_{\tau}) - \frac{[\sigma_{i,\tau}(\theta_{\tau})]^2}{2 \theta_{0,\tau} \theta_{i,\tau}} \right] + \gamma \xi_{\tau+1} [(\theta_{0,t}, \tau) > t] + \chi_{\tau+1} [(\theta_{0,t}, \tau) > t] = 0
\]

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Proof. Agent $\theta$'s expected payoff from reporting truthfully is
\[\rho(n^* - 1)\left[W_i(\theta_H, \theta'_{-i}) - W^e_i(\theta_H, \theta'_{-i})\right] \geq \rho(n^*)\left[W^e_i(\theta) - W_i(\theta)\right].\]

Proof. We have
\[
\begin{align*}
\rho(n^* - 1)\left[W_i(\theta_H, \theta'_{-i}) - W^e_i(\theta_H, \theta'_{-i})\right] - \rho(n^*)\left[W^e_i(\theta) - W_i(\theta)\right] &= \rho(n^* - 1)\left(W_i(\theta_H, \theta'_{-i}) - W^e_i(\theta_H, \theta'_{-i})\right) - \frac{n-n^*}{n}\left(W^e_i(\theta) - W_i(\theta)\right) \\
&= \frac{\rho(n^* - 1)}{n^*}\left(n^nW_i(\theta_H, \theta'_{-i}) - W^e_i(\theta_H, \theta'_{-i})\right) - (n-n^*)\left(W^e_i(\theta) - W_i(\theta)\right) \\
&= \frac{\rho(n^* - 1)}{n^*}\left[n^*W_i(\theta_H, \theta'_{-i}) + (n-n^*)W_i(\theta)\right] - \left[n^*W^e_i(\theta_H, \theta'_{-i}) + (n-n^*)W^e_i(\theta)\right] \geq 0.
\end{align*}
\]

We note that $n^*$ agents have type $\theta_H$ and $n-n^*$ agents have type $\theta_L$ in the type profile $\theta$. The inequality comes from the fact that $[n^*W_i(\theta_H, \theta'_{-i}) + (n-n^*)W_i(\theta)]$ is the aggregate payoff when the constrained-efficient equity-effort rule $(s^*, \sigma^*)$ is used, whereas $[n^*W^e_i(\theta_H, \theta'_{-i}) + (n-n^*)W^e_i(\theta)]$ is the aggregate payoff when the equal sharing rule is used.

The following lemma claims that an agent who has type $\theta_L$ always prefers to report truthfully in expectation.

Lemma D.2 For each $i \in N$, we have $E_{\theta_{-i}}\bar{W}_i(\theta_L, \theta_{-i}; \theta_L) \geq E_{\theta_{-i}}\bar{W}_i(\theta_L, \theta_{-i}; \theta_H)$.

Proof. Agent $i$'s expected payoff from reporting truthfully is
\[
E_{\theta_{-i}}\bar{W}_i(\theta_L, \theta_{-i}; \theta_L) = \sum_{n'=0}^{n-1} \rho(n')\mu(\theta_L)^{n-n'-1}\mu(\theta_H)^{n'}\left[W_i(\theta_L, \xi_{-i}(n')) + (1 - (n-n')s_i[\theta_L; \xi_{-i}(n')])w_i[\theta_L, \xi_{-i}(n')]\right].
\]
Agent $i$’s expected payoff from reporting $\theta_H$ is

$$\mathbb{E}_{\theta_i} \tilde{W}_i(\theta_L, \theta_{i-1}; \theta_H) = \sum_{n' = 0}^{n-1} \rho(n') \mu(\theta_L)^{n-n'-1} \mu(\theta_H)^n \left[ \tilde{W}_i[\theta_L, \xi_{-i}(n') ; \theta_H] - (n-n'-1) s_i^* [\theta_H, \xi_{-i}(n')] w_i^* [\theta_L, \xi_{-i}(n'+1)] \right] .$$

