Network Formation and Information Acquisition*

Bernard Herskovic† João Ramos‡
NYU NYU

December 15, 2014

Abstract

We develop a social network formation model in which agents form links in order to acquire information. Agents acquire information personally and may obtain additional information by connecting to other agents. The relative position of an agent in the social network is relevant to form expectations about actions of other players. In our setting, information is not perfectly substitutable and the information of an "opinion maker", an agent that has more connections, is more informative about the average action than the information from other players. We show that, when players are choosing their connections, (i) it is always preferable to connect to opinion makers, and (ii) opinion makers have less incentives to form links. These two results alone are sufficient to characterize the endogenous shape of the network. Any strict equilibrium of the network formation game generates a hierarchical network structure.

Keywords: Network Formation, Information Acquisition, Coordination.

JEL Codes: D83, D85.

*First draft: May 15th, 2013. VERY PRELIMINARY. We would like to thank David Pearce, Alberto Bisin, Boyan Jovanovic, Laurent Mathevet, Chris Woolnough and participants of NYU Micro Lunch Workshop and NRET. Updates: `https://files.nyu.edu/bh922/public/NetworkInfo_HerskovicRamos.pdf`.

†Email:herskovic@nyu.edu.
‡Email:ramos@nyu.edu.

1 Introduction

Social networks are embedded in our social and economic lives. Individual decisions concerning career choices, for whom to vote, whether to buy a new product or whether or not to commit a crime are often influenced by friends and acquaintances. In particular, the structure of social and economic networks determines the transmission of information such as job opportunities, and accordingly plays a central role in the likelihood of succeeding professionally. Individuals with different positions in the network can experience very different economic outcomes: a more connected person learns about more job openings through her friends and contacts, placing her in a prime position to obtain a better offer. However, more than a person's relative position in the network matters for economic outcomes. The whole architecture of the network may have implications for the societal welfare. The goal of this paper is to understand how social networks are formed and what type of network is observed in equilibrium. We focus on how people choose their connections in order to acquire the necessary information to make decisions. We develop a network formation game in which agents connect to each other in order to acquire information, and we show that, under general conditions, any strict equilibrium of the network formation game generates a hierarchical network structure. Furthermore, we discuss the implications of such type of structures to welfare.

A hierarchical network is characterized by the existence of different tiers of informational importance in the society. An individual that belong to the top tier is very influential, as her information is observed by members of all other tiers. A second tier individual's information is observed by all members of tiers below her. This category of network architectures is more general than core-periphery, which has been documented by the literature on network formation and information acquisition (Bala and Goyal 2000, Galeotti and Goyal 2010). A hierarchical network allows the existence of multiple social groups, ranked by the degree of which they may influence economic outcomes, as opposed to only two social groups, namely core and periphery.

We develop a two-stage game, in which there is an unknown state of the world and each agent receive an equally informative independent signal of it. Each agents is a source of information that can be tapped via costly social connections. In the first stage players form links (and a resulting network) with the intent of acquiring information. In the second stage players choose an action. In the network formation stage, each player may unilaterally decide to pay a cost and acquire information from another agent by unilaterally forming a connection. We make two assumptions on how the information propagates along the network linkages. First, informational flow is one-way: a player connecting to another agent acquire her information, but the reverse is not true. The agent being connected to does not acquire the information from the player forming the connection, unless he forms another linkage himself. Second, the information acquired by a player through the network is limited to her immediate connections. A player does not acquire information from other agents' connections. Our network formation model is a non-cooperative game, and a strategy consists of choosing whom a player connects to. The links formed by all agents collectively define the network.

In the second stage, after forming their connections, players use the information obtained through his social network to choose an action.[1] A player wants to be as close as possible both to the unknown state of the world and to the average action of the economy.

The introduction of this beauty contest element to the payoff generates complementarity and has two immediate effects on the incentives in the network formation game. First, information is not perfectly substitutable. All players have the same precision in their private information; thus, to learn about the state of the world, it is irrelevant from which agent one acquires information. However, this is not true to predict the average action. Some agents' private information may be more informative than others, depending on their position in the network. These agents are *opinion makers* in that their information have more impact on the average action than other players. The second implication of our payoff structure is that there is strategic complementarity between the connection decisions of the agents.[2] When a player connects to an opinion maker, she reinforces the opinion maker's role in the network. Her action is influenced by the opinion maker's information which increases the opinion maker's impact on the average action.

We prove two key monotonicity results that drive the decisions of forming connections. The first one states that a player would rather connect to an agent who is observed by more players, an opinion maker, than to an agent who is observed by fewer players. We interpret this result as a consequence of the fact that an opinion maker's private information is more informative about the average action of the economy than the information from other players. The second monotonicity result states that an opinion maker has less incentives to form links. The intuition is that an opinion maker's own information is more informative about the average action, which disincentivizes the acquisition of additional information.

Our main result is to characterize the network structure common to all strict Nash equilibria.[3] We show that all strict Nash equilibria in this model have a hierarchical network

[1]In the formal description of the game, this second stage refers to periods 2 and 3. After the network is formed, the information is revealed in period 2, that is, players' signal about the state of the world is realized. Agents choose an action in period 3 and then collect their payoffs.

[2]Bulow, Geanakoplos, and Klemperer (1985)

[3]To focus on strict equilibria is quite standard on the literature, see Bala and Goyal (2000) and Galeotti

and this result rely solely on our two monotonicity results. A network is said to be hierarchical if the set of agents can be partitioned in L subsets, which are ranked in levels or tiers. Agents in a lower level connect to all agents of the upper levels, and players in the same group either connect to each other or they do not connect to one another at all. Notice that agents in the lower ranked level sponsor the importance of those in an upper level, by bearing the cost of linkage formation.

In addition, we investigate the implications of such network structure for efficiency and fluctuations of the average action. First, we discuss the aggregation of idiosyncratic shocks and show that independent idiosyncratic shocks do generate aggregate fluctuations. Second, we present examples in which the equilibrium is not efficient. Not only one equilibrium can be Pareto dominated by another equilibrium, but also even the best equilibrium may be Pareto dominated by non-equilibrium configurations and outcomes.

On a technical level, it is worth noting that the proof for the main result is quite different than usual proofs in the literature. This comes as a result of the information structure specified; the fact that a link provides a unilateral informational flow and requires only a unilateral decision, vastly simplifies the decision problem. Also, the validity of our main result extends far beyond the set-up presented, given that the proof depends only on the two monotonicity results.

The rest of the paper is organized as follows. In the next pages we discuss the related literature. In the next section, we present and solve the model. Section 3 has our main result, we characterize hierarchical networks in equilibrium and discuss implications of such architectures for efficiency and aggregation. The last section concludes.

Related Literature More than anything else, our paper is a contribution to the theory of network formation and information acquisition. In particular to the theory of one-sided link formation. Bala and Goyal (2000) initiated this literature by proposing a simple connections model. Each player gain from observing (connecting to) other players. A player, a, can observe another player, b, and by doing so observes as well all other players observed by b. Our result reinforces Bala and Goyal's first result, that networks are either connected or empty. However, our result is in strike opposition to other results in the paper. First, Bala and Goyal characterize equilibrium networks in a model where information has no decay as it‘s being passed from individual to individual, and show that the only strict equilibria are the wheel [4] network or the empty one. In our model, the wheel is never an equilibrium. Second, Bala and Goyal conclude that the equilibria network is efficient. We show that not

and Goyal (2010)

[4] A wheel network, as it‘s name indicates, is a network in which each individual observes only one other player. Player 1 observes player 2, who observes 3,..., who observes player n who observes player 1.

only the equilibria networks can be Pareto ranked, the best one may not be Pareto efficient.

Our paper is closer to Galeotti and Goyal (2010). The authors provide a model of information acquisition through social networks that aims to explain the "law of the few": an empirical observation that most individuals acquire their information from a small subset of people. Their results are a consequence of two characteristics of their model: (i) information sources are perfectly substitutable, and (ii) it is possible to costly form links with players that acquire information. The authors characterize core-periphery networks [5] as an equilibrium outcome. In some sense we generalize Galeotti and Goyal (2010) result, since a core-periphery network is a particular type of hierarchical architecture. Our model, however, differs from theirs in a couple of very important dimension. First, we consider a unilateral flow of information, that is when agent 1 is looking at agent 2 in our model, agent 2 does not obtain any information from agent 1. Second, we ignore personal acquisition of information. Third, and more importantly, in our model the complementarity in payoff space makes information sources to be non-substitutable. While Galleoti and Goyal result is a direct consequence of the concavity in payoffs with respect to information and linearity of costs of acquiring it, our model's result comes from such complementarity.

Our paper is related to the literature that combines both network formation and strategic interaction, for instance Calvó-Armengol and Zenou (2004), Goyal and Vega-Redondo (2005), and Königa, Tessoneb, and Zenouc (2014). Closer to our results, Königa, Tessoneb, and Zenouc (2014) constructs a dynamic network formation model, aiming to explain nestedness: the neighborhood of a node is contained in the neighborhood of a node with higher degree. The concept of nestedness in network connections can be understood as a definition of hierarchy of nodes—a more connected node has more importance since it is connected to everyone a less connected node is. In their paper, the hierarchy is a result of both strategic link formation and exogenous link destruction: links connecting central players have a lower probability of being broken. In contrast, our result is a consequence of strictly strategic choices of the players.

Another strand of literature our paper contributes to is the literature on granular networks. In our model, idiosyncratic shocks (i.e. the privately public signal) don't aggregate in equilibrium and how certain agents perceive the public signal matters. Specifically, some agents will have more influence on the equilibrium outcome than others. This is related to the literature of origins of aggregate fluctuations (Gabaix 2011, Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi 2012), but we presented it in a game-theoretic perspective. Typically, in

[5] A core-periphery network is a network in which the set of players can be partitioned in two, the core and the periphery. All members of the periphery are connected to all members of the core. All members of the core are connected to each other and members of the periphery are not connected to each other.

this literature, the sparsity of the network is capable of generating aggregate fluctuations from idiosyncratic shocks. That is true in our paper as well, however, in our paper, the network sparsity is endogenous as agents choose to form their connections. This result extends Jovanovic (1987) classical paper on complementarities and aggregate fluctuations to an endogenous network formation model. Jovanovic (1987) shows that even with independent individual shocks aggregate risk can be generated in games with payoff complementarities. Notice however that our result is not a direct implication of his, since the network formation process could in principle eliminate such complementarity.

2 Model

The economy is populated by n agents, let $N = \{1, 2, ..., n\}$ be the set of players. At the beginning of the game a state of the world is drawn by nature,

$$\theta \sim N(0, \sigma_e^2).$$

Each agent receives a private signal of it.

This is a 3-period model. In the first period, agents simultaneously decide to form unilateral connections (whom to observe). The benefit of connecting to an agent is that if agent i connects to agent j, agent i is able to observe $j's$ private signal. However making connections is costly and the cost is carried by the linking part of the connection. We make the assumption that the cost depends only on the number of links. Agents do not observe the resulting network.

In the second period, given the true state of the world, private signals are realized. Thus, the signal[6]

$$e_i \sim N(\theta, \sigma_e^2),$$

is observed by the i^{th} agent in the model as well as by any other agent that chose to observe agent i in the first period.

In the final period, agents choose an action, $a_i \in \mathbb{R}$, in order to minimize a linear combination of the distance between her action and the state of the world, and the distance between her action and the average action of the economy. To insure that both concerns are present, we parametrize $r \in (0, 1)$. The payoff function for a given action is given by:

$$\Pi(G, \sigma_i, a_i) = -\left[r\left(a_i - \theta\right)^2 + (1 - r)\left(a_i - \overline{a}\right)^2\right] - c(K_i)$$

[6] Observe that both the state of the nature and the signal have the same standard error in their respective normal distributions. This has no severe implications but simplifies the notation used.

where K_i is the number of connections agent i decided to form, while $c(K_i)$ is the cost of forming K_i connections and $\overline{a} = \frac{\sum_{i=1}^{n} a_i}{n}$ is the average action. The assumptions of normality of the signal structure and quadratic payoffs are standard in the organizational economics literature[7].

2.1 Solving the Model

We proceed to solve the model by backward induction. The game is organized in three stages. In the first agents choose the links they want to make. A strategy here is a vector $g_i = (g_{i,1}, g_{i,2}, ..., g_{i,n})$, where $g_{i,j} \in 0, 1$ for each $j \in N, \quad j \neq i$ [8]. Then the shocks are realized and they observe the private signal of each member of their own network. Finally, agents choose an action. We start by solving for the optimal action, given the informational set of an agent. Agents cannot search for information in any other way other than through their network. That is, agents acquire information about the state of the world only by checking with the members of their social network what signal they have received. We say an agent i is looking at an agent j if and only if $g_{i,j} = 1$. That implies that the information set agent i has can be described by $\mathbb{I}_i = \{e_j\}_{j:g_{ij}=1}$

2.1.1 Optimal Action

The agent chooses an action in order to minimize the linear combination of the distances between (i) her action and the average action and (ii) her action and the true state of the world.

Agent i's problem

$$\max_{a_i} -\mathbb{E}\left[r\left(a_i - \theta\right)^2 + (1-r)\left(a_i - \overline{a}\right)^2 |\mathbb{I}_i\right] - c\left(K_i\right) \tag{1}$$

Observe that there exists a finite number of agents, n, and thus agent i recognizes her influence over the average action. Thus, before proceeding in solving the problem we have to rephrase the maximization problem to include such influence:

$$\max_{a_i} -\mathbb{E}\left[r\left(a_i - \theta\right)^2 + (1-r)\left(a_i - \frac{a_i}{n} - \frac{(n-1)\overline{a}_{-i}}{n}\right)^2 |\mathbb{I}_i\right] - c\left(K_i\right) \tag{2}$$

where $\overline{a}_{-i} = \frac{\sum_{j\neq i} a_j}{n-1}$ is the average action of all players excluding player i. And thus, the

[7]see Garicano and Pratt (2011) for a survey and Calvó-Armegol, Martí and Pratt (2012) for an interesting interpretation

[8]Throughout the paper we restrict our attention to pure strategies

final version of the individual problem is given by:

$$\max_{a_i} -\mathbb{E}\left[r\left(a_i-\theta\right)^2+(1-r)\left(\frac{n-1}{n}\right)^2\left(a_i-\overline{a}_{-i}\right)^2|\mathbb{I}_i\right]-c\left(K_i\right) \tag{3}$$

applying a monotonic transformation to the payoff we have:

$$\max_{a_i} -\mathbb{E}\left[r'\left(a_i-\theta\right)^2+(1-r')\left(a_i-\overline{a}_{-i}\right)^2|\mathbb{I}_i\right]-c\left(K_i\right) \tag{4}$$

where $r'=\frac{r}{r+(1-r)\left(\frac{n-1}{n}\right)^2}$

The first order necessary condition for a solution gives us the optimal action:

$$a_i=r'\mathbb{E}\left[\theta|\mathbb{I}_i\right]+(1-r')\,\mathbb{E}\left[\overline{a}_{-i}|\mathbb{I}_i\right] \tag{5}$$

All what is left to solve this stage of the game is to characterize $\mathbb{E}\left[\theta|\mathbb{I}_i\right]$ and $\mathbb{E}\left[\overline{a}_{-i}|\mathbb{I}_i\right]$. The following Proposition will help characterize the second one, while the first one is trivial.

