Revealed preferences over risk and uncertainty

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Abstract: Consider a finite data set where each observation consists of a bundle of contingent consumption chosen by an agent from a constraint set of such bundles. We develop a general procedure for testing the consistency of this data set with a broad class of models of choice under risk and under uncertainty. Unlike previous work, we do not require that the agent has a convex preference, so we allow for risk loving and elation seeking behavior. Our procedure can also be extended to calculate the magnitude of violations from a particular model of choice, using an index first suggested by Afriat (1972, 1973). We then apply this index to evaluate different models (including expected utility and disappointment aversion) in the data collected by Choi *et al.* (2007). We show that more than half of all subjects exhibiting choice behavior consistent with utility maximization are also consistent with models of expected utility and disappointment aversion.

Keywords: expected utility, rank dependent utility, maxmin expected utility, variational preferences, generalized axiom of revealed preference

JEL classification numbers: C14, C60, D11, D12, D81

1. Introduction

Let $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$ be a finite set, where $p^t \in \mathbb{R}^{\bar{s}}_{++}$ and $x^t \in \mathbb{R}^{\bar{s}}_{+}$. We interpret $\mathcal{O}$ as a set of observations, where x^t is the observed bundle of $\bar{s}$ goods chosen by an agent (the *demand bundle*) at the price vector p^t. A utility function $U : \mathbb{R}^{\bar{s}}_{+} \to \mathbb{R}$ is said to *rationalize* $\mathcal{O}$ if, at every observation t, x^t is the bundle that maximizes U in the budget set

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$B^t = \{x \in \mathbb{R}_+^{\bar{s}} : p^t \cdot x \leqslant p^t \cdot x^t\}$. For any data set that is rationalizable by a locally non-satiated utility function, its revealed preference relations must satisfy a no-cycling condition called the generalized axiom of revealed preference (GARP). Afriat's (1967) Theorem shows that any data set that obeys GARP will in turn be rationalizable by a utility function that is continuous, concave, and increasing in all dimensions. This result is very useful because it provides a non-parametric test of utility maximization that can be easily implemented in observational and experimental settings. It is known that GARP holds if and only if there is a solution to a set of linear inequalities constructed from the data; much applied work using Afriat's Theorem tests GARP by checking for a solution to this linear program.[1]

It is both useful and natural to develop tests, similar to the one developed by Afriat, for alternative hypotheses on agent behavior. Our objective in this paper is to develop a procedure that is useful for testing models of choice under risk and under uncertainty. Retaining the formal setting described in the previous paragraph, we can interpret $\bar{s}$ as the number of states of the world, with x^t a bundle of contingent consumption, and p^t the state prices faced by the agent. In a setting like this, we can ask what conditions on the data set are necessary and sufficient for it to be consistent with an agent who is maximizing an expected utility function. This means that x^t maximizes the agent's expected utility, compared to other bundles in the budget set. Assuming that the probability of state s is commonly known to be π_s, this involves recovering a Bernoulli utility function $u : \mathbb{R}_+ \to \mathbb{R}$, which we require to be increasing and continuous, so that, for each $t = 1, 2, \ldots, T$,

$$\sum_{s=1}^{\bar{s}} \pi_s u(x_s^t) \geqslant \sum_{s=1}^{\bar{s}} \pi_s u(x_s) \text{ for all } x \in B^t. \tag{1}$$

In the case where the state probabilities are subjective and unknown to the observer, it would be necessary to recover both u and $\{\pi_s\}_{s=1}^{\bar{s}}$ so that (1) holds.

In fact, tests of this sort have already been developed by Varian (1983) and Green and Srivastava (1986). The tests developed by these authors involve solving a set of inequalities that are derived from the data; there is consistency with expected utility maximization if and only if a solution to these inequalities exists. However, these results (and later generalizations

[1] For proofs of Afriat's Theorem, see Afriat (1967), Diewert (1973), Varian (1982), and Fostel, Scarf, and Todd (2004). The term GARP is from Varian (1982); Afriat refers to the same property as *cyclical consistency.*

and variations, including those on other choice models under risk or uncertainty[2]) rely on two crucial assumptions: the agent's utility function is concave and the budget set B^t takes the classical form defined above, where prices are linear and markets are complete. These two assumptions guarantee that the first order conditions are necessary *and* sufficient for optimality and can in turn be converted to a necessary and sufficient test. The use of concavity to simplify the formulation of revealed preference tests is well known and can be applied to models of choice in other contexts (see Diewert (2012)).