For each $n^* \in \{1, \ldots, n-1\}$, let

$$f(n^*) = \sum_{n' = 0}^{n^*} \rho(n') \mu(\theta_L)^{n-n'-1} \mu(\theta_H)^n \left[ W_i[\theta_L, \xi_{-i}(n')] - \tilde{W}_i[\theta_L, \xi_{-i}(n') ; \theta_H] \right]$$

$$+ \left( 1 - (n-n') s_i^* [\theta_L, \xi_{-i}(n')] \right) w_i^* [\theta_L, \xi_{-i}(n')] + (n-n'-1) s_i^* [\theta_H, \xi_{-i}(n')] w_i^* [\theta_L, \xi_{-i}(n'+1)] .$$

Then $\mathbb{E}_{\theta_i} \tilde{W}_i(\theta_L, \theta_{i-1}; \theta_H) - \mathbb{E}_{\theta_i} W_i(\theta_L, \theta_{i-1}; \theta_H) = f(n-1)$. We show $f(n-1) \geq 0$ by induction.

**Step 1.** We have

$$f(1) = \mu(\theta_L)^{n-2} \left\{ (n-1) \mu(\theta_L) + (n-2) \mu(\theta_H) \left[ W_i[\theta_L, \xi_{-i}(0)] - \tilde{W}_i[\theta_L, \xi_{-i}(0) ; \theta_H] \right] \right. $$

$$+ \left. \left[ \rho(1) \mu(\theta_L)^{n-2} \mu(\theta_H) \left[ W_i[\theta_L, \xi_{-i}(1)] - \tilde{W}_i[\theta_L, \xi_{-i}(1) ; \theta_H] \right] \right. $$

$$\left. + s_i^* [\theta_H, \xi_{-i}(0)] w_i^* [\theta_L, \xi_{-i}(1)] + (n-2) s_i^* [\theta_H, \xi_{-i}(1)] w_i^* [\theta_L, \xi_{-i}(2)] \right\}$$

$$= \mu(\theta_L)^{n-2} \left\{ (n-1) \mu(\theta_L) + \rho(1) \mu(\theta_H) \left[ W_i[\theta_L, \xi_{-i}(0)] - \tilde{W}_i[\theta_L, \xi_{-i}(0) ; \theta_H] \right] \right. $$

$$\left. - \rho(1) \mu(\theta_H) \left[ W_i[\theta_L, \xi_{-i}(1)] - \tilde{W}_i[\theta_L, \xi_{-i}(1) ; \theta_H] \right] \right\}$$

$$+ \rho(1) \mu(\theta_L)^{n-2} \mu(\theta_H) \left\{ (n-2) s_i^* [\theta_H, \xi_{-i}(1)] w_i^* [\theta_L, \xi_{-i}(2)] - \left( W_i[\theta_L, \xi_{-i}(1); \theta_H] - W_i^*[\theta_L, \xi_{-i}(1)] \right) \right\} .$$

**Step 2.** Suppose $n \geq 3$. Fix $n^* \in \{1, \ldots, n-2\}$.

Let $g(n^*) = \sum_{n' = 0}^{n^*-1} \mu(\theta_L)^{n-n'-2} \mu(\theta_H)^n \left\{ \rho(n')(n-n'-1) \mu(\theta_L) + \rho(n'+1)(n'+1) \mu(\theta_H) \right\}$

$$s_i^* [\theta_H, \xi_{-i}(n')] w_i^* [\theta_L, \xi_{-i}(n'+1)]$$

$$- \rho(n') \mu(\theta_H) \left( \tilde{W}_i[\theta_L, \xi_{-i}(n') ; \theta_H] - W_i^*[\theta_L, \xi_{-i}(n')] \right)$$

$$- \rho(n'+1) \mu(\theta_H) \left( W_i^*[\theta_L, \xi_{-i}(n'+1)] - W_i[\theta_L, \xi_{-i}(n'+1)] \right) .$$

Suppose $f(n^*) = g(n^*) + \rho(n^*) \mu(\theta_L)^{n-n'-1} \mu(\theta_H)^n \left\{ (n-n'-1) s_i^* [\theta_H, \xi_{-i}(n')] w_i^* [\theta_L, \xi_{-i}(n'+1)] \right.$

$$- \left( \tilde{W}_i[\theta_L, \xi_{-i}(n'); \theta_H] - W_i^*[\theta_L, \xi_{-i}(n'+1)] \right) \right\} .$$