Proposition 1. *For each network formed in the first stage of the game, there exists a unique equilibrium in linear strategies in the second stage of the game.*

The proof of the proposition is long and tedious. It is presented in the Appendix of the paper. It follows the traditional path: we guess and verify a linear equilibrium. The Proposition guarantees the existence and uniqueness of a linear strategy equilibrium. The restriction on the strategy space clearly restricts the set of equilibria, however it is a standard assumption in the literature using the normal-quadratic approach. Even though there no known counterexamples exists, it is not obvious to show that only linear equilibria exist[9].

Moving back to the problem at hand, each signal follows a normal distribution, thus the expected value of state is given by

$$\begin{aligned}\mathbb{E}\left[\theta|\mathbb{I}_i\right] &= \mathbb{E}\left[\theta|\{e_j\}_{j:g_{ij}=1}\right]\\ &= \frac{0\times\sigma_e^{-2}+\sum_{j=1}^n g_{ij}e_i\sigma_e^{-2}}{\sigma_e^{-2}+\sum_{j=1}^n g_{ij}\sigma_e^{-2}}\\ &= \frac{\sum_j g_{i,j}e_j}{K+1}\end{aligned}$$

Using the linear equilibrium, we have that each action is a linear combination of the signals in the economy, that is $a_i=\sum_{j=1}^n\gamma_{i,j}e_j$. This guarantees that the average action and the

[9] Angeletos and Pavan (2009), Calvó-Armengol, Marti and Pratt (2012) and Dewan and Myatt (2008)

average action not including my own action are linear combinations of the signals as well, $\overline{a} = \sum_{j=1}^{n} \beta_j e_j$ and $\overline{a}_{-i} = \sum_{j=1}^{n} \beta_{-i,j} e_j$.

Lastly, the expected value of average action not including my own action is given by

$$\begin{aligned}
\mathbb{E}\left[\overline{a_{-i}}|\mathbb{I}_i\right] &= \textstyle\sum_{j=0}^{n} \beta_{-i,j} \left(g_{i,j} e_j + (1-g_{i,j}) \frac{1}{K+1} \sum_{s=0}^{n} g_{i,s} e_s\right) \\
&= \textstyle\sum_{j=0}^{n} \beta_{-i,j} g_{i,j} e_j + \sum_{j=0}^{n} \beta_{-i,j} (1-g_{i,j}) \frac{\sum_{s=0}^{n} g_{i,s} e_s}{K+1} \\
&= \textstyle\sum_{j=0}^{n} g_{i,j} \underbrace{\left[\beta_{-i,j} + \frac{\sum_{s=0}^{n} (1-g_{i,s}) \beta_{-i,s}}{K+1}\right]}_{M_j} e_j \\
&= \textstyle\sum_{j=0}^{n} g_{i,j} M_j e_j
\end{aligned}$$

2.1.2 Network Formation

At the moment the agent decides on which connections she wants to form, she has not observed any signal yet. Thus the first step is to compute the ex-ante expected payoff. Once more we will have to consider that the agent knows his own action when forming an expectation about the average action. Also, we have to consider the ex-ante correlation between signals given by the state of the world. We leave this computations to the appendix. The resulting ex-ante payoff function is given by:

$$\begin{aligned}
\text{Payoff} = -\sigma_{ep}^2 \Bigg[\sum_{j=1}^{n} g_{i,j} \Bigg[&-(1-r') \underbrace{\beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} (1-g_{i,s}) \beta_{-i,s} + \frac{1}{K+1} \sum_{s=1}^{n} \beta_{-i,s}\right)}_{U_j^a} \\
&+ \left(r'(1-r')^2 + (1-r')r'^2\right) \underbrace{\left(M_j - \frac{1}{K+1}\right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1}\right)\right)}_{U_j^b} \Bigg] \\
&+ (1-r') \sum_{j=1}^{n} \beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} \beta_{-i,s}\right) + r' \frac{1}{K+1} \Bigg] - c(G)
\end{aligned}$$

another way of writing it is

$$\text{Payoff} = -\sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j} \left[-(1-r')U_j^a + \left(r'(1-r')^2 + (1-r')r'^2 \right) U_j^b \right] + \text{constant} - c(G)$$

Monotonicity Results

The main features of the ex-ante payoff function can be summarized in two Lemmas:

Lemma 1. $\forall i \quad g_{i,h} = 1 \implies g_{i,f} = 1, \ \forall f : \bar{K}_f \geq \bar{K}_h$

Lemma 2. $\bar{K}_f \geq \bar{K}_h \implies K_f \leq K_h \ \ \forall f, h$

Both lemmas are monotonicity results. The first one states that if a player i is connected to a player h, then it must be connected to any other person observed by more people than h is. The reason is that by switching the link from h to such more watched player, player i is acquiring the same amount of information about the real state of the economy while increasing the amount of information she is obtaining about the average action. That is, there exists a monotonicity of individual preferences on whom to connect to. One always wants to observe the players that are more observed, because those are the "opinion makers".

The second lemma states that if player f is being observed by more players than player h, then it cannot be observing more players than h is. Again the intuition is very simple here, if a player is being observed by more players, her own signal is more informative of the average action, and thus she has less incentives to form links than a less observed player.

Finally it is worth noting that the intuition behind both lemmas is more general than the set-up presented so far, and it does not depend on the normal-linear structure. It depends only on the complementarity on the payoff.

Proof of Lemma 1

The proof of Lemma 1 relies heavily on Fact 1, which is proven in the appendix.

Fact 1. *Let's compare agent i's ex-ante payoff,* $\Pi_i(G_i)$*, in two distinct situations. In both situations agent i is connected to an arbitrary group of agents, call it A. In the first situation i is also connected to f but not to h* ($g_{i,h} = 0$ *and* $g_{i,f} = 1$)*. The second situation presents the opposite. Agent i is connected to h but not to f* ($g_{i,f} = 0$ *and* $g_{i,h} = 1$)*. Assume that agents f and h can be ranked by their impact on the average action, that is* $\beta_{-i,f} > \beta_{-i,h}$*. For any set A,* $\Pi_i(A, f) - \Pi_i(A, h) > 0$

All we need to show to prove Lemma 1 is that the more looked an agent j is higher $\beta_{-i,j}$ is, for all i. The first step is to establish a relationship between $\beta_{-i,j}$ and β_j. More specifically, we'll show that $\beta_f > \beta_h$ implies that $\beta_{-i,f} > \beta_{-i,h}$ for every i that satisfies one of the following situations: (i) $g_{if} = g_{if} = 0$, (ii) $g_{if} = g_{if} = 1$, or (iii) $g_{if} = 1$ and $g_{if} = 0$. The fourth case is a trivial implication of Fact 1, thus it is not discussed below.

The loading $\beta_{-i,j}$ can be written as

$$\beta_{-i,j} = \frac{n}{n-1}\beta_j - \frac{1}{n-1}\gamma_{ij}$$

where

$$\begin{aligned}
\gamma_{ij} &= \bar{r}\frac{g_{ij}}{K_i+1} + (1-\bar{r})g_{ij}\beta_j + (1-\bar{r})\frac{g_{ij}}{K_i+1} - (1-\bar{r})\sum_{s=0}^{n}\beta_s g_{is}\frac{g_{ij}}{K_i+1} \\
&= \frac{g_{ij}}{K_i+1} + (1-\bar{r})\left[g_{ij}\beta_j - \sum_{s=0}^{n}\beta_s g_{is}\frac{g_{ij}}{K_i+1}\right] \\
&= g_{ij}\left[\frac{1}{K_i+1} + (1-\bar{r})\left[\beta_j - \frac{\sum_{s=0}^{n}\beta_s g_{is}}{K_i+1}\right]\right]
\end{aligned}$$

is the weight that agent i gives to the jth signal. A detailed derivation of the individual weights is provided in the Appendix.

Substituting in the $\beta_{-i,j}$ expression,

$$\begin{aligned}
\beta_{-i,j} &= \frac{n}{n-1}\beta_j - \frac{1}{n-1}\gamma_{ij} \\
&= \frac{n}{n-1}\beta_j - \frac{g_{ij}}{n-1}\left[\frac{1}{K_i+1} + (1-\bar{r})\left[\beta_j - \frac{\sum_{s=0}^{n}\beta_s g_{is}}{K_i+1}\right]\right] \\
&= \frac{n}{n-1}\beta_j - \frac{g_{ij}}{n-1}\left[\frac{1}{K_i+1} + (1-\bar{r})\left[\beta_j - \frac{\sum_{s=0}^{n}\beta_s g_{is}}{K_i+1}\right]\right]
\end{aligned}$$

If $g_{if} = g_{ih} = 0$, then $\beta_{-i,f} - \beta_{-i,h} = \frac{n}{n-1}(\beta_f - \beta_h) > 0$.

If $g_{if} = g_{ih} = 1$, then $\beta_{-i,f} - \beta_{-i,h} = \left(\frac{n}{n-1} - \frac{1-\bar{r}}{n-1}\right)(\beta_f - \beta_h) > 0$.

If $g_{if} = 0$ and $g_{ih} = 1$, then $\beta_{-i,f} - \beta_{-i,h} = \frac{n}{n-1}(\beta_f - \beta_h) + \frac{1-(1-\bar{r}\sum_{s=0}^{n}\beta_s g_{is}}{(n-1)(K_i+1)} + \frac{1-\bar{r}}{N-1}\beta_h > 0.$

Thus, we have that $\beta_f \geq \beta_h \implies \beta_{-i,f} \geq \beta_{-i,h} \forall i$. All what's left to show of Lemma 1 is that a more looked at individual has a higher influence on the average action, that is $\bar{K}_f \geq \bar{K}_h \implies \beta_f \geq \beta_h$.

Proposition. *Monotonicity of β_j Agent f's private signal has a weakly higher impact in the average action than agent h's has if, and only if, agent f is weakly more looked at than agent* *h. That is:*

$$\beta_f \geq \beta_g \iff \bar{K}_f \geq \bar{K}_h$$

Proof. First of all, note that the first fact already guarantees us the only if part, we need to show only the if part. As usual in economics, assume not. Assume $\beta_f < \beta_g$ & $\bar{K}_f \geq \bar{K}_h$. Using the fact 1 once more, we have that we can restrict the above to $\beta_f < \beta_h$ & $\bar{K}_f = \bar{K}_h$. We will proceed to construct a contradiction to it. From the fact that all players use the beta ranking to decide who to look, we have that $g_{i,h} = g_{i,f}\ \forall i$.

Let's now consider the formula for β_j showed in appendix.

$$\beta_j = \frac{\bar{r}}{n}\sum_{i=1}^{n}\frac{g_{ij}}{K_i} + \frac{(1-\bar{r})}{n}\left[\beta_j\overline{K}_j + \sum_{i=1}^{n}\frac{g_{ij}}{K_i} - \sum_{i=1}^{n}\sum_{s=0}^{n}\frac{\beta_s g_{is} g_{ij}}{K_i}\right]$$

$$\beta_f - \beta_h = \frac{\bar{r}}{n}\sum_{i=1}^{n}\frac{g_{if} - g_{ih}}{K_i} + \frac{(1-\bar{r})}{n}\left[\beta_f\overline{K}_f - \beta_h\overline{K}_h + \sum_{i=1}^{n}\frac{g_{if} - g_{ih}}{K_i} - \sum_{i=1}^{n}\sum_{s=0}^{n}\frac{\beta_s g_{is}(g_{if} - g_{ih})}{K_i}\right]$$

$$= \frac{(1-\bar{r})}{n}\left[(\beta_f - \beta_h)\overline{K}_f\right]$$

Thus $\beta_f - \beta_h = 0$ □

Proof of Lemma 2

The proof of this lemma will be done by considering different cases, depending on whether agents h and f are looking at each other or not. In all cases we consider $\bar{K}_f \geq \bar{K}_h$, that is agent f is more attractive to looks than agent h. As general notation, let H be the set of players that agent h is looking at excluding agent h itself and agent f, and define F similarly for agent f. As defined above, let agent i's ex-ante payoff be $\Pi_i(G_i)$. Observe also that the payoff function depends on the agent only through the fact that the agent is looking at itself. That is, we can define a payoff function that is independent of the player, $\Pi_i(G_i) = \Pi(G_i, i)$. This leads us to our first observation, two agents that are looking to the same set of players (including themselves) must have the same payoff.

case 1 In the first case, let's consider agent f looks at agent h but not the other way around, that is $g_{f,h} = 1$ and $g_{h,f} = 0$. From agent f revealed preference, we have that $\Pi(F, f, h) > \Pi(H, f)$ and from agent h revealed preference $\Pi(H, h) > \Pi(F, f, h)$. This is a contradiction, since by Fact 1 we have that $\Pi(H, f) > \Pi(H, h)$. Thus we have that if a more looked agent is looking at a less looked agent, it must be the case that the less looked one is retributing the look.

case 2 In the second case, let's consider agents f and h are looking at each other, $g_{f,h} = 1$ and $g_{h,f} = 1$. From revealed preferences, we must have that both agents are having the same payoff, otherwise the agent with lowest payoff could always mimic the high payoff one. This guarantees that they are looking at the same set of agents.