Our contribution in this paper is to develop a testing procedure that has the following features: **(i)** it is potentially adaptable to test for different models of choice under risk and uncertainty, and not just the expected utility model; **(ii)** it is a 'pure' test of a given model as such and does not require the *a priori* exclusion of phenomena, such as risk loving or elation seeking behavior or reference point effects, that lead to a non-concave u or (more generally) non-convex preferences over contingent consumption; **(iii)** it is applicable to situations with complex budgetary constraints and can be employed even when there is market incompleteness or when there are non-convexities in the budget set due to non-linear pricing or other practices;[3] and **(iv)** it can be easily adapted to measure 'near' rationalizability (using the indices developed by Afriat (1972, 1973) and Varian (1990)) in cases where the data set is not exactly rationalizable by a particular model.

In the case of objective expected utility maximization, a data set is consistent with this model if and only if there is a solution to a set of linear inequalities. In the case of, for example, subjective expected utility, rank dependent utility, or maxmin expected utility, our test involves solving a finite set of bilinear inequalities that is constructed from the data. These problems are decidable, in the sense that there is a known algorithm that can determine in a finite number of steps whether or not a solution exists. Non-linear tests are not new to the revealed preference literature; for example, they appear in tests of weak separability (Varian, 1983), in tests of maxmin expected utility and other models developed in Bayer *et al.* (2013), and also in Brown and Matzkin's (1996) test of Walrasian general equilibrium. The computational demands of solving such problems can in general be a serious obstacle

[2] See Diewert (2012), Bayer *et al.* (2013)), Kubler, Selden, and Wei (2014), Echenique and Saito (2014), Chambers, Liu, and Martinez (2014), and Chambers, Echenique, and Saito (2015).

[3] For an extension Afriat's Theorem to non-linear budget constraints, see Forges and Minelli (2009).

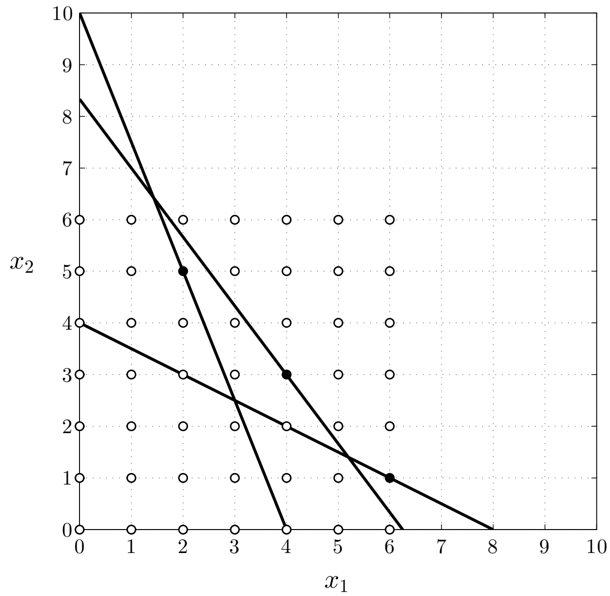


Figure 1: Constructing the finite lattice

to implementation, but some are computationally manageable if they possess certain special features and/or if the number of observations is small.[4] In the case of the tests that we develop, they simplify dramatically and are implementable in practice when there are only two states (though they remain non-linear). The two-state case, while special, is common in applied theoretical settings and laboratory experiments.