We have

$$f(n^* + 1) = g(n^*) + \rho(n^*) \mu(\theta_L)^{n-n'-1} \mu(\theta_H)^n \left\{ (n-n'-1) s_i^* [\theta_H, \xi_{-i}(n')] w_i^* [\theta_L, \xi_{-i}(n'+1)] \right.$$

$$- \left( \tilde{W}_i[\theta_L, \xi_{-i}(n'); \theta_H] - W_i^*[\theta_L, \xi_{-i}(n'+1)] \right) \right\}$$

$$+ \rho(n^* + 1) \mu(\theta_L)^{n-n'-2} \mu(\theta_H)^{n+1} \left[ W_i[\theta_L, \xi_{-i}(n+1)] - W_i[\theta_L, \xi_{-i}(n+1); \theta_H] \right]$$

$$+ \left( 1 - (n-n'-1) s_i^* [\theta_L, \xi_{-i}(n+1)] \right) w_i^* [\theta_L, \xi_{-i}(n+1)] + (n-n'-2) s_i^* [\theta_H, \xi_{-i}(n+1)] w_i^* [\theta_L, \xi_{-i}(n+2)] \right\} .$$
\[ g(n^* + 1) + \rho(n^* + 1)\mu(\theta_L) n^*-2\mu(\theta_H) n^*+1 \left\{ (n - n^* - 2)s^*_i [\theta_H, \xi_{i-1}(n^* + 1)] w^*_i [\theta_L, \xi_{i-1}(n^* + 2)] \right. \\
- \left. \left( \tilde{W}_i [\theta_L, \xi_{i-1}(n^* + 1); \theta_H] - W^*_i [\theta_L, \xi_{i-1}(n^* + 1)] \right) \right\}. \]

**Step 3.** Note that \( \tilde{W}_i [\theta_L, \xi_{i-1}(n-1); \theta_H] = W^*_i [\theta_L, \xi_{i-1}(n-1)] \). It follows from Steps 1 and 2 that \( f(n - 1) = g(n - 1) \). By construction of \( w^* \), we have \( f(n - 1) \geq 0 \).

The following lemma claims that an agent who has type \( \theta_H \) always prefers to report truthfully in expectation.

**Lemma D.3** For each \( i \in N \), we have \( \mathbb{E}_{\theta_{-i}} \tilde{W}_i (\theta_H, \theta_{-i}; \theta_H) \geq \mathbb{E}_{\theta_{-i}} \tilde{W}_i (\theta_H, \theta_{-i}; \theta_L) \).

**Proof.** Agent \( i \)'s expected payoff from reporting truthfully is
\[
\mathbb{E}_{\theta_{-i}} \tilde{W}_i (\theta_H, \theta_{-i}; \theta_H) = \\
\sum_{n' = 0}^{n^* - 1} \rho(n') \mu(\theta_L) n'^{-1} \mu(\theta_H) n' \left[ W_i [\theta_H, \xi_{i-1}(n')] - (n - n' - 1)s^*_i [\theta_H, \xi_{i-1}(n')] w^*_i [\theta_L, \xi_{i-1}(n' + 1)] \right].
\]

Agent \( i \)'s expected payoff from reporting \( \theta_L \) is
\[
\mathbb{E}_{\theta_{-i}} \tilde{W}_i (\theta_H, \theta_{-i}; \theta_L) = \\
\sum_{n' = 0}^{n^* - 1} \rho(n') \mu(\theta_L) n'^{-1} \mu(\theta_H) n' \left[ \tilde{W}_i [\theta_H, \xi_{i-1}(n') + \theta_L] + \left( 1 - (n - n')s^*_i [\theta_L, \xi_{i-1}(n')] \right) w^*_i [\theta_L, \xi_{i-1}(n')] \right].
\]

For each \( n^* \in \{1, \ldots, n - 1\} \), let
\[
f(n) = \sum_{n' = 0}^{n^*} \rho(n') \mu(\theta_L) n'^{-1} \mu(\theta_H) n' \left[ W_i [\theta_H, \xi_{i-1}(n')] - \tilde{W}_i [\theta_H, \xi_{i-1}(n')] + \left( 1 - (n - n')s^*_i [\theta_L, \xi_{i-1}(n')] \right) w^*_i [\theta_L, \xi_{i-1}(n')] \right].
\]

Then \( \mathbb{E}_{\theta_{-i}} \tilde{W}_i (\theta_H, \theta_{-i}; \theta_H) - \mathbb{E}_{\theta_{-i}} \tilde{W}_i (\theta_H, \theta_{-i}; \theta_L) = f(n - 1) \). We show \( f(n - 1) \geq 0 \) by induction.