Thus, if agent f is looking at agent h, Lemma 2 is satisfied.

case 3 In the third case we will consider that agent f is not looking at agent h. We will assume that the Lemma do not hold, agent f is looking at more players than agent h, and show the contradiction. If f is looking at more players than h, by the first Lemma we have $F \supset H$. Rewrite $F = H \cup F_1$ and $F_1 \cap H = \emptyset$. From agent h revealed preference we have $\Pi(H,h) > \Pi(H,F_1,h)$. From agent f revealed preference $\Pi(H,F_1,f) > \Pi(H,f)$. Subtracting the two inequalities, we have $\Pi(H,f) - \Pi(H,h) < \Pi(H,F_1,f) - \Pi(H,F_1,h)$. Contradiction, due to Fact 2 stated below.

Fact 2. *Consider a set A and a set A' such that $A \subset A'$, where $f,h \notin A'$. Also, by the first fact, if $\beta_{-i,j} \in A'$, but $\beta_{-i,j} \notin A$, we know that $\beta_{-i,j} \leq \beta_{-i,k} \forall k \in A$. Then* $[\Pi_i(A,f) - \Pi_i(A,h)] - [\Pi_i(A',f) - \Pi_i(A',h)] > 0$

The proof of Fact 2 is provided in the appendix.

3 Network Shape Characterization

Definition. *HDN A network is considered a Hierarchical Directed Network (HDN) if, and only if, there exists a partition $\{A_n\}_{\{}1,2,...,N\}$, such that:*

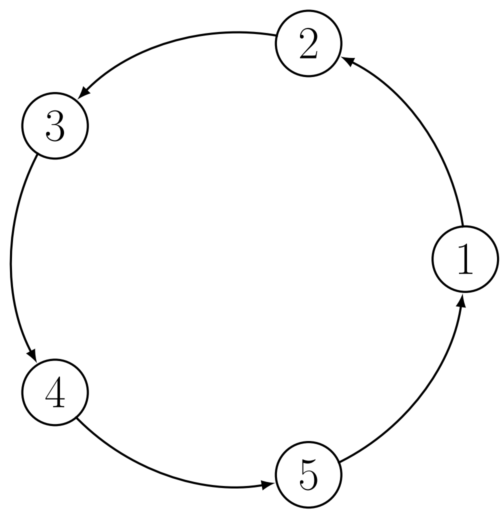
i) $i \in A_1$ *if, and only if,* $\forall j \notin A_1, g_{j,i} = 1$ *and if there exists* $i,j \in A_1, g_{i,j} = 1 \implies g_{l,m} = 1 \forall l,m \in A_1$

ii) $i \in A_n$ *if, and only if,* $\forall j \notin \cup_{k=1}^{n} A_k, g_{j,i} = 1$ *and if there exists* $i,j \in A_n, g_{i,j} = 1 \implies g_{l,m} = 1 \forall l,m \in A_n$

iii) $i \in A_n,\ g_{i,j} = 1$ *and* $j \notin \cup_{j=1}^{n-1} A_j \implies g_{l,m} = 1\ \forall l,m \in A_n$

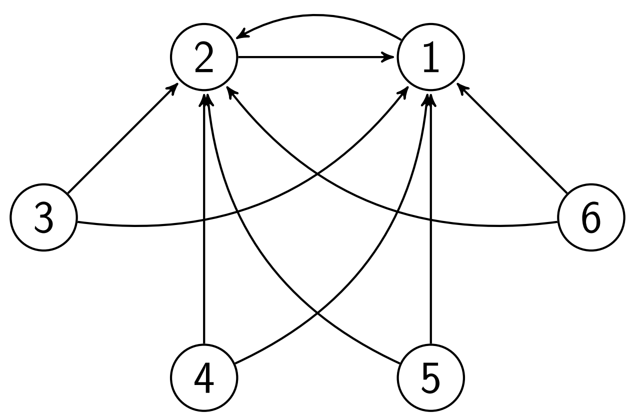
To help understand the definition it is useful to present a couple of examples. Consider the wheel network, pictured above considering 5 players. Notice that it violates the definition of hierarchical, since if we consider all players to be in the same level, we have that the level is neither full nor empty and if we consider the players to be in different levels, not all players from a level below are looking to upper levels.

Figure 1: Wheel Network



On the other hand, consider the periphery sponsored core-periphery network, with 6 players: 2 in the center and 4 in the periphery, depicted below. Such network satisfies the hierarchical definition. Consider the partition that places players 1 and 2 in one group and the other players in a second group. Notice that all groups are either empty or full and that all players in the second group do observe all players in the first group.

Figure 2: Core-Periphery Network



Theorem. *If the two lemma hold, any strict equilibrium of the network formation game described above generates a Hierarchical Directed Network structure*

Before we proceed to the proof of the theorem, a couple of words about the phrasing of it. It is intentional that we phrase the conditions for the theorem to hold in terms of endogenous objects (Lemma 1 and Lemma 2). Lemma 1 and Lemma 2 are fairly general in our model, and so within the model that is done with no loss to the reader. However such particular phrasing should make a very important point clear, the theorem is far more general than the set up we present here, and for the proof nothing other than both lemmas is needed.

Proof. Assume the network formation game is in equilibrium. Define the following sets, $i \in B_1 \iff \bar{K}_i \geq \bar{K}_k \forall k$.

step 1 If $G = \emptyset$ the proof is done, since the empty network is a hierarchical network.

step 2 If G is not empty, then B_1 is not empty/ Let k_1 be an element of B_1. Let's focus on constructing B_1.

We will start by showing that if one considers $l \notin B_1$, then it must be the case that $g_{l,k_1} = 1$. That is, if l is not among the set of most connected players it must be observing k_1. The structure of the argument will be based on contradiction. As usual, assume not.

If $g_{l,j} = 1$ for some j we are done, since we can use Lemma 1 and arrive at a contradiction already. Player l should be observing player k_1. Observe that this holds true even if $j \in B_1$.

But what if $g_{l,j} = 0 \ \forall j$? Then we know that:

1. $g_{l,k} = 0 \ \forall k : \ \bar{K}_k \geq \bar{K}_l$
2. Let B_N be the group of the least observed players. $l \notin B_N$, otherwise by Lemma 2 we would have an empty network.
3. $l \notin B_1 \implies \bar{K}_l < \bar{K}_{k_1}$, thus $\exists j_1 : \ g_{j_1,l} = 0 \ and \ g_{j_1,k_1} = 1$
4. By Lemma 2, it must be that $\bar{K}_{j_1} < \bar{K}_l$
5. By the same arguments as before, it must be that $\exists j_2 : \ g_{j_2,j_1} = 0 \ and \ g_{j_2,l} = 1$
6. If we proceed by induction, we will finally conclude that $\exists j_N : \ g_{j_N,j_N-1} = 0 \ and \ g_{j_N,j_N-2} = 1$. Observe that since this is the last step in the induction, it must be that $j_N \in B_N$.
7. However, if $g_{j_N,j_N-1} = 0$, then $\bar{K}_{j_N-1} = 0$. And thus, $j_N - 1 \in B_N$. Contradiction.

In order to finish constructing the set B_1, we still need to show that if one element of the set B_1 is being observed by any other element of B_1, it must be being observed by all elements in B_1 and also it must observe all players in B_1.

1. If $k_1 \in B_1$ is being observed by one other player in B_1 it must be the case that all players in B_1 are being observed by one other member in B_1. The reason for this is that they all receive the same number of looks from players outside B_1 and they also have the same number of total players looking to them.
2. If all players are looking to someone, from Lemma 1, we already have that all players are looking to everyone else in B_1.
3. What if only a subset of players from B_1 are looking to someone in B_1? Call this subset $S_{looking}$ and let the complement be $S_{blind} = B_1 \backslash S_{looking}$. By Lemma 1, every member of $S_{looking}$ is looking to everyone else in B_1.
4. Observe then that the number of looks an element of $S_{looking}$ receives is strict smaller than the umber of looks an element of S_{blind} receives. The reason for that is that a player in S_{blind} receives from other players in B_1 $\bar{K}_{S_{looking}} + 1$ looks, where $\bar{K}_{S_{looking}}$ is the cardinality of the set $S_{looking}$ and the extra one is from himself. While a player in $S_{looking}$ receives $\bar{K}_{S_{looking}}$ looks.
   Contradiction

Step 3 The next step is to construct the second level, B_2. Begin by ignoring any looks cast by the first level and simply follow the same steps detailed above. By induction we can proceed to construct B_2, B_3,...

□

3.1 Equilibrium Payoffs

A couple of smaller results help us establish a second network characterization result. Any equilibrium network, and thus individual ex-ante payoffs, can be recovered from the in and out degree matrix, $(K, \bar{K}) = \{(K_i, \bar{K}_i)\}\ \forall i$. First we characterize the looking down an individual can do.

Proposition. *Looking Down*

In any equilibrium:

$$i \in A_n \ \&\ j \in A_{k>n} \ \text{and}\ g_{i,j} = 1 \implies \begin{array}{l} a)\forall j \in \cup_1^n A_k,\ g_{i,j} = 1 \\ b)\forall k \neq j \in \cup_{n+1}^N A_s,\ g_{i,l} = 0 \\ c) j \in A_{n+1} \end{array}$$

Proof. Item (a) is a direct implication of Lemma 1, while item (c) is a direct implication of Lemma 1 and item (b). All what is need is to show item (b).

Item (b) Again, the proof will be done by contradiction. Suppose not, thus $g_{i,j} = 1\&\ g_{i,l} = 1\ j, l \in A_{n+1}$. Start by observing that level k cannot be full. Members of $n+1$ cannot be looking at each other, otherwise players j and l would not be members of $n+1$ level, but of n level. The next step is to count the number of looks by each agent, let's start with agent j $K_j = \bar{N} \equiv$ cardinality of $[\cup_1^n A_k]$, while $K_i = \bar{N} - 1 + 2$. Contradiction to Lemma 2. □

Proposition. *Equilibrium Payoffs depend only on the in and out degree matrix, $(K, \bar{K})$.*

Proof. As a proof we present the algorithm to construct G from the pair of vectors $(K, \bar{K})$. It is quite a simple algorithm, with only four steps.

1 Rank players by $\bar{K}$. The first group will be those with highest $\bar{K}$. Those are the members of A_1.

2 Let K_1 be value of K for the members of the set A_1. If $K_1 \leq \#A_1 - 1$, then A_2 will be the set of players with higher $\bar{K}$, excluding A_1.

3 If $K_1 = \#A_1$ then subtract $\#A_1$ from $\bar{K}$ for the player with higher $\bar{K}$, excluding A_1. Using this modified vector, A_2 will be the set of players with higher $\bar{K}$, excluding A_1.

4 repeat the steps

□

3.2 Equilibria Examples

In this section we provide a couple of equilibria examples, depending on the parameter values specified. In all the equilibria computations we use a linear cost function, $c(K_i) = c.K_i$. A parametrized model is characterized by three parameters, the number of players, n, how much an agent cares about the true state of the world, r, and the linear cost of forming links, c. Something that should be clear at this point is the multiplicity of equilibria in this game. Not only for a given equilibrium network configuration we have the multiplicity of selecting different players for different roles in the network, but also different configurations can be equilibria. Here another aspect of Lemma 1 should be clear, it greatly simplifies equilibrium computations. With no lemma 1, to check whether a certain network configuration is an equilibrium or not one would have to compare, for any agent, the equilibrium-candidate payoff with every possible unilateral deviation. There are up to 2^n possible network configurations, thus one would have to compare $n2^n$ payoffs. Using Lemma 1 one only needs to check two deviations for each player, making an extra link (with the agent that has the highest $\beta_{-i,k}$ that the player is not connected) and breaking a link (with the agent that has the lowest $\beta_{-i,k}$ that the player is connected).

First, we present all equilibria networks for a couple of parameters. Starting with a set of parameters we constructed all possible hierarchical networks and checked whether each one was an equilibrium or not. We present below the set of possible equilibria configurations, as well as the associated ex-ante payoff for each agent in each equilibrium network configuration. Even though we have to consider only the hierarchical networks, this approach faces computing limitations. For instance for a set of 10 players, we check close to 10 thousand networks. The main conclusion to be taken from such examples is that multiplicity of equilibria is pervasive in this model. Another conclusion to be taken from this exercise is that core-periphery networks are usually the equilibrium configuration. After this exercise we focus on networks with larger number of players and provide one equilibrium example in a network with 100 players.

The notation used in the graphs below deserves a bit of clarification. The number associated with a node represents the number of players in that tier, A black node represents the fact that all agents in that tier are looking at each other (full tier), while a white node represents that they are not (empty tier). Finally, the notation used to indicate the case in which all members of a tier decide to look at one member of a lower tier is a dotted arrow from a tier to a lower one.

3.2.1 First Parametrization

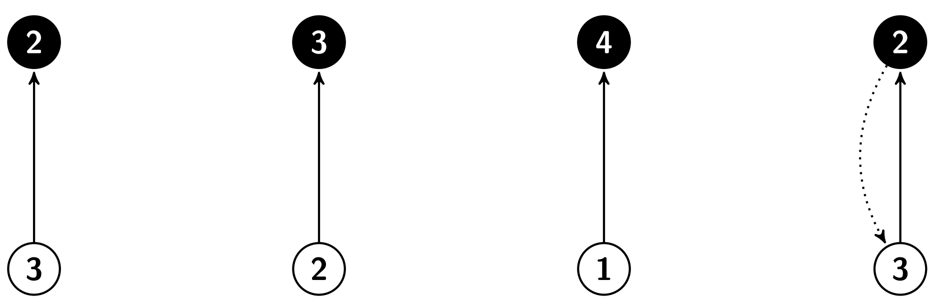
Here we present a model with eight players. We set how much an agent cares about the true state of the world to $r = 0.5$, and the linear cost of forming links, $c = 0.1$. The two equilibria have a periphery sponsored core-periphery structure, with the difference between them being how many members are in the core. In the first equilibrium there is only one individual in the core (with ex-ante payoff of -0.2961) and seven in the periphery (with ex-ante payoff of -0.3204). And in the second equilibrium there two in the core (with ex-ante payoff of -0.2925) and six in the periphery (with ex-ante payoff of -0.3591).