1.1 The lattice test procedure

We now give a brief description of our test. Given a data set $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$, we define the discrete consumption set $\mathcal{X} = \{x' \in \mathbb{R}_+ : x' = x_s^t \text{ for some } t, s\} \cup \{0\}$. Besides zero, the consumption set $\mathcal{X}$ contains those levels of consumption that were chosen at some observation and at some state. Since $\mathcal{O}$ is finite, so is $\mathcal{X}$, and its product $\mathcal{L} = \mathcal{X}^{\bar{s}}$ forms a grid of points in $\mathbb{R}_+^{\bar{s}}$; in formal terms, $\mathcal{L}$ is a finite lattice. For example, consider the data set depicted in Figure 1, where $x^1 = (2,5)$ at $p^1 = (5,2)$, $x^2 = (6,1)$ at $p^2 = (1,2)$, and $x^3 = (4,3)$ at $p^3 = (4,3)$. In this case, $\mathcal{X} = \{0,1,2,3,4,5,6\}$ and $\mathcal{L} = \mathcal{X} \times \mathcal{X}$.

Suppose we would like to test whether the data set is consistent with expected utility maximization given objective probabilities $\{\pi_s\}_{s=1}^{\bar{s}}$ that are known to us. Clearly, a *necessary*

[4] It is not uncommon to perform tests on fewer than 20 observations. This is partly because revealed preference tests do not in general account for errors, which are unavoidable across many observations.

condition for this to hold is that we can find a set of numbers $\{\bar{u}(r)\}_{r\in\mathcal{X}}$ with the following properties: (i) $\bar{u}(r'') > \bar{u}(r')$ whenever $r'' > r'$, and (ii) for every $t = 1, 2, \ldots, T$,

$$\sum_{s=1}^{\bar{s}} \pi_s \bar{u}(x_s^t) \geqslant \sum_{s=1}^{\bar{s}} \pi_s \bar{u}(x_s) \text{ for all } x \in B^t \cap \mathcal{L}, \tag{2}$$

with the inequality strict whenever $x \in B^t \cap \mathcal{L}$ and x is in the interior of the budget set B^t. Since $\mathcal{X}$ is finite, the existence of $\{\bar{u}(r)\}_{r\in\mathcal{X}}$ with properties (i) and (ii) can be straightforwardly ascertained by solving a family of linear inequalities. Our main result says that *if a solution can be found, then there is a continuous and increasing utility function* $u : R_+ \to R$ *that extends* $\{\bar{u}(r)\}_{r\in\mathcal{X}}$ *and satisfies (1).* Returning to the example depicted in Figure 1, suppose we know that $\pi_1 = \pi_2 = 0.5$. Our test requires that we find $\bar{u}(r)$, for $r = 0, 1, 2, \ldots, 6$, such that the expected utility of the chosen bundle (x_1^t, x_2^t) is (weakly) greater than that of any lattice points within the corresponding budget set B^t. One could check that these requirements are satisfied for $\bar{u}(r) = r$, for $r = 0, 1, \ldots, 6$, so we conclude that the data set is consistent with expected utility maximization. A detailed explanation of our testing procedure and its application to the expected utility model is found in Section 2. Section 3 shows how this procedure can be applied to test for other models of choice under risk and uncertainty, including rank dependent utility, choice acclimating personal equilibrium, maxmin expected utility, and variational preferences.

1.2 Empirical implementation

To illustrate the use of these tests, we implement them in Section 5 on a data set obtained from the portfolio choice experiment in Choi *et al.* (2007). In this experiment, each subject was asked to purchase Arrow-Debreu securities under different budget constraints. There were two states of the world and it was commonly known that states occurred either symmetrically (each with probability 1/2) or asymmetrically (one with probability 1/3 (2/3) and the other with probability 2/3 (1/3)). We test the following models on these data, in decreasing order of generality: utility maximization, disappointment aversion (Gul, 1991), and expected utility. These three models were also examined by Choi *et al.* (2007). To briefly summarize their procedure, Choi *et al.* (2007) first performed GARP tests on the data; subjects who passed or came very close to passing (and were therefore deemed to be consistent with utility maximization) were then fitted to a parameterized model of disappointment aversion.