**Step 1.** We have
\[
f(1) = \mu(\theta_L)^n \left\{ W_i [\theta_H, \xi_{i-1}(0)] - \tilde{W}_i [\theta_H, \xi_{i-1}(0); \theta_L] + \rho(1) \mu(\theta_L)^{-2} \mu(\theta_H) \left( W_i [\theta_H, \xi_{i-1}(1)] - \tilde{W}_i [\theta_H, \xi_{i-1}(1); \theta_L] \right) \\
- s^*_i [\theta_H, \xi_{i-1}(0)] w^*_i [\theta_L, \xi_{i-1}(1)] - (n - 2)s^*_i [\theta_H, \xi_{i-1}(1)] w^*_i [\theta_L, \xi_{i-1}(2)] \right\} \\
= \mu(\theta_L)^n \left\{ - [(n - 1)\mu(\theta_L) + \rho(1) \mu(\theta_H)] s^*_i [\theta_H, \xi_{i-1}(0)] w^*_i [\theta_L, \xi_{i-1}(1)] \\
+ \rho(1) \mu(\theta_L)^{-2} \mu(\theta_H) \left( W_i [\theta_H, \xi_{i-1}(0)] - W^*_i [\theta_H, \xi_{i-1}(0)] \right) \right\} \right\} \\
= \mu(\theta_L)^n \left\{ - [(n - 1)\mu(\theta_L) + \rho(1) \mu(\theta_H)] s^*_i [\theta_H, \xi_{i-1}(0)] w^*_i [\theta_L, \xi_{i-1}(1)] \\
+ \rho(1) \mu(\theta_L)^{-2} \mu(\theta_H) \left( W_i [\theta_H, \xi_{i-1}(0)] - W^*_i [\theta_H, \xi_{i-1}(0)] \right) \right\} \right\}.
\]

\[\text{When } n' = n - 1, \xi_{i-1}(n') \text{ is undefined. However, for notational convenience, let } w^*_i [\theta_L, \xi_{i-1}(n)] = 0.\]
Step 2. Suppose \( n \geq 3 \). Fix \( n^* \in \{1, \ldots, n-2\} \).

Let \( g(n^*) \equiv \sum_{n'=0}^{n^*-1} \mu(\theta_L)^{n-n'-2} \mu(\theta_H)^{n'} \left\{ -[\rho(n')(n-n'-1)\mu(\theta_L)+\rho(n'+1)(n'-1)\mu(\theta_H)]s_i^* \left[ \theta_H, \xi_{i-n}(n') \right] w_i^* \left[ \theta_L, \xi_{i-n}(n'+1) \right] +\rho(n')\mu(\theta_L) \left( W_i [\theta_H, \xi_{i-n}(n')] - W_i^* [\theta_H, \xi_{i-n}(n')] \right) +\rho(n'+1)\mu(\theta_H) \left( W_i^* [\theta_H, \xi_{i-n}(n'+1)] - W_i [\theta_H, \xi_{i-n}(n'+1); \theta_L] \right) \right\} \).

Suppose \( f(n^*) = g(n^*) + \rho(n^*)\mu(\theta_L)\mu(\theta_H)^{n-n^*-1} \mu(\theta_H)^{n^*} \left\{ -(n-n^*-1)s_i^* \left[ \theta_H, \xi_{i-n}(n^*) \right] w_i^* \left[ \theta_L, \xi_{i-n}(n^*+1) \right] +W_i [\theta_H, \xi_{i-n}(n^*)] - W_i^* [\theta_H, \xi_{i-n}(n^*)] \right\} +\rho(n^*+1)\mu(\theta_L)\mu(\theta_H)^{n-n^*-2} \mu(\theta_H)^{n^*+1} \left\{ W_i [\theta_H, \xi_{i-n}(n^*+1)] - W_i^* [\theta_H, \xi_{i-n}(n^*+1); \theta_L] - \left( (n-n^*+2)s_i^* \left[ \theta_H, \xi_{i-n}(n^*+1) \right] w_i^* \left[ \theta_L, \xi_{i-n}(n^*+2) \right] - (n^*+1)s_i^* \left[ \theta_H, \xi_{i-n}(n^*) \right] w_i^* \left[ \theta_L, \xi_{i-n}(n^*+1) \right] \right\} \equiv g(n^*+1) + \rho(n^*+1)\mu(\theta_L)\mu(\theta_H)^{n-n^*-2} \mu(\theta_H)^{n^*+1} \left\{ -(n-n^*-2)s_i^* \left[ \theta_H, \xi_{i-n}(n^*+1) \right] w_i^* \left[ \theta_L, \xi_{i-n}(n^*+2) \right] +W_i [\theta_H, \xi_{i-n}(n^*+1)] - W_i^* [\theta_H, \xi_{i-n}(n^*+1)] \right\} \). 