Figure 3: First Parametrization

3.2.2 Second Parametrization

Here we present a model with five players. We set how much an agent cares about the true state of the world to $r = 0.3337$, and the linear cost of forming links, $c = 0.036$. There are four equilibria, out of which three have a periphery sponsored core-periphery structure, with the difference between them being how many members are in the core. In the first equilibrium there are two individuals in the core (with ex-ante payoff of -0.1853) and three in the periphery (with ex-ante payoff of -0.2003). In the second equilibrium there are three in the core (with ex-ante payoff of -0.1826) and two in the periphery (with ex-ante payoff of -0.2074). And in the third there are four in the core (with ex-ante payoff of -0.1960) and one in the periphery (with ex-ante payoff of -0.2251). The fourth equilibrium has a modified core-periphery structure, in which there are two individuals in the core (with ex-ante payoff of -0.1906), but those two individuals not only chose to look at each other but they also look at one player in the "periphery". The remaining two members are in the periphery (with ex-ante payoff of -0.2081).

Figure 4: Second Parametrization



3.2.3 Third Parametrization

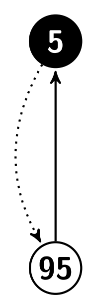
Again we present a model with five players. We set how much an agent cares about the true state of the world to $r = 0.44$, and the linear cost of forming links, $c = 0.08$. There are two equilibria. In the first equilibrium there are two individuals in the core (with ex-ante payoff of -0.2877) and three in the periphery (with ex-ante payoff of -0.3021). The second equilibrium has a modified core-periphery structure, in which there is one individual in the core (with ex-ante payoff of -0.2673), and she chose to look at one player in the "periphery". The remaining members are in the periphery (with ex-ante payoff of -0.3162).

Figure 5: Third Parametrization

3.2.4 100 Players

Here we present a model with 100 players. We set how much an agent cares about the true state of the world $r = 0.8$, and the linear cost of forming links, $c = 0.0146$. Instead of searching for all equilibria given those parameters, we simply verify that a given network is indeed an equilibrium.

Figure 6: 100 Players



3.3 Implications of the Network Structure

3.3.1 Aggregate Shocks as a result of Individual Shocks

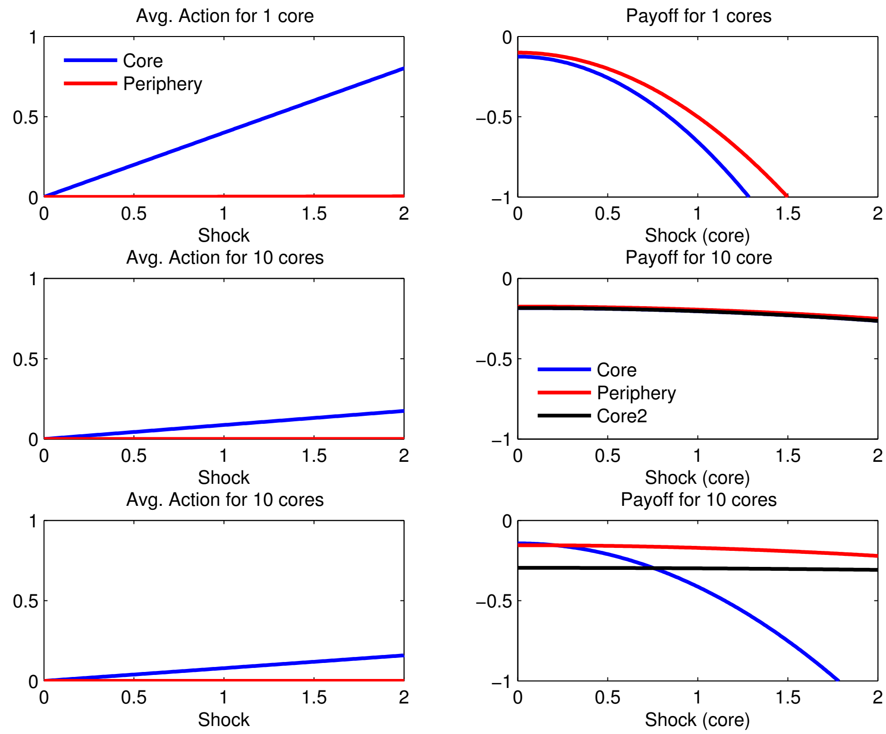
One interesting implication of the hierarchical network structure is the no aggregation of information. The signals of players in the network core, the "opinion makers", will affect the actions and the payoff of all other players relatively more than the signal of players in the outer layers of the hierarchy.

Consider the last stage of our network formation game with 100 players, and let's compare how different network structures affect the information aggregation. To this end, we'll consider three different networks: (i) star network with one player in the core and all the remaining 99 in the periphery only looking at the core, (ii) core-periphery with 10 players in the core looking at each other and all 90 remaining players in the periphery only looking at the core, (iii) core-periphery with 10 players in the core not looking at each other and all 90 remaining players in the periphery only looking at the core.

For each of these networks, we will compute the payoff of each agent as well as the average action assuming that all observed signals are all zeros, except for one particular agent. The first graph in figure 7 (top graph on the left) show how the average action changes when the core signal changes (blue line) and when periphery signal changes (red line). Clearly, the core' signal highly affects the average, while the periphery's signal doesn't. The second graph (top graph, on the right) show how the core and periphery payoff varies with the core's signal. As the core's signal moves away from the true state, the payoff of all agents decreases considerably.

Like the first and the second, the third graph (middle graph, on the left) and the fourth graph (middle graph, on the right) report the payoffs and average actions, but for the network (ii). Adding more players to the core decreases the core's influence on the equilibrium outcome and there is more aggregation. The last 2 graphs follow the same reasoning, but for the last network with the core players not looking at each other. The effect on the average action is similar to the second network, however the effect on the payoff is rather different. The periphery is looking at all core players, so their payoffs are not highly affects by the signal of a particular core player. Core players are not looking at each other, so the payoff of a core player is not affect by the signal of another core player (black line), however it is highly affected by their own signal (blue line). This example shows that under sparse networks (e.g. star network), information aggregation may not happen. Since only hierarchical networks happen in equilibrium and hierarchical networks are typically sparse, aggregate flutuation as a result of independent shocks is an equilibrium outcome of our model.

Figure 7: Aggregate Fluctuations



3.3.2 Equilibrium Inefficiency

The goal of this section is to present examples that indicate that efficiency is not an tribute of equilibria in this model.

First, observe that for certain parameters some equilibrium Pareto dominates others. For instance, take a second look at the second parametrization provided above. The first and second equilibria Pareto dominates the third and the fourth. Also, even though we cannot rank the first and second equilibria, the fourth equilibrium dominates the third one.

A second interesting observation is that the sum of payoffs can be higher in a non-equilibrium network configuration than in the best equilibrium. For instance consider the third parametrization provided above. The sum of payoffs in the first equilibrium is -1.4831 and in the second -1.4817, while if we were to consider a star-shaped network the sum of payoffs would be -1.4782. The question then is how to redistribute the payoffs surplus generated by the non-equilibrium network? That has to be done by agents choosing a non-optimal weight on some agents signal, as a way to reward other agents for making/breaking a connection.

Finally, we present an example in which one non-dominated equilibrium is Pareto dominated by a non equilibrium configuration. We consider a model with thirty players. We set how much an agent cares about the true state of the world to $r = 0.44$, and the linear cost of forming links, $c = 0.08$. There are two equilibria. In the first equilibrium there are two individuals in the core (with ex-ante payoff of -0.2963) and twenty eight in the periphery (with ex-ante payoff of -0.3441). The sum of payoffs is -10.2270. The second equilibrium has a modified core-periphery structure, in which there is one individual in the core (with ex-ante payoff of -0.3194), and she chose to look at one player in the "periphery". The remaining members are in the periphery (with ex-ante payoff of -0.3208). The sum of payoffs is -9.6226. Notice that no equilibrium Pareto dominates the other.

Consider a star-shaped network, notice that it is not an equilibrium. However, not only the sum of payoff of such network configuration is higher, but we are able to choose the γ weights that each player will use to compute their actions in such a way that a star-shaped network with modified actions Pareto dominates the second equilibrium. Start by observing that player 1 is the one that wants to deviate from the star to equilibrium. Observe also that player 2 (the one being observed by player 1 in the equilibrium configuration) also has a smaller payoff in the star network than in the equilibrium. Those are the players that need to be rewarded [10]. All others are the one's that will have to pay. The idea is to diminish the importance of one's own signal in one's action for players 3 to 30, increasing the importance of the prior. This rewards players 1 and 2 since it increases their ability to predict the average action. The final payoffs in this star network using modified γ is that player 1 is earning 0.002 more than in the second equilibrium, player 2 0.0003 and all other players 0.0017. Thus the non equilibria star architecture Pareto dominates the equilibrium.

Figure 8: Equilibria Inefficiency

[10] We have chosen the cost so that player 1's losses are quite small so we can focus on rewarding player 2.

4 Final Remarks

We have developed a framework to understand how people choose to form connections in order to inform themselves about a situation. We considered a case in which people need to be informed not only about the state of the world but also about how are other people perceiving it and acting on it. Our analysis provides a clear characterization of equilibria architecture and discuss implications of it for aggregate fluctuations and efficiency.

There are interesting extensions to be made to our current setting. Immediate and straightforward extensions include analyzing the value of the public information, and evaluating whether a more precise public information improves agents payoffs. Another possible extension is to consider the limiting case, as the number of players grows. In this case, the hierarchy cannot have infinite levels, otherwise the cost beard by the first tier would be infinite, and, for the same reason, the number of players in the tiers other than the first has to be finite. For future research, an insightful extension would be to change how information flows along the network linkages, by allowing players to partially acquire information from agents that are not their immediate connections.

References

Acemoglu, D., V. M. Carvalho, A. Ozdaglar, and A. Tahbaz-Salehi (2012): "The network origins of aggregate fluctuations," *Econometrica*, 80(5), 1977–2016.

Bala, V., and S. Goyal (2000): "A noncooperative model of network formation," *Econometrica*, 68(5), 1181–1229.

Bandiera, O., and I. Rasul (2006): "Social networks and technology adoption in northern mozambique*," *The Economic Journal*, 116(514), 869–902.

Bulow, J. I., J. D. Geanakoplos, and P. D. Klemperer (1985): "Multimarket oligopoly: Strategic substitutes and complements," *The Journal of Political Economy*, pp. 488–511.

Calvó-Armengol, A. (2004): "Job contact networks," *Journal of economic Theory*, 115(1), 191–206.

Calvó-Armengol, A., J. De Marti, and A. Prat (2011): "Communication and influence," *Unpublished Manuscript, London School of Economics.*

Calvo-Armengol, A., and M. O. Jackson (2004): "The effects of social networks on employment and inequality," *The American Economic Review*, 94(3), 426–454.

Calvó-Armengol, A., E. Patacchini, and Y. Zenou (2009): "Peer effects and social networks in education," *The Review of Economic Studies*, 76(4), 1239–1267.

Calvó-Armengol, A., and Y. Zenou (2004): "Social Networks and Crime Decisions: The Role of Social Structure in Facilitating Delinquent Behavior," *International Economic Review*, 45(3), 939–958.

Conley, T. G., and C. R. Udry (2010): "Learning about a new technology: Pineapple in Ghana," *The American Economic Review*, pp. 35–69.

De Weerdt, J. (2002): *Risk-sharing and endogenous network formation*, no. 2002/57. WIDER Discussion Papers//World Institute for Development Economics (UNU-WIDER).

De Weerdt, J., and S. Dercon (2006): "Risk-sharing networks and insurance against illness," *Journal of Development Economics*, 81(2), 337–356.

Durlauf, S. N. (2004): "Neighborhood effects," *Handbook of regional and urban economics*, 4, 2173–2242.

Fafchamps, M., and S. Lund (2003): "Risk-sharing networks in rural Philippines," *Journal of development Economics*, 71(2), 261–287.

Foster, A. D., and M. R. Rosenzweig (1995): "Learning by doing and learning from others: Human capital and technical change in agriculture," *Journal of political Economy*, pp. 1176–1209.

Gabaix, X. (2011): "The granular origins of aggregate fluctuations," *Econometrica*, 79(3), 733–772.

Galeotti, A., and S. Goyal (2010): "The law of the few," *The American Economic Review*, 100(4), 1468–1492.

Galeotti, A., S. Goyal, and J. Kamphorst (2006): "Network formation with heterogeneous players," *Games and Economic Behavior*, 54(2), 353–372.

Garicano, L., and A. Prat (2010): "Organizational economics with cognitive costs," .

Glaeser, E. L., B. Sacerdote, and J. A. Scheinkman (1996): "Crime and social interactions," *The Quarterly Journal of Economics*, 111(2), 507–548.

Goyal, S., and F. Vega-Redondo (2005): "Network formation and social coordination," *Games and Economic Behavior*, 50(2), 178–207.

Granovetter, M. (1973): "The strength of weak ties," *American journal of sociology*, 78(6), l.

Jackson, M. O. (2010): *Social and economic networks*. Princeton University Press.

Jovanovic, B. (1987): "Micro shocks and aggregate risk," *The Quarterly Journal of Economics*, 102(2), 395–409.

Königa, M. D., C. J. Tessoneb, and Y. Zenouc (2014): "Nestedness in Networks: A Theoretical Model and Some Applications," *Theoretical Economics*.

Munshi, K. (2004): "Social learning in a heterogeneous population: technology diffusion in the Indian Green Revolution," *Journal of Development Economics*, 73(1), 185–213.