The tools developed in this paper allow us to test all three models using a common non-parametric approach. Given that there are 50 observations on every subject, it is not empirically meaningful to simply carry out *exact* tests, because nearly every subject is likely to fail every test. What is required is a way of measuring how close each subject is to being consistent with a particular model of behavior. In the case of utility maximization, Choi *et al.* (2007) measured this gap using the *critical cost efficiency index* (CCEI). This index was first proposed by Afriat (1972, 1973) who also showed how GARP can be modified to calculate the index. We extend this approach by calculating CCEIs for all three models of interest. Not all revealed preference tests can be straightforwardly adapted to perform CCEI calculations; the fact that it is possible to modify our tests for this purpose is one of its important features and is due to the fact that our tests can be performed on non-convex budget sets. We explain this in greater detail in Section 4, which discusses CCEI. We also determine the *power* (Bronars, 1987) of each model, i.e. the probability of a random data set being consistent with a particular model (at a given efficiency threshold). This information allows us to rank the performance of each model using the Selten index (1991). This index balances the success of a model in predicting observations (which favors utility maximization since it is the most permissive) with the specificity of its predictions (which favors expected utility since it is the most restrictive).

Our main findings are as follows. **(i)** In the context of the Choi *et al.* (2007) experiment, all three models are very sharp in the sense that the probability of a randomly drawn data set being consistent with the model is close to zero. **(ii)** Measured by the Selten index, the best performing model is utility maximization, followed by disappointment aversion, and then expected utility. In other words, the greater success that utility maximization has in explaining a set of observations more than compensates for its relative lack of specificity. That said, all three models have very considerable success in explaining the data; for example, at an efficiency level of 0.9, the pass rates of the three models are 81%, 54%, and 52%. **(iii)** Conditioning on agents who pass GARP (at or above some efficiency level, say 0.9), both disappointment aversion and expected utility remain very precise, i.e., the probability of a randomly drawn data set satisfying disappointment aversion (and hence expected utility), conditional on it passing GARP, is also very close to zero. **(iv)** On the other hand, more than half of the subjects who pass GARP are also consistent with disappointment aversion and expected utility, which gives clear support for these models in explaining the behavior of agents who are (in the first place) maximizing some increasing utility function.

2. Testing the model on a lattice

We assume that there is a finite set of states, denoted by $S = \{1, 2, \ldots, \bar{s}\}$. The contingent consumption space is $\mathbb{R}^{\bar{s}}_+$; for a typical consumption bundle $x \in \mathbb{R}^{\bar{s}}_+$, the sth entry, x_s, specifies the consumption level in state s.[5] We assume that there are T observations in the data set $\mathcal{O}$, where $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$. This means that the agent is observed to choose the bundle x^t from the set $B^t \subset \mathbb{R}^{\bar{s}}_+$. We assume that B^t is compact and that $x^t \in \partial B^t$, where ∂B^t denotes the *upper boundary* of B^t. An element $y \in B^t$ is in ∂B^t if there is no $x \in B^t$ such that $x > y$.[6,7] The most important example of B^t is the standard budget set when markets are complete, i.e., when $B^t = \{x \in \mathbb{R}^{\bar{s}}_+ : p^t \cdot x \leqslant p^t \cdot x^t\}$, with $p^t \gg 0$ the vector of state prices. We also allow for the market to be incomplete. Suppose that the agent's contingent consumption is achieved through a portfolio of securities and that the asset prices do not admit arbitrage; then it is well known that there is some $p^t \gg 0$ such that

$$B^t = \{x \in \mathbb{R}^s_+ : p^t \cdot x \leqslant p^t \cdot x^t\} \cap \{Z + \omega\},$$

where Z is the span of assets available to the agent and ω is his endowment of contingent consumption. Note that the budget set B^t and the contingent consumption bundle x^t will both be known to the observer so long as he can observe the asset prices and the agent's holding of securities, the asset payoffs in every state, and the agent's endowment of contingent consumption, ω.