Step 3. Note that \( W_i [\theta_H, \xi_{i-n}(n^*+1)] = W_i^* [\theta_H, \xi_{i-n}(n^*+1)] \). It follows from Steps 1 and 2 that \( f(n-1) = g(n-1) \). By construction of \( w^* \), we have \( f(n-1) \geq 0 \). 

E. A Voting Mechanism

For each \( i \in N \), let \( A_i^* = \begin{bmatrix} a_{i1}^* & a_{i2}^* & \cdots & a_{iL}^* \\ a_{i1}^* & a_{i2}^* & \cdots & a_{iL}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{iK1}^* & a_{iK2}^* & \cdots & a_{iKL}^* \end{bmatrix} \).

Lemma E.1 For each \( i \in N \), all entries of \( A_i^* \) are distinct and \( \tilde{A}_i^* = \tilde{A}_j^* \) for each \( j \neq i \).

Proof.

Step 1. We show that all entries of \( A_i^* \) are distinct. Take two entries \( a_{kl}^{is} \) and \( a_{kl'}^{is} \). By construction, \( a_{kl}^{is} \equiv \alpha^s(\theta) \) and \( a_{kl'}^{is} \equiv \alpha^s(\theta') \) for some \( \theta \neq \theta' \). Let \( \theta_j \in N \) such that \( \theta_j \neq \theta_j' \). Without loss of generality, assume \( \theta_j > \theta_j' \). If \( s^s(\theta) \neq s^s(\theta') \), then it is clear
that $\alpha^*(\theta) \neq \alpha^*(\theta')$. If $s^*(\theta) = s^*(\theta')$, then $\sigma^*_j(\theta) < \sigma^*_j(\theta')$ (by Lemma 8.3), which implies $\alpha^*(\theta) \neq \alpha^*(\theta')$.

**Step 2.** By construction, $\tilde{A}_i^* = \{\alpha^*(\theta) \mid \theta \in \Theta^n\}$ for each $i \in N$, which implies $\tilde{A}_i^* = \tilde{A}_j^*$ for $i \neq j$.

**Proof for Theorem 8.1.**

We construct a Nash equilibrium $\lambda^*$ of voting mechanism $\mathcal{V}^*$ as follows. For each $i \in N$ and each $\theta_i \in \Theta_i$,

- $\lambda_i^*(\theta_i)[\alpha^*(\theta_i, \theta_{-i})] = 1$ for each $\theta_{-i} \in \Theta_{-i}$,

- $\lambda_i^*(\theta_i, \nu) \in \arg\max_{\nu \in \mathcal{V}} v_i[\xi^\pi(\nu), e_i, \xi^\nu_i(\nu), \theta_i]$ for each $\nu \in \mathcal{V}$.

**Step 1.** Fix $\theta \in \Theta$. We show that $\xi^*\lambda_i^*(\theta_1), \ldots, \lambda_i^*(\theta_n) = \alpha^*(\theta)$. By construction, $\tilde{A}^* = \{\alpha^*(\theta') \mid \theta' \in \Theta^n\}$ and

$$\sum_{i \in N} \lambda_i^*(\theta_i)[\alpha^*(\theta')] = |\{i \in N \mid \theta_i = \theta_i'\}|.$$

It follows that $\alpha^*(\theta) = \arg\max_{a \in \tilde{A}^*} \sum_{i \in N} \lambda_i^*(\theta_i)[a]$.