Appendix

A Proof of Proposition 1

Let $\overline{a} \equiv \frac{1}{n}\sum_{j=1}^{n} a_j$ be the average action, and let $\overline{a}_{-i} \equiv \frac{1}{n-1}\sum_{j\neq i} a_j = \frac{n}{n-1}\overline{a} - \frac{1}{n-1}a_i$ be the average action without agent i. We will verify the following guess:

$$\overline{a} = \sum_{j=0}^{n} \beta_j e_j$$

From the first order condition, agent i's optimal action satisfies:

$$\begin{aligned}
a_i &= r'\mathbb{E}\left[\theta|\mathbb{I}_i\right] + (1-r')\,E\left[\overline{a}_{-i}|\mathbb{I}_i\right] \\
a_i &= r'\mathbb{E}\left[\theta|\mathbb{I}_i\right] + (1-r')\,E\left[\tfrac{n}{n-1}\overline{a} - \tfrac{1}{n-1}a_i|\mathbb{I}_i\right] \\
a_i &= \underbrace{\frac{1}{1+\frac{1-r'}{n-1}}r'}_{\equiv\bar{r}}\,\mathbb{E}\left[\theta|\mathbb{I}_i\right] + \tfrac{1}{1+\frac{1-r'}{n-1}}\,(1-r')\,\tfrac{n}{n-1}E\left[\overline{a}|\mathbb{I}_i\right] \\
a_i &= \bar{r}\mathbb{E}\left[\theta|\mathbb{I}_i\right] + (1-\bar{r})E\left[\overline{a}|\mathbb{I}_i\right] \\
a_i &= \overline{re}_i + (1-\bar{r})\left(\textstyle\sum_{j=0}^{n}\beta_j g_{ij} e_j + \sum_{j=0}^{n}\beta_j (1-g_{ij})\overline{e}_i\right)
\end{aligned}$$

Taking the average yields

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} a_i &= \bar{r}\underbrace{\frac{1}{n}\sum_{i=1}^{n}\overline{e}_i}_{\equiv\overline{\overline{e}}} + (1-\bar{r})\,\tfrac{1}{n}\textstyle\sum_i \left(\sum_{j=0}^{n}\beta_j g_{ij} e_j + \sum_{j=0}^{n}\beta_j (1-g_{ij})\overline{e}_i\right) \\
&= \bar{r}\overline{\overline{e}} + (1-\bar{r})\,\tfrac{1}{n}\textstyle\sum_{i=1}^{n}\sum_{j=0}^{n}\beta_j g_{ij} e_j + (1-\bar{r})\,\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j=0}^{n}\beta_j \overline{e}_i - (1-\bar{r})\,\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j=0}^{n}\beta_j g_{ij}\overline{e}_i \\
&= \bar{r}\overline{\overline{e}} + (1-\bar{r})\,\tfrac{1}{n}\textstyle\sum_{j=0}^{n}\beta_j \underbrace{\left(\sum_{i=1}^{n} g_{ij}\right)}_{\equiv\overline{K}_j} e_j + (1-\bar{r})\sum_{j=0}^{n}\beta_j \underbrace{\frac{1}{n}\sum_{i=1}^{n}\overline{e}_i}_{\equiv\overline{\overline{e}}} - (1-\bar{r})\,\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j=0}^{n}\beta_j g_{ij}\overline{e}_i \\
&= \bar{r}\overline{\overline{e}} + (1-\bar{r})\,\tfrac{1}{n}\textstyle\sum_{j=0}^{n}\beta_j \overline{K}_j e_j + (1-\bar{r})\sum_{j=0}^{n}\beta_j\overline{\overline{e}} - (1-\bar{r})\,\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j=0}^{n}\beta_j g_{ij}\overline{e}_i \\
&= \bar{r}\overline{\overline{e}} + (1-\bar{r})\,\tfrac{1}{n}\beta'\mathrm{diag}(\overline{K})\overrightarrow{e} + (1-\bar{r})\,\beta'\mathbf{1}\overline{\overline{e}} - (1-\bar{r})\,\tfrac{1}{n}\beta' G'\overrightarrow{\overline{e}} \\
&= \bar{r}\tfrac{1}{n}\mathbf{1}'\overrightarrow{\overline{e}} + (1-\bar{r})\,\tfrac{1}{n}\beta'\mathrm{diag}(\overline{K})\overrightarrow{e} + (1-\bar{r})\,\beta'\mathbf{1}\tfrac{1}{n}\mathbf{1}'\overrightarrow{\overline{e}} - (1-\bar{r})\,\tfrac{1}{n}\beta' G'\overrightarrow{\overline{e}} \\
&= \bar{r}\tfrac{1}{n}\mathbf{1}'\mathrm{diag}(K)^{-1}G\overrightarrow{e} + (1-\bar{r})\,\tfrac{1}{n}\beta'\mathrm{diag}(\overline{K})\overrightarrow{e} \\
&\quad + (1-\bar{r})\,\beta'\mathbf{1}\tfrac{1}{n}\mathbf{1}'\mathrm{diag}(K)^{-1}G\overrightarrow{e} - (1-\bar{r})\,\tfrac{1}{n}\beta' G'\mathrm{diag}(K)^{-1}G\overrightarrow{e} \\
&= \left[\bar{r}\tfrac{1}{n}\mathbf{1}'\mathrm{diag}(K)^{-1}G + (1-\bar{r})\,\tfrac{1}{n}\beta'\mathrm{diag}(\overline{K})\right. \\
&\quad \left. + (1-r)\,\beta'\mathbf{1}\tfrac{1}{n}\mathbf{1}'\mathrm{diag}(K)^{-1}G - (1-\bar{r})\,\tfrac{1}{n}\beta' G'\mathrm{diag}(K)^{-1}G\right]\overrightarrow{e} \\
\overline{a} &= \tfrac{1}{n}\left[\bar{r}\mathbf{1}'\mathrm{diag}(K)^{-1}G + (1-\bar{r})\,\beta'\left(\mathrm{diag}(\overline{K}) + \mathbf{1}\mathbf{1}'\mathrm{diag}(K)^{-1}G - G'\mathrm{diag}(K)^{-1}G\right)\right]\overrightarrow{e}
\end{aligned}
$$

where $\overrightarrow{\overline{e}} = (\overline{e}_1, \ldots, \overline{e}_n)'$ is n by 1, $\overrightarrow{e} = (e_0, \ldots, e_n)'$ is $n+1$ by 1, and G is n by $n+1$. To keep all in matrix notation, note that

$$
\begin{aligned}
K_{n\times 1} &= G\mathbf{1} \\
\overline{K}_{n+1\times 1} &= G'\mathbf{1} \\
\overrightarrow{\overline{e}}_{n\times 1} &= \mathrm{diag}(K)^{-1}G\overrightarrow{e} \\
\overline{\overline{e}} &= \tfrac{1}{n}\mathbf{1}'\overrightarrow{\overline{e}}
\end{aligned}
$$

Thus, we have verified the guess. Finally, applying the method of undetermined coefficients yields

$$\begin{aligned}
\beta' &= \tfrac{1}{n}\left[\bar{r}\mathbf{1}'\text{diag}(K)^{-1}G + (1-\bar{r})\,\beta'\left(\text{diag}(\overline{K}) + \mathbf{1}\mathbf{1}'\text{diag}(K)^{-1}G - G'\text{diag}(K)^{-1}G\right)\right] \\
n\beta' &= \bar{r}\mathbf{1}'\text{diag}(K)^{-1}G + (1-\bar{r})\,\beta'\left(\text{diag}(\overline{K}) + \mathbf{1}\mathbf{1}'\text{diag}(K)^{-1}G - G'\text{diag}(K)^{-1}G\right) \\
\beta' &= \bar{r}\mathbf{1}'\text{diag}(K)^{-1}G\left[nI - (1-\bar{r})\left(\text{diag}(\overline{K}) + \mathbf{1}\mathbf{1}'\text{diag}(K)^{-1}G - G'\text{diag}(K)^{-1}G\right)\right]^{-1}
\end{aligned}$$

We can verify that the average action loadings sum to 1. Starting from the equation above and post-multiplying by a vector of ones.

$$\begin{aligned}
n\beta' &= \bar{r}\mathbf{1}'\text{diag}(K)^{-1}G + (1-\bar{r})\,\beta'\left(\text{diag}(\overline{K}) + \mathbf{1}\mathbf{1}'\text{diag}(K)^{-1}G - G'\text{diag}(K)^{-1}G\right) \\
n\beta'\mathbf{1} &= \bar{r}\mathbf{1}'\text{diag}(K)^{-1}G\mathbf{1} + (1-\bar{r})\,\beta'\left(\text{diag}(\overline{K})\mathbf{1} + \mathbf{1}\mathbf{1}'\text{diag}(K)^{-1}G\mathbf{1} - G'\text{diag}(K)^{-1}G\mathbf{1}\right) \\
n\beta'\mathbf{1} &= \bar{r}n + (1-\bar{r})\,\beta'\left(\overline{K} + \mathbf{1}n - G'\mathbf{1}\right) \\
n\beta'\mathbf{1} &= \bar{r}n + (1-\bar{r})\,\beta'\left(G'\mathbf{1} + \mathbf{1}n - G'\mathbf{1}\right) \\
n\beta'\mathbf{1} &= \bar{r}n + (1-\bar{r})\,\beta'\mathbf{1}n \\
\therefore \beta'\mathbf{1} &= 1
\end{aligned}$$

The action of each agent is a linear combination of signals

$$\begin{aligned}
a_i &= \bar{r}\mathbb{E}\left[\theta|\mathbb{I}_i\right] + (1-\bar{r})E\left[\overline{a}|\mathbb{I}_i\right] \\
a_i &= \bar{r}\overline{e}_i + (1-\bar{r})\left(\textstyle\sum_{j=0}^{n}\beta_j g_{ij} e_j + \sum_{j=0}^{n}\beta_j(1-g_{ij})\overline{e}_i\right)
\end{aligned}$$

which in vector notation becomes

$$\begin{aligned}
\overrightarrow{a} &= \bar{r}\,\overrightarrow{\overline{e}} + (1-\bar{r})\left(G\text{diag}(\beta)\overrightarrow{e} + \overrightarrow{\overline{e}} - \text{diag}(\beta'G')\,\overrightarrow{\overline{e}}\right) \\
&= \bar{r}\text{diag}(K)^{-1}G\overrightarrow{e} + (1-\bar{r})\,G\text{diag}(\beta)\overrightarrow{e} + (1-\bar{r})\,\text{diag}(K)^{-1}G\overrightarrow{e} - (1-r)\,\text{diag}(\beta'G')\text{diag}(K)^{-1}G\overrightarrow{e} \\
&= \left[\bar{r}\text{diag}(K)^{-1}G + (1-\bar{r})\,G\text{diag}(\beta) + (1-\bar{r})\,\text{diag}(K)^{-1}G - (1-r)\,\text{diag}(\beta'G')\text{diag}(K)^{-1}G\right]\overrightarrow{e}
\end{aligned}$$

Hence, we have

$$\overrightarrow{a} = \Gamma\overrightarrow{e}$$

where

$$\Gamma_{n\times n+1}\equiv \bar{r}\text{diag}(K)^{-1}G+(1-\bar{r})\,G\text{diag}(\beta)+(1-\bar{r})\,\text{diag}(K)^{-1}G-(1-r)\,\text{diag}(\beta'G')\text{diag}(K)^{-1}G$$

βs in sum notation:

$$\begin{aligned}
n\beta' &= \bar{r}\mathbf{1}'\text{diag}(K)^{-1}G+(1-\bar{r})\,\beta'\left(\text{diag}(\overline{K})+\mathbf{1}\mathbf{1}'\text{diag}(K)^{-1}G-G'\text{diag}(K)^{-1}G\right)\\
n\beta_j &= \bar{r}\sum_{i=1}^{n}\frac{g_{ij}}{K_i}+(1-\bar{r})\left[\beta_j\overline{K}_j+\sum_{i=1}^{n}\frac{g_{ij}}{K_i}-\sum_{i=1}^{n}\sum_{s=0}^{n}\frac{\beta_s g_{is}g_{ij}}{K_i}\right]\\
\beta_j &= \frac{\bar{r}}{n}\sum_{i=1}^{n}\frac{g_{ij}}{K_i}+\frac{(1-\bar{r})}{n}\left[\beta_j\overline{K}_j+\sum_{i=1}^{n}\frac{g_{ij}}{K_i}-\sum_{i=1}^{n}\sum_{s=0}^{n}\frac{\beta_s g_{is}g_{ij}}{K_i}\right]
\end{aligned}$$

where $\overline{K}_j=\sum_{i=1}^{n}g_{ij}$ and $K_i=\sum_{j=0}^{n}g_{ij}$.

B Proof of the two monotonicity results

Here we present the construction of the expected payoff of the agent once she acknowledges her own influence on the average action. Such construction is useful for both the proof of Lemma 1 and Lemma 2.

$$\begin{aligned}
\text{Payoff} &= -\mathbb{E}\left[r\left(a_i-\theta\right)^2+(1-r)\left(a_i-\overline{a}\right)^2|\mathbb{G}_{-i}\right]-c\left(G\right)\\
&= -\mathbb{E}\left[r\left(a_i-\theta\right)^2+(1-r)\left(\tfrac{n-1}{n}\right)^2\left(a_i-\overline{a_{-i}}\right)^2|\mathbb{G}_{-i}\right]-c\left(G\right)\\
&= -\left(r+(1-r)\left(\tfrac{n-1}{n}\right)^2\right)\mathbb{E}\left[\frac{r}{r+(1-r)\left(\frac{n-1}{n}\right)^2}\left(a_i-\theta\right)^2+\left(\frac{(1-r)\left(\frac{n-1}{n}\right)^2}{r+(1-r)\left(\frac{n-1}{n}\right)^2}\right)\left(a_i-\overline{a_{-i}}\right)^2|\mathbb{G}_{-i}\right]-c\left(G\right)
\end{aligned}$$

Using a monotonic transformation of the payoff:

$$\begin{aligned}
\text{Payoff} &= -\mathbb{E}\left[\frac{r}{r+(1-r)\left(\frac{n-1}{n}\right)^2}\left(a_i-\theta\right)^2|\mathbb{G}_{-i}\right]-\mathbb{E}\left[\left(\frac{(1-r)\left(\frac{n-1}{n}\right)^2}{r+(1-r)\left(\frac{n-1}{n}\right)^2}\right)\left(a_i-\overline{a_{-i}}\right)^2|\mathbb{G}_{-i}\right]-c\left(G\right)\\
&= -\frac{r}{r+(1-r)\left(\frac{n-1}{n}\right)^2}\mathbb{E}\left[\left(a_i-\theta\right)^2|\mathbb{G}_{-i}\right]-\left(\frac{(1-r)\left(\frac{n-1}{n}\right)^2}{r+(1-r)\left(\frac{n-1}{n}\right)^2}\right)\mathbb{E}\left[\left(a_i-\overline{a_{-i}}\right)^2|\mathbb{G}_{-i}\right]-c\left(G\right)\\
\text{Payoff} &= -r'\underbrace{\mathbb{E}\left[\left(a_i-\theta\right)^2|\mathbb{G}_{-i}\right]}_{\text{First Term}}-(1-r')\underbrace{\mathbb{E}\left[\left(a_i-\overline{a_{-i}}\right)^2|\mathbb{G}_{-i}\right]}_{\text{Second Term}}-c\left(G\right)
\end{aligned}$$

where $\overline{a_{-i}}$ is the average action played by the other agents, ignoring agent i's action. We will proceed by working with the two terms separately. Before that, we compute the agents optimal action and a couple of helpful elements.