Let $\{\phi(\cdot, t)\}_{t=1}^T$ be a collection of functions, where $\phi(\cdot, t) : \mathbb{R}^{\bar{s}}_+ \to \mathbb{R}$ is increasing in all of its arguments[8] and continuous. The data set $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ is said to be *rationalizable*

[5] Our results do depend on the realization in each state being one-dimensional (which can be interpreted as a monetary payoff, but not a bundle of goods). This case is the one most often considered in applications and experiments and is also the assumption in a number of recent papers, including Kubler, Selden, and Wei (2014), Echenique and Saito (2014), and Chambers, Echenique, and Saito (2015). The papers by Varian (1983), Green and Srivastava (1986), Bayer *et al.* (2013), and Chambers, Liu, and Martinez (2014) allow for multi-dimensional realizations but, unlike this paper, they also require the convexity of the agent's preference over contingent consumption and linear budget sets.

[6] For vectors $x, y \in \mathbb{R}^{\bar{s}}$, $x > y$ if $x \neq y$ and $x_i \geqslant y_i$ for all i. If $x_i > y_i$ for all i, we write $x \gg y$.

[7] For example, if $B^t = \{(x, y) \in \mathbb{R}^2_+ : (x, y) \leqslant (1, 1)\}$, then $(1, 1) \in \partial B^t$ but $(1, 1/2) \notin \partial B^t$.

[8] By this we mean that $\phi(x, t) > \phi(x', t)$ if $x > x'$.

by $\{\phi(\cdot, t)\}_{t=1}^T$ if there exists a continuous and increasing function $u : \mathbb{R}_+ \to \mathbb{R}_+$ (which we shall call the *Bernoulli utility function*) such that

$$\phi(\mathbf{u}(x^t), t) \geqslant \phi(\mathbf{u}(x), t) \text{ for all } x \in B^t, \tag{3}$$

where $\mathbf{u}(x) = (u(x_1), u(x_2), \ldots, u(x_{\bar{s}}))$. In other words, there is some Bernoulli utility function u under which x^t is an optimal choice in B^t, assuming that the agent is maximizing $\phi(\mathbf{u}(x), t)$. Many of the basic models of choice under risk and uncertainty can be described within this framework, with different models leading to different functional forms for $\phi(\cdot, t)$. Of course, this includes expected utility, as we show in the example below. For some of these models (such as rank dependent utility (see Section 3.1)), $\phi(\cdot, t)$ can be a non-concave function, in which case the agent's preference over contingent consumption may be non-convex, even if u is concave.

Example: Suppose that both the observer and the agent know that the probability of state s at observation t is $\pi_s^t > 0$. If the agent is maximizing expected utility,

$$\phi(u_1, u_2, \ldots, u_{\bar{s}}, t) = \sum_{s=1}^{\bar{s}} \pi_s^t u_s, \tag{4}$$

and (3) requires that

$$\sum_{s=1}^{\bar{s}} \pi_s^t u(x_s^t) \geqslant \sum_{s=1}^{\bar{s}} \pi_s^t u(x_s) \text{ for all } x \in B^t, \tag{5}$$

i.e., the expected utility of x^t is greater than that of any other bundle in B^t. When there exists a continuous and increasing function u such that (5) holds, we say that the data set is *EU-rationalizable with probability weights* $\{\pi^t\}_{t=1}^T$, where $\pi^t = (\pi_1^t, \pi_2^t, \ldots, \pi_{\bar{s}}^t)$.

If $\mathcal{O}$ is rationalizable by $\{\phi(\cdot, t)\}_{t=1}^T$, then since the objective function $\phi(\mathbf{u}(\cdot), t)$ is strongly increasing in x, we must have

$$\phi(\mathbf{u}(x^t), t) \geqslant \phi(\mathbf{u}(x), t) \text{ for all } x \in \underline{B}^t \tag{6}$$

where $\underline{B}^t = \{y \in \mathbb{R}_+^{\bar{s}} : y \leqslant x \text{ for some } x \in B^t\}$. Furthermore, the inequality in (6) is strict whenever $x \in \underline{B}^t \backslash \partial \underline{B}^t$ (where $\partial \underline{B}^t$ refers to the upper boundary of $\underline{B}^t$). We define

$$\mathcal{X} = \{x' \in \mathbb{R}_+ : x' = x_s^t \text{ for some } t,\ s\} \cup \{0\}.$$