**Step 2.** We show that $\lambda^*$ is a Nash equilibrium of $\mathcal{V}^*$. Let $\theta \in \Theta$ be true type profile. Fix $i \in N$.

Let $\nu^* = [\lambda_1^*(\theta_1), \ldots, \lambda_n^*(\theta_n)]$. By definition, agent $i$’s payoff given strategy profile $\lambda^*$ is

$$\phi_i(\lambda^*, \theta) = \varphi_i[\nu^*, (\lambda_j^*(\theta_j, \nu^*))_{j \in N}, \theta_i] = u_i[\xi^\pi(\nu^*), (\lambda_j^*(\theta_j, \nu^*))_{j \in N}, \theta_i].$$

By Step 1, $\xi^\pi(\nu^*) = [s^*(\theta), w^*(\theta)]$. By construction, for each $j \in N$,

$$\lambda_j^*(\theta_j, \nu^*) \in \arg\max_{e_j \in \mathbb{R}_+} u_j[s^*(\theta), w^*(\theta), e_j, \sigma^*_j(\theta), \theta_j].$$

By definition, $\lambda_j^*(\theta_j, \nu^*) = \sigma_j^*(\theta)$. It follows that

$$\phi_i(\lambda^*, \theta) = u_i[s^*(\theta), w^*(\theta), \sigma^*(\theta), \theta_i].$$

Suppose agent $i$ chooses some strategy $\lambda'_i \in \Lambda_i$ while other agents choose $\lambda^*_{-i}$. Let $\nu' = [\lambda'_i(\theta_i), (\lambda_j^*(\theta_j))_{j \neq i}]$. By construction, for each $\theta'_i \in \Theta$,

$$\sum_{j \in N} \nu'_j[\alpha^*(\theta'_j, \theta_{-i})] = n - 1 + \lambda'_i(\theta_i)[\alpha^*(\theta'_i, \theta_{-i})].$$
Since $\lambda_i'(\theta_i)[\alpha^*(\theta'_i, \theta_{-i})] \in \{0, 1\}$ for each $\theta'_i \in \Theta$ and $\sum_{\theta'_i \in \Theta} \lambda_i'(\theta_i)[\alpha^*(\theta'_i, \theta_{-i})] = 1$, there is a unique $a \in \{\alpha^*(\theta'_i, \theta_{-i}) \mid \theta'_i \in \Theta\}$ such that $\sum_{j \in N} \nu'_j(a) = n$. For each $\theta' \in \Theta^n$ such that $\theta'_{-i} \neq \theta_{-i}$,

$$\sum_{j \in N} \nu'_j[\alpha^*(\theta')] = |\{j \in N \setminus \{i\} \mid \theta_j = \theta'_j\}| + \lambda_i'(\theta_i)[\alpha^*(\theta')] \leq n - 1.$$ 

Hence, $\xi^*(\nu') = \alpha^*(\theta'_i, \theta_{-i})$ for some $\theta'_i \in \Theta$. By construction of $\lambda^*$, for each $j \in N \setminus \{i\}$,

$$\lambda_j^*(\theta_j, \nu') \in \arg \max_{e_j \in \mathbb{R}_+} u_j[s^*(\theta'_i, \theta_{-i}), w^*(\theta'_i, \theta_{-i}), e_j, \sigma^*_j(\theta'_i, \theta_{-i}), \theta_j].$$

By definition of $\sigma^*$, $\lambda_j^*(\theta_j, \nu') = \sigma_j^*(\theta'_i, \theta_{-i})$. It follows that

$$\phi_i(\lambda_i', \lambda_{-i}^*, \theta) = \phi_i[\nu', \lambda'_i(\theta_i, \nu'), (\lambda_j^*(\theta_j, \nu'))_{j \neq i, \theta_i}] = u_i[s^*(\theta'_i, \theta_{-i}), w^*(\theta'_i, \theta_{-i}), \lambda_i(\theta_i, \nu'), \sigma^*_j(\theta'_i, \theta_{-i}), \theta_i].$$

By ex post incentive compatibility, for each $\theta_i \in \Theta$, we have $\phi_i(\lambda^*, \theta) \geq \phi_i(\lambda'_i, \lambda_{-i}^*, \theta)$.  

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