Optimal Action

Agent Optimization F.O.C.

$$\begin{aligned} 0 &= \mathbb{E}\left[r'\left(a_i-\theta\right)+\left(1-r'\right)\left(a_i-\overline{a_{-i}}\right)|\mathbb{I}_i\right]-c\left(G\right) \\ a_i &= r'\mathbb{E}\left[\theta|\mathbb{I}_i\right]+(1-r')\mathbb{E}\left[\overline{a_{-i}}|\mathbb{I}_i\right] \end{aligned}$$

Expected Value of State

$$\mathbb{E}\left[\theta|\mathbb{I}_i\right] = \frac{\sum_{j=0}^{n} g_{i,j}e_j}{K+1}$$

Expected Value of Average Action

$$\begin{aligned} \mathbb{E}\left[\overline{a_{-i}}|\mathbb{I}_i\right] &= \sum_{j=0}^{n}\beta_{-i,j}\left(g_{i,j}e_j+\left(1-g_{i,j}\right)\tfrac{1}{K+1}\sum_{s=0}^{n} g_{i,s}e_s\right) \\ &= \sum_{j=0}^{n}\beta_{-i,j}g_{i,j}e_j+\sum_{j=0}^{n}\beta_{-i,j}\left(1-g_{i,j}\right)\tfrac{\sum_{s=0}^{n} g_{i,s}e_s}{K+1} \\ &= \sum_{j=0}^{n} g_{i,j}\underbrace{\left[\beta_{-i,j}+\frac{\sum_{s=0}^{n}\left(1-g_{i,s}\right)\beta_{i,s}}{K+1}\right]}_{M_j}e_j \\ &= \sum_{j=0}^{n} g_{i,j}M_je_j \end{aligned}$$

First Term Start by observing that $\mathbb{E}\left[a_i-\theta|\mathbb{G}_i\right]=0$, since $\sum_{j=0}^{n} g_{i,j}M_j=1$. Thus we have:

$$\mathbb{E}\left[\left(a_i-\theta\right)^2|\mathbb{G}_{-i}\right]=\mathbb{V}\left[\mathbb{E}\left[\left(a_i-\theta|\mathbb{I}_i\right)\right]|\mathbb{G}_{-i}\right]+\mathbb{E}\left[\mathbb{V}\left(a_i-\theta|\mathbb{I}_i\right)|\mathbb{G}_{-i}\right]$$

substituting the optimal action

$$=\mathbb{V}\left[\sum_{j=0}^{n} g_{i,j}\left(\frac{r'}{K+1}+\left(1-r'\right)M_j-\frac{1}{K+1}\right)e_j|\mathbb{G}_{-i}\right]+\frac{\sigma_{ep}^2}{K+1}$$

colecting the terms and using the fact that $e_0=0$

$$=\mathbb{V}\left[\sum_{j=1}^{n} g_{i,j}\left(1-r'\right)\left(-\frac{1}{K+1}+M_j\right)e_j|\mathbb{G}_{-i}\right]+\frac{\sigma_{ep}^2}{K+1}$$

Using the known correlation between error terms of the signals

$$= (1-r')^2 \left[\sum_{j=1}^{n} g_{i,j} \left(M_j - \frac{1}{K+1} \right)^2 \left(\frac{\sigma_{ep}^2}{K+1} + \sigma_{ep}^2 \right) + \right.$$

$$\left. \sum_{j=1}^{n} \sum_{s \neq j} g_{i,j} g_{i,s} \left(M_j - \frac{1}{K+1} \right) \left(M_s - \frac{1}{K+1} \right) \frac{\sigma_{ep}^2}{K+1} \right] + \frac{\sigma_{ep}^2}{K+1}$$

colecting the terms

$$= (1-r')^2 \left[\sum_{j=1}^{n} g_{i,j} \left(M_j - \frac{1}{K+1} \right)^2 \sigma_{ep}^2 + \right.$$

$$\left. \sum_{j=1}^{n} \sum_{s=1}^{n} g_{i,j} g_{i,s} \left(M_j - \frac{1}{K+1} \right) \left(M_s - \frac{1}{K+1} \right) \frac{\sigma_{ep}^2}{K+1} \right] + \frac{\sigma_{ep}^2}{K+1}$$

Finally

$$\mathbb{E}\left[(a_i - \theta)^2 | \mathbb{G}_{-i} \right] = (1-r')^2 \sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j} \left(M_j - \frac{1}{K+1} \right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1} \right) \right) +$$

$$\frac{\sigma_{ep}^2}{K+1}$$

Second Term Let's now work on the second term.

$$\mathbb{E}\left[(a_i - \overline{a_{-i}})^2 | \mathbb{G}_{-i} \right] = \mathbb{V}\left[\mathbb{E}\left[(a_i - \overline{a_{-i}} | \mathbb{I}_i) \right] | \mathbb{G}_{-i} \right] + \mathbb{E}\left[\mathbb{V}\left(a_i - \overline{a_{-i}} | \mathbb{I}_i \right) | \mathbb{G}_{-i} \right]$$

substing the optimal action

$$= \mathbb{V}\left[\sum_{j=0}^{n} g_{i,j} \left(\frac{r'}{K+1} + (1-r') M_j - M_j \right) e_j \right] + \mathbb{E}\left[\mathbb{V}\left[\sum_{j=0}^{n} (1-g_{i,j}) \beta_{-i,j} e_j \right] \right]$$

colecting the terms and using the fact that $e_0 = 0$

$$= \mathbb{V}\left[\sum_{j=1}^{n} g_{i,j} r' \left(\frac{1}{K+1} - M_j\right) e_j\right] + \mathbb{E}\left[\mathbb{V}\left[\sum_{j=1}^{n} \left(1 - g_{i,j}\right) \beta_{-i,j} e_j\right]\right]$$

Using the known correlation between error terms of the signals

$$= r'^2 \Bigg[\sum_{j=1}^{n} g_{i,j} \left(M_j - \frac{1}{K+1}\right)^2 \left(\frac{\sigma_{ep}^2}{K+1} + \sigma_{ep}^2\right) +$$

$$\sum_{j=1}^{n} \sum_{s \neq j} g_{i,j} g_{i,s} \left(M_j - \frac{1}{K+1}\right) \left(M_s - \frac{1}{K+1}\right) \frac{\sigma_{ep}^2}{K+1}\Bigg] +$$

$$\Bigg[\sum_{j=1}^{n} \left(1 - g_{i,j}\right) \beta_{-i,j}^2 \left(\frac{\sigma_{ep}^2}{K+1} + \sigma_{ep}^2\right) +$$

$$\sum_{j=1}^{n} \sum_{s \neq j} \left(1 - g_{i,j}\right) \left(1 - g_{i,s}\right) \beta_{-i,j} \beta_{-i,s} \frac{\sigma_{ep}^2}{K+1}\Bigg]$$

colecting the terms we have

$$\mathbb{E}\left[\left(a_i - \overline{a_{-i}}\right)^2 | \mathbb{G}_{-i}\right] = r'^2 \sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j} \left(M_j - \frac{1}{K+1}\right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1}\right)\right)$$

$$+ \sigma_{ep}^2 \Bigg[\sum_{j=1}^{n} \left(1 - g_{i,j}\right) \beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} \left(1 - g_{i,s}\right) \beta_{-i,s}\right)\Bigg]$$

working only on the second part, we can rewrite it as

$$= r'^2 \sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j} \left(M_j - \frac{1}{K+1}\right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1}\right)\right)$$

$$- \sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j} \beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} \left(1 - g_{i,s}\right) \beta_{-i,s}\right)$$

$$+ \sigma_{ep}^2 \sum_{j=1}^{n} \beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} \left(1 - g_{i,s}\right) \beta_{-i,s}\right)$$

simplifying some more

$$= r'^2 \sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j} \left(M_j - \frac{1}{K+1} \right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1} \right) \right)$$

$$- \sigma_{ep}^2 \Bigg[\sum_{j=1}^{n} g_{i,j} \beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} (1 - g_{i,s}) \beta_{-i,s} \right)$$

$$+ \sum_{j=1}^{n} \left(\beta_{-i,j}^2 + \frac{1}{K+1} \beta_{-i,j} \left(\sum_{s=1}^{n} \beta_{-i,s} - \sum_{s=1}^{n} g_{i,s} \beta_{-i,s} \right) \right) \Bigg]$$

Which finally gives us

$$\mathbb{E} \left[(a_i - \overline{a_{-i}})^2 | \mathbb{G}_{-i} \right] = r'^2 \sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j} \left(M_j - \frac{1}{K+1} \right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1} \right) \right)$$

$$- \sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j} \beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s} (1 - g_{i,s}) \beta_{-i,s} + \frac{1}{K+1} \sum_{s=1}^{n} \beta_{-i,s} \right)$$

$$+ \sigma_{ep}^2 \sum_{j=1}^{n} \left(\beta_{-i,j}^2 + \frac{1}{K+1} \beta_{-i,j} \sum_{s=1}^{n} \beta_{-i,s} \right)$$

Third Step Lets combine the two terms above and rewrite the payoff function. Observe that the last part of the first and second term is constant and do not depend on the connections the individual have made, $g_{i,j}$.

$$\text{Payoff} = - \left[r' \underbrace{\mathbb{E} \left[(a_i - \theta)^2 | \mathbb{G}_{-i} \right]}_{\text{First Term}} + (1 - r') \underbrace{\mathbb{E} \left[(a_i - \overline{a_{-i}})^2 | \mathbb{G}_{-i} \right]}_{\text{Second Term}} \right] - c(G)$$

using the calculations above

$$= -\Bigg[r'\left(1-r'\right)^2 \sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j}\left(M_j - \frac{1}{K+1}\right)\left(M_j - \frac{1}{K+1} + \frac{1}{K+1}\sum_{s=1}^{n} g_{i,s}\left(M_s - \frac{1}{K+1}\right)\right) +$$

$$r'\frac{\sigma_{ep}^2}{K+1} + (1-r')r'^2\sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j}\left(M_j - \frac{1}{K+1}\right)\left(M_j - \frac{1}{K+1} + \frac{1}{K+1}\sum_{s=1}^{n} g_{i,s}\left(M_s - \frac{1}{K+1}\right)\right)$$

$$-(1-r')\sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j}\beta_{-i,j}\left(\beta_{-i,j} + \frac{1}{K+1}\sum_{s}^{n}\left(1-g_{i,s}\right)\beta_{-i,s} + \frac{1}{K+1}\sum_{s=1}^{n}\beta_{-i,s}\right)$$

$$+(1-r')\sigma_{ep}^2 \sum_{j=1}^{n} \beta_{-i,j}\left(\beta_{-i,j} + \frac{1}{K+1}\beta_{-i,j}\sum_{s=1}^{n}\beta_{-i,s}\right)\Bigg] - c\left(G\right)$$

collecting terms

$$= -\sigma_{ep}^2\Bigg[r'\left(1-r'\right)^2 \sum_{j=1}^{n} g_{i,j}\left(M_j - \frac{1}{K+1}\right)\left(M_j - \frac{1}{K+1} + \frac{1}{K+1}\sum_{s=1}^{n} g_{i,s}\left(M_s - \frac{1}{K+1}\right)\right) +$$

$$+(1-r')r'^2 \sum_{j=1}^{n} g_{i,j}\left(M_j - \frac{1}{K+1}\right)\left(M_j - \frac{1}{K+1} + \frac{1}{K+1}\sum_{s=1}^{n} g_{i,s}\left(M_s - \frac{1}{K+1}\right)\right)$$

$$-\left(1-r'\right) \sum_{j=1}^{n} g_{i,j}\beta_{-i,j}\left(\beta_{-i,j} + \frac{1}{K+1}\sum_{s=1}^{n}\left(1-g_{i,s}\right)\beta_{-i,s} + \frac{1}{K+1}\sum_{s=1}^{n}\beta_{-i,s}\right)$$

$$+\left(1-r'\right) \sum_{j=1}^{n} \beta_{-i,j}\left(\beta_{-i,j} + \frac{1}{K+1}\beta_{-i,j}\sum_{s=1}^{n}\beta_{-i,s}\right) + r'\frac{1}{K+1}\Bigg] - c\left(G\right)$$

separating terms that depend on conections and terms that do not

$$= -\sigma_{ep}^2 \Bigg[\sum_{j=1}^{n} g_{i,j} \Bigg[r' \left(1-r'\right)^2 \left(M_j - \frac{1}{K+1} \right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1} \right) \right) +$$

$$+ (1-r')r'^2 \left(M_j - \frac{1}{K+1} \right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1} \right) \right)$$

$$- (1-r')\beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} \left(1-g_{i,s}\right) \beta_{-i,s} + \frac{1}{K+1} \sum_{s=1}^{n} \beta_{-i,s} \right) \Bigg]$$

$$+ (1-r') \sum_{j=1}^{n} \beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \beta_{-i,j} \sum_{s=1}^{n} \beta_{-i,s} \right) + r' \frac{1}{K+1} \Bigg] - c\left(G\right)$$

thus we have

$$\text{Payoff} = -\sigma_{ep}^2 \Bigg[\sum_{j=1}^{n} g_{i,j} \Bigg[- (1-r')\beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} \left(1-g_{i,s}\right) \beta_{-i,s} + \frac{1}{K+1} \sum_{s=1}^{n} \beta_{-i,s} \right)$$

$$+ \left(r' \left(1-r'\right)^2 + (1-r')r'^2 \right) \left(M_j - \frac{1}{K+1} \right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1} \right) \right) \Bigg]$$

$$+ (1-r') \sum_{j=1}^{n} \beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \beta_{-i,j} \sum_{s=1}^{n} \beta_{-i,s} \right) + r' \frac{1}{K+1} \Bigg] - c\left(G\right)$$

and finnally:

$$\text{Payoff} = -\sigma_{ep}^2 \Bigg[\sum_{j=1}^{n} g_{i,j} \Bigg[- (1-r')\, \underbrace{\beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} (1-g_{i,s})\, \beta_{-i,s} + \frac{1}{K+1} \sum_{s=1}^{n} \beta_{-i,s} \right)}_{U_j^a}$$

$$+ \left(r'\,(1-r')^2 + (1-r')r'^2 \right) \underbrace{\left(M_j - \frac{1}{K+1} \right) \left(M_j - \frac{1}{K+1} + \frac{1}{K+1} \sum_{s=1}^{n} g_{i,s} \left(M_s - \frac{1}{K+1} \right) \right)}_{U_j^b} \Bigg]$$

$$+ (1-r') \sum_{j=1}^{n} \beta_{-i,j} \left(\beta_{-i,j} + \frac{1}{K+1} \sum_{s=1}^{n} \beta_{-i,s} \right) + r' \frac{1}{K+1} \Bigg] - c\,(G)$$

another way of writing it is

$$\text{Payoff} = -\sigma_{ep}^2 \sum_{j=1}^{n} g_{i,j} \left[- (1-r')U_j^a + \left(r'\,(1-r')^2 + (1-r')r'^2 \right) U_j^b \right] + cte - c\,(G)$$

Fourth Step Let‘s compare agent i's payoff in two distinct situations. In both situations agent i is connected to an arbitrary group of agents, call it A. In the first situation i is also connected to f but not to h ($g_{i,h} = 0$ and $g_{i,f} = 1$). The second situation presents the opposite. Agent i is connected to h but not to f ($g_{i,f} = 0$ and $g_{i,h} = 1$). Assume that agents f and h can be ranked by their impact on the average action, that is $\beta_{-i,f} > \beta_{-i,h}$.

$$\text{Payoff(1)-Payoff(2)} = \sigma_{ep}^2 \sum_{j \in A} \left[(1-r') \underbrace{\left(U_j^a(1) - U_j^a(2) \right)}_{\text{First Term}} - \left(r'\,(1-r')^2 + (1-r')r'^2 \right) \underbrace{\left(U_j^b(1) - U_j^b(2) \right)}_{\text{Second Term}} \right]$$

$$+ \sigma_{ep}^2 \left((1-r') \underbrace{\left(U_f^a(1) - U_h^a(2) \right)}_{\text{Third Term}} - \left(r'\,(1-r')^2 + (1-r')r'^2 \right) \underbrace{\left(U_f^b(1) - U_h^b(2) \right)}_{\text{Fourth Term}} \right)$$

$$\text{Payoff(1)-Payoff(2)} = \sigma_{ep}^2 \Bigg[(1-r') \Bigg(\sum_{j \in A} \left(U_j^a(1) - U_j^a(2) \right) + \left(U_f^a(1) - U_h^a(2) \right) \Bigg)$$

$$- \left(r' \left(1-r'\right)^2 + (1-r')r'^2 \right) \Bigg(\sum_{j \in A} \left(U_j^b(1) - U_j^b(2) \right) + \left(U_f^b(1) - U_h^b(2) \right) \Bigg) \Bigg]$$

working with the first term

$$\begin{aligned}
U_j^a(1) - U_j^a(2) &= \frac{\beta_{-i,j}}{K+1} \left(\sum_{s=1}^{n} \left((1 - g_{i,s}(1)) - (1 - g_{i,s}(2)) \beta_{-i,s} \right) \right) \\
&= \frac{\beta_{-i,j}}{K+1} \left(\beta_{-i,h} - \beta_{-i,f} \right) \\
&= -\frac{\beta_{-i,f} - \beta_{-i,h}}{K+1} \beta_{-i,j} \\
&= -\frac{M_f(1) - M_h(2)}{K} \beta_{-i,j} \\
&= -\left(M_f(1) - M_h(2) \right) \frac{\beta_{-i,j}}{K}
\end{aligned}$$

observe that although the term depends on which individual j we are talking about the dependence will disapear once we sum on all j's

$$\sum_{j \in A} \left(U_j^a(1) - U_j^a(2) \right) = -\frac{M_f(1) - M_h(2)}{K} \sum_{s \in A} \beta_{-i,s}$$

working with the third term

$$
\begin{aligned}
U_f^a(1) - U_h^a(2) &= \beta_{-i,f}^2 - \beta_{-i,h}^2 + \frac{(\beta_{-i,f} - \beta_{-i,h})}{K+1}\left(2\sum_{s=1}^{n}\beta_{-i,s} - \sum_{s\in A}\beta_{-i,s}\right) - \frac{\beta_{-i,f}^2 - \beta_{-i,h}^2}{K+1} \\
&= \frac{(\beta_{-i,f} - \beta_{-i,h})}{K+1}\left(K\beta_{-i,f} + K\beta_{-i,h} + 2\sum_{s=1}^{n}\beta_{-i,s} - \sum_{s\in A}\beta_{-i,s}\right) \\
&= \frac{(M_f(1) - M_h(2))}{K}\left(K(\beta_{-i,f} + \beta_{-i,h}) + 2\sum_{s=1}^{n}\beta_{-i,s} - \sum_{s\in A}\beta_{-i,s}\right)
\end{aligned}
$$

working the first and the third term

$$
U_f^a(1) - U_h^a(2) + \sum_{j\in A}\left(U_j^a(1) - U_j^a(2)\right) = \frac{M_f(1) - M_h(2)}{K}\left(K(\beta_{-i,f} + \beta_{-i,h}) + 2\left(\sum_{s=1}^{n}\beta_{-i,s} - \sum_{s\in A}\beta_{-i,s}\right)\right)
$$

Working with the second term will be more complicated. We apologize to the reader and ask to bear with us a little bit longer

$$
\begin{aligned}
U_j^b &= \left(M_j - \frac{1}{K+1}\right)\left(M_j - \frac{1}{K+1} + \frac{1}{K+1}\sum_{s=1}^{n} g_{i,s}\left(M_s - \frac{1}{K+1}\right)\right) \\
&= \tilde{M}_j\left(\tilde{M}_j + \frac{1}{K+1}\sum_{s=1}^{n} g_{i,s}\tilde{M}_s\right) \\
&= \tilde{M}_j^{\,2} + \frac{1}{K+1}\sum_{s=1}^{n} g_{i,s}\tilde{M}_j\tilde{M}_s
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_j &= M_j - \frac{1}{K+1} \\
&= \beta_{-i,j} + \frac{1}{K+1}\sum_{l=1}^{n}(1-g_{i,l})\beta_{-i,l} - \frac{1}{K+1} \\
&= \beta_{-i,j} + \frac{1}{K+1}\sum_{l=0}^{n}(1-g_{i,l})\beta_{-i,l} - \frac{1}{K+1} \\
&= \beta_{-i,j} + \frac{1}{K+1}\sum_{l=0}^{n}\beta_{-i,l} - \frac{1}{K+1}\sum_{l=0}^{n}g_{i,l}\beta_{-i,l} - \frac{1}{K+1} \\
&= \beta_{-i,j} - \frac{1}{K+1}\sum_{l=0}^{n}g_{i,l}\beta_{-i,l} \\
\tilde{M}_j(1) &= \beta_{-i,j} - \frac{1}{K+1}\sum_{l=0}^{n}g_{i,l}(1)\beta_{-i,l} \\
\tilde{M}_j(1) &= \beta_{-i,j} - \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l} - \frac{1}{K+1}\beta_{-i,f}
\end{aligned}
$$

where $\tilde{A} = A \cup 0$

$$
\begin{aligned}
U_j^b(1) - U_j^b(2) &= \tilde{M}_j^{\,2}(1) - \tilde{M}_j^{\,2}(2) + \frac{1}{K+1}\sum_{s=1}^{n}\Big(g_{i,s}(1)\tilde{M}_j(1)\tilde{M}_s(1) - g_{i,s}(2)\tilde{M}_j(2)\tilde{M}_s(2)\Big) \\
&= \tilde{M}_j^{\,2}(1) - \tilde{M}_j^{\,2}(2) + \frac{1}{K+1}\left(\tilde{M}_j(1)\sum_{s=1}^{n} g_{i,s}(1)\tilde{M}_s(1) - \tilde{M}_j(2)\sum_{s=1}^{n} g_{i,s}(2)\tilde{M}_s(2)\right) \\
&= \tilde{M}_j^{\,2}(1) - \tilde{M}_j^{\,2}(2) + \frac{1}{K+1}\Big(\tilde{M}_j(1)(-\tilde{M}_0(1)) - \tilde{M}_j(2)(-\tilde{M}_0(2))\Big) \\
&= \tilde{M}_j^{\,2}(1) - \tilde{M}_j^{\,2}(2) + \frac{1}{K+1} \\
&\quad \left(\Big(\tilde{M}_j(1) - \tilde{M}_j(2)\Big)\left(-\beta_{-i,0} + \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) + \frac{1}{K+1}\beta_{-i,f}\tilde{M}_j(1) - \frac{1}{K+1}\beta_{-i,h}\tilde{M}_j(2)\right) \\
&= \tilde{M}_j^{\,2}(1) - \tilde{M}_j^{\,2}(2) + \frac{1}{K+1}\Big(\tilde{M}_j(1) - \tilde{M}_j(2)\Big)\left(-\beta_{-i,0} + \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) \\
&\quad + \frac{1}{(K+1)^2}\left(\beta_{-i,f} - \beta_{-i,h}\right)\left(\beta_{-i,j} - \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) + \frac{1}{(K+1)^3}\left(-\beta_{-i,f}^2 + \beta_{-i,h}^2\right) \\
&= \Big(\tilde{M}_j(1) - \tilde{M}_j(2)\Big)\left(\tilde{M}_j(1) + \tilde{M}_j(2) - \frac{1}{K+1}\beta_{-i,0} + \frac{1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) \\
&\quad + \frac{1}{(K+1)^2}\left(\beta_{-i,f} - \beta_{-i,h}\right)\left(\beta_{-i,j} - \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l} - \frac{1}{K+1}\left(\beta_{-i,f} + \beta_{-i,h}\right)\right) \\
&= \Big(\tilde{M}_j(1) - \tilde{M}_j(2)\Big)\left(\tilde{M}_j(1) + \tilde{M}_j(2) - \frac{1}{K+1}\beta_{-i,0} + \frac{1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) \\
&\quad - \frac{1}{(K+1)}\Big(\tilde{M}_j(1) - \tilde{M}_j(2)\Big)\left(\beta_{-i,j} - \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l} - \frac{1}{K+1}\left(\beta_{-i,f} + \beta_{-i,h}\right)\right)
\end{aligned}
$$

$$U_j^b(1) - U_j^b(2) = \left(\tilde{M}_j(1) - \tilde{M}_j(2)\right)\left(\tilde{M}_j(1) + \tilde{M}_j(2) - \frac{1}{K+1}\beta_{-i,0} + \frac{1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l}\right)$$

$$- \frac{1}{(K+1)}\left(\tilde{M}_j(1) - \tilde{M}_j(2)\right)\left(\beta_{-i,j} - \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l} - \frac{1}{K+1}\left(\beta_{-i,f} + \beta_{-i,h}\right)\right)$$

$$= \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K}\left(-\tilde{M}_j(1) - \tilde{M}_j(2) + \frac{1}{K+1}\beta_{-i,0} - \frac{1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l}\right)$$

$$+ \frac{1}{(K+1)}\frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K}\left(\beta_{-i,j} - \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l} - \frac{1}{K+1}\left(\beta_{-i,f} + \beta_{-i,h}\right)\right)$$

$$= \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K}\Bigg(- \tilde{M}_j(1) - \tilde{M}_j(2) + \frac{1}{K+1}\beta_{-i,0} - \frac{1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l}$$

$$+ \frac{1}{(K+1)}\beta_{-i,j} - \frac{1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l} - \frac{1}{(K+1)^2}\left(\beta_{-i,f} + \beta_{-i,h}\right)\Bigg)$$

$$= \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K}\Bigg(- 2\beta_{-i,j} + \frac{2}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l} + \frac{1}{K+1}\left(\beta_{-i,f} + \beta_{-i,h}\right) + \frac{1}{K+1}\beta_{-i,0}$$

$$- \frac{1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l} + \frac{1}{(K+1)}\beta_{-i,j} - \frac{1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l} - \frac{1}{(K+1)^2}\left(\beta_{-i,f} + \beta_{-i,h}\right)\Bigg)$$

$$= \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K}\Bigg(- \frac{2K+1}{(K+1)}\beta_{-i,j} + \frac{2K}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l} + \frac{K}{(K+1)^2}\left(\beta_{-i,f} + \beta_{-i,h}\right) + \frac{1}{K+1}$$

observe that although the term depends on which individual j we are talking about the dependence will disapear once we sum on all j's

$$\sum_{j\in A}\left(U_j^b(1) - U_j^b(2)\right) = \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K}$$

$$\Bigg(- \frac{2K+1}{(K+1)}\sum_{s\in A}\beta_{-i,s} + \frac{2K(K-1)}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l} + \frac{K(K-1)}{(K+1)^2}\left(\beta_{-i,f} + \beta_{-i,h}\right) + \frac{K-1}{K+1}\beta$$

Working with the fourth term

$$
\begin{aligned}
U_f^b(1) - U_h^b(2) &= \tilde{M_f}^2(1) - \tilde{M_h}^2(2) + \frac{1}{K+1}\sum_{s=1}^{n}\left(g_{i,s}(1)\tilde{M}_f(1)\tilde{M}_s(1) - g_{i,s}(2)\tilde{M}_h(2)\tilde{M}_s(2)\right) \\
&= \tilde{M_f}^2(1) - \tilde{M_h}^2(2) + \frac{1}{K+1}\left(\tilde{M}_f(1)\sum_{s=1}^{n} g_{i,s}(1)\tilde{M}_s(1) - \tilde{M}_h(2)\sum_{s=1}^{n} g_{i,s}(2)\tilde{M}_s(2)\right) \\
&= \tilde{M_f}^2(1) - \tilde{M_h}^2(2) + \frac{1}{K+1}\left(\tilde{M}_f(1)(-\tilde{M}_0(1)) - \tilde{M}_h(2)(-\tilde{M}_0(2))\right) \\
&= \tilde{M_f}^2(1) - \tilde{M_h}^2(2) + \frac{1}{K+1} \\
&\quad \left(\left(\tilde{M}_f(1) - \tilde{M}_h(2)\right)\left(-\beta_{-i,0} + \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) + \frac{1}{K+1}\beta_{-i,f}\tilde{M}_f(1) - \frac{1}{K+1}\beta_{-i,h}\tilde{M}_h(2) \right. \\
&= \tilde{M_f}^2(1) - \tilde{M_h}^2(2) + \frac{1}{K+1}\left(\tilde{M}_f(1) - \tilde{M}_h(2)\right)\left(-\beta_{-i,0} + \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) \\
&\quad + \frac{1}{(K+1)^2}\left(\beta_{-i,f}^2 - \beta_{-i,h}^2\right) + \frac{1}{(K+1)^2}\left(\beta_{-i,f} - \beta_{-i,h}\right)\left(-\frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) + \frac{1}{(K+1)^3}\left(-\right. \\
&= \tilde{M_f}^2(1) - \tilde{M_h}^2(2) + \frac{1}{K+1}\left(\tilde{M}_f(1) - \tilde{M}_h(2)\right)\left(-\beta_{-i,0} + \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) \\
&\quad + \left(\beta_{-i,f} - \beta_{-i,h}\right)\left(\left(\frac{1}{(K+1)^2} - \frac{1}{(K+1)^3}\right)\left(\beta_{-i,f} + \beta_{-i,h}\right) - \frac{1}{(K+1)^3}\sum_{l\in\tilde{A}}\beta_{-i,l}\right) \\
&= \left(\tilde{M}_f(1) - \tilde{M}_h(2)\right)\left(\tilde{M}_f(1) + \tilde{M}_h(2) + \frac{1}{K+1}\left(-\beta_{-i,0} + \frac{1}{K+1}\sum_{l\in\tilde{A}}\beta_{-i,l}\right)\right) \\
&\quad + \frac{1}{(K+1)^3}\left(\beta_{-i,f} - \beta_{-i,h}\right)\left(K\left(\beta_{-i,f} + \beta_{-i,h}\right) - \sum_{l\in\tilde{A}}\beta_{-i,l}\right)
\end{aligned}
$$

$$
\begin{aligned}
U_f^b(1) - U_h^b(2) &= \left(\tilde{M}_f(1) - \tilde{M}_h(2)\right) \left(\tilde{M}_f(1) + \tilde{M}_h(2) - \frac{1}{K+1}\beta_{-i,0} + \frac{1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} \right) \\
&\quad + \frac{1}{(K+1)^2} \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K} \left(K \left(\beta_{-i,f} + \beta_{-i,h}\right) - \sum_{l \in \tilde{A}} \beta_{-i,l} \right) \\
&= \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K} \left(K\tilde{M}_f(1) + K\tilde{M}_h(2) - \frac{K}{K+1}\beta_{-i,0} + \frac{K}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} \right) \\
&\quad + \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K} \left(\frac{K}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h}\right) - \frac{1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} \right) \\
&= \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K} \left(\frac{K^2}{K+1} \left(\beta_{-i,f} + \beta_{-i,h}\right) - \frac{2K}{K+1} \sum_{l \in \tilde{A}} \beta_{-i,l} - \frac{K}{K+1}\beta_{-i,0} + \frac{K}{(K+1)^2} \sum_{l \in \tilde{A}} \beta \right. \\
&\quad + \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K} \left(\frac{K}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h}\right) - \frac{1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} \right) \\
U_f^b(1) - U_h^b(2) &= \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K} \left(\frac{K^3 + K^2 + K}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h}\right) - \frac{2K^2 + K + 1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} - \frac{K}{K+1}\beta_{-i,0} \right)
\end{aligned}
$$

Using the second and fourth term:

$$
\begin{aligned}
U_f^b(1) - U_h^b(2) + \sum_{j \in A} \left(U_j^b(1) - U_j^b(2) \right) &= \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K} \\
&\left(- \frac{2K+1}{(K+1)} \sum_{s \in A} \beta_{-i,s} + \frac{2K(K-1)}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} + \frac{K(K-1)}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h} \right) + \frac{K-1}{K+1} \beta \right. \\
&\left. + \left(\frac{K^3+K^2+K}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h} \right) - \frac{2K^2+K+1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} - \frac{K}{K+1} \beta_{-i,0} \right) \right) \\
&= \frac{\tilde{M}_f(1) - \tilde{M}_h(2)}{K} \\
&\left(- \frac{2K+1}{(K+1)} \sum_{s \in A} \beta_{-i,s} - \frac{3K+1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} + \frac{K^2(K+2)}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h} \right) - \frac{1}{K+1} \beta_{-} \right. \\
&= \frac{M_f(1) - M_h(2)}{K} \\
&\left(- \frac{2K+1}{(K+1)} \sum_{s \in A} \beta_{-i,s} - \frac{3K+1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} + \frac{K^2(K+2)}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h} \right) - \frac{1}{K+1} \beta_{-} \right.
\end{aligned}
$$

and finnaly the payoff difference is

$$
\begin{aligned}
\Pi(1) - \Pi(2) &= Payoff(1) - Payoff(2) \\
&= \sigma_{ep}^2 \frac{M_f(1) - M_h(2)}{K} \left((1-r') \left(K(\beta_{-i,f} + \beta_{-i,h}) + 2 \left(\sum_{s=1}^{n} \beta_{-i,s} - \sum_{s \in A} \beta_{-i,s} \right) \right) \right. \\
&\quad - \left(r'(1-r')^2 + (1-r')r'^2 \right) \\
&\quad \left. \left(- \frac{2K+1}{(K+1)} \sum_{s \in A} \beta_{-i,s} - \frac{3K+1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} + \frac{K^2(K+2)}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h} \right) - \frac{1}{K+1} \beta_{-i,0} \right) \right) \\
&= \sigma_{ep}^2 \frac{M_f(1) - M_h(2)}{K} \left((1-r') \left(K(\beta_{-i,f} + \beta_{-i,h}) + 2 \left(\sum_{s=1}^{n} \beta_{-i,s} - \sum_{s \in A} \beta_{-i,s} \right) \right) \right. \\
&\quad + \left(r'(1-r')^2 + (1-r')r'^2 \right) \\
&\quad \left. \left(\frac{2K+1}{(K+1)} \sum_{s \in A} \beta_{-i,s} + \frac{3K+1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} - \frac{K^2(K+2)}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h} \right) + \frac{1}{K+1} \beta_{-i,0} \right) \right)
\end{aligned}
$$

C First Fact

Proposition. *Fact One: For any set A,* $\Pi(1) - \Pi(2) > 0$

Proof. Let's begin by rewriting the payoff difference.

$$\Pi(1) - \Pi(2) = \sigma^2_{ep}\frac{M_f(1) - M_h(2)}{K}\Bigg((1-r')\Bigg((K+2)(\beta_{-i,f} + \beta_{-i,h}) + 2\Bigg(\sum_{s=1}^{n}\beta_{-i,s} - (\beta_{-i,f} + \beta_{-i,h}) - \sum_{s\in A}$$

$$+ \left(r'(1-r')^2 + (1-r')r'^2\right)$$

$$\left(\frac{2K+1}{(K+1)}\sum_{s\in A}\beta_{-i,s} + \frac{3K+1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l} - \frac{K^2(K+2)}{(K+1)^2}(\beta_{-i,f} + \beta_{-i,h}) + \frac{1}{K+1}\beta_{-i,0}\right)\Bigg)$$

since $1 - r' > (r'(1-r')^2 + (1-r')r'^2)$

$$\Pi(1) - \Pi(2) \geq \sigma^2_{ep}\frac{M_f(1) - M_h(2)}{K}\Bigg((1-r')2\left(\sum_{s=1}^{n}\beta_{-i,s} - (\beta_{-i,f} + \beta_{-i,h}) - \sum_{s\in A}\beta_{-i,s}\right)$$

$$(1-r')\left[(K+2) - \frac{K^2(K+2)}{(K+1)^2}\right](\beta_{-i,f} + \beta_{-i,h})$$

$$+ \left(r'(1-r')^2 + (1-r')r'^2\right)\left(\frac{2K+1}{(K+1)}\sum_{s\in A}\beta_{-i,s} + \frac{3K+1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l} + \frac{1}{K+1}\beta_{-i,0}\right)\Bigg)$$

$$= \sigma^2_{ep}\frac{M_f(1) - M_h(2)}{K}\Bigg((1-r')2\left(\sum_{s=1}^{n}\beta_{-i,s} - (\beta_{-i,f} + \beta_{-i,h}) - \sum_{s\in A}\beta_{-i,s}\right)$$

$$(1-r')\left[\frac{(K+2)(K+1)^2}{(K+1)^2} - \frac{K^2(K+2)}{(K+1)^2}\right](\beta_{-i,f} + \beta_{-i,h})$$

$$+ \left(r'(1-r')^2 + (1-r')r'^2\right)\left(\frac{2K+1}{(K+1)}\sum_{s\in A}\beta_{-i,s} + \frac{3K+1}{(K+1)^2}\sum_{l\in\tilde{A}}\beta_{-i,l} + \frac{1}{K+1}\beta_{-i,0}\right)\Bigg)$$

Notice that all terms in the final expression are positive, thus so it's its sum. □

D Second Fact

Proposition. *Fact Two: Consider a set A and a set A′ such that* $A \subset A'$, *where* $f, h \notin A'$. *Also, by the first fact proven above, if* $\beta_{-i,j} \in A'$, *but* $\beta_{-i,j} \notin A$, *we know that* $\beta_{-i,j} \leq \beta_{-i,k} \forall k \in A$. *Then* $[\Pi(1) - \Pi(2)]_A - [\Pi(1) - \Pi(2)]_{A'} > 0$

Proof. Once more we will begin by rewriting the payoff difference.

$$
\begin{aligned}
\Pi(1) - \Pi(2) = \sigma_{ep}^2 \frac{M_f(1) - M_h(2)}{K} & \Bigg((1 - r') \Bigg((K+2)(\beta_{-i,f} + \beta_{-i,h}) + 2 \Bigg(\sum_{s=1}^{n} \beta_{-i,s} - (\beta_{-i,f} + \beta_{-i,h}) - \sum_{s \in A} \beta \\
& + \left(r'(1-r')^2 + (1-r')r'^2 \right) \\
& \left(\frac{2K+1}{(K+1)} \sum_{s \in A} \beta_{-i,s} + \frac{3K+1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} - \frac{K^2(K+2)}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h} \right) + \frac{1}{K+1} \beta_{-i,0} \right) \Bigg) \\
= \sigma_{ep}^2 \frac{\beta_{-i,f} - \beta_{-i,h}}{K+1} & \Bigg((1 - r') \Bigg((K+2)(\beta_{-i,f} + \beta_{-i,h}) + 2 \Bigg(\sum_{s=1}^{n} \beta_{-i,s} - (\beta_{-i,f} + \beta_{-i,h}) - \sum_{s \in A} \beta_{-i,} \\
& + \left(r'(1-r')^2 + (1-r')r'^2 \right) \\
& \left(\frac{2K+1}{(K+1)} \sum_{s \in A} \beta_{-i,s} + \frac{3K+1}{(K+1)^2} \sum_{l \in \tilde{A}} \beta_{-i,l} - \frac{K^2(K+2)}{(K+1)^2} \left(\beta_{-i,f} + \beta_{-i,h} \right) + \frac{1}{K+1} \beta_{-i,0} \right) \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\Pi(1)-\Pi(2) = {} & \sigma_{ep}^2 \left(\beta_{-i,f}-\beta_{-i,h}\right) \Bigg((1-r') \\
& \left(\frac{K+2}{K+1}(\beta_{-i,f}+\beta_{-i,h}) + \frac{2}{K+1} \left(\sum_{s=1}^{n} \beta_{-i,s} - (\beta_{-i,f}+\beta_{-i,h}) - \sum_{s\in A} \beta_{-i,s} \right)\right) \\
& + \left(r'(1-r')^2 + (1-r')r'^2 \right) \\
& \left(\frac{2K+1}{(K+1)^2} \sum_{s\in A} \beta_{-i,s} + \frac{3K+1}{(K+1)^3} \sum_{l\in \tilde{A}} \beta_{-i,l} - \frac{K^2(K+2)}{(K+1)^3} \left(\beta_{-i,f}+\beta_{-i,h}\right) + \frac{1}{(K+1)^2}\beta_{-i,0} \right) \Bigg) \\
= {} & \sigma_{ep}^2 \left(\beta_{-i,f}-\beta_{-i,h}\right) \Bigg((1-r') \\
& \left(\frac{K+2}{K+1}(\beta_{-i,f}+\beta_{-i,h}) + \frac{2}{K+1} \left(\sum_{s=1}^{n} \beta_{-i,s} - (\beta_{-i,f}+\beta_{-i,h}) - \sum_{s\in A} \beta_{-i,s} \right)\right) \\
& + \left(r'(1-r')^2 + (1-r')r'^2 \right) \\
& \left(\frac{2K+1}{(K+1)^2} \sum_{s\in A} \beta_{-i,s} + \frac{3K+1}{(K+1)^3} \sum_{l\in A} \beta_{-i,l} - \frac{K^2(K+2)}{(K+1)^3} \left(\beta_{-i,f}+\beta_{-i,h}\right) + \frac{4K+2}{(K+1)^3}\beta_{-i,0} \right) \Bigg) \\
= {} & \sigma_{ep}^2 \left(\beta_{-i,f}-\beta_{-i,h}\right) \Bigg((1-r') \\
& \left(\frac{K+2}{K+1}(\beta_{-i,f}+\beta_{-i,h}) + \frac{2}{K+1} \left(\sum_{s=1}^{n} \beta_{-i,s} - (\beta_{-i,f}+\beta_{-i,h}) - \sum_{s\in A} \beta_{-i,s} \right)\right) \\
& + \left(r'(1-r')^2 + (1-r')r'^2 \right) \\
& \left(\frac{(2K+1)(K+1)+3K+1}{(K+1)^2} \frac{\sum_{s\in A}\beta_{-i,s}}{K+1} - \frac{K^3+2K^2}{(K+1)^3} \left(\beta_{-i,f}+\beta_{-i,h}\right) + \frac{4K+2}{(K+1)^3}\beta_{-i,0} \right) \Bigg)
\end{aligned}
$$

Finally observe that each term is decreasing in the cardinality of the set A. □