Besides zero, $\mathcal{X}$ contains those levels of consumption that were chosen at some observation and at some state. Since the data set is finite, so is $\mathcal{X}$. Given $\mathcal{X}$, we may construct $\mathcal{L} = \mathcal{X}^{\bar{s}}$, which consists of a finite grid of points in $\mathbb{R}^{\bar{s}}_{+}$; in formal terms, $\mathcal{L}$ is a finite lattice. Let $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ be the restriction of the Bernoulli utility function u to $\mathcal{X}$. Given our observations, the following must hold:

$$\phi(\bar{\mathbf{u}}(x^t), t) \geqslant \phi(\bar{\mathbf{u}}(x), t) \text{ for all } x \in \underline{B}^t \cap \mathcal{L} \text{ and} \tag{7}$$

$$\phi(\bar{\mathbf{u}}(x^t), t) > \phi(\bar{\mathbf{u}}(x), t) \text{ for all } x \in \left(\underline{B}^t \backslash \partial \underline{B}^t\right) \cap \mathcal{L}, \tag{8}$$

where $\bar{\mathbf{u}}(x) = (\bar{u}(x_1), \bar{u}(x_2), \ldots, \bar{u}(x_{\bar{s}})$. Our main theorem says that the converse is also true.

Theorem 1. *Suppose that for some data set $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ and collection of functions $\{\phi(\cdot, t)\}_{t=1}^T$ that are continuous and increasing in all dimensions, there is an increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ that satisfies conditions (7) and (8). Then there is an increasing and continuous function $u : \mathbb{R}_+ \to \mathbb{R}_+$ that extends $\bar{u}$ and guarantees the rationalizability of $\mathcal{O}$ by $\{\phi(\cdot, t)\}_{t=1}^T$.*[9]

The intuition for this result ought to be strong. Given $\bar{u}$ satisfying (7) and (8), we can define the step function $\hat{u} : \mathbb{R}_+ \to \mathbb{R}_+$ where $\hat{u}(r) = \bar{u}([r])$, with $[r]$ being the largest element of $\mathcal{X}$ weakly lower than r, i.e., $[r] = \max\{r' \in \mathcal{X} : r' \leqslant r\}$. Notice that $\phi(\hat{\mathbf{u}}(x^t), t) = \phi(\bar{\mathbf{u}}(x^t), t)$ and, for any $x \in \underline{B}^t$, $\phi(\hat{\mathbf{u}}(x), t) = \phi(\bar{\mathbf{u}}([x]), t)$, where $[x] = ([x_1], [x_2], \ldots, [x_{\bar{s}}])$ in $\underline{B}^t \cap \mathcal{L}$. Clearly, if $\bar{u}$ obeys (7) and (8) then $\mathcal{O}$ is rationalized by $\{\phi(\cdot, t)\}_{t=1}^T$ and $\hat{u}$ (in the sense that (3) holds). This falls short of the claim in the theorem only because $\hat{u}$ is neither continuous nor strictly increasing; the proof in the Appendix shows how one could in fact construct a Bernoulli utility function with these added properties.

2.1 Testing the expected utility model

We wish to check whether $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ is EU-rationalizable with probability weights $\{\pi^t\}_{t=1}^T$, in the sense defined in the previous example. By Theorem 1, EU-rationalizability holds if and only if there is a collection of real numbers $\{\bar{u}(r)\}_{r \in \mathcal{X}}$ such that

$$0 \leqslant \bar{u}(r') < \bar{u}(r) \text{ whenever } r' < r, \tag{9}$$

[9] The increasing assumptions on ϕ and $\bar{u}$ ensure that we may confine ourselves to checking (7) and (8) for undominated elements of $\underline{B}^t \cap \mathcal{L}$, i.e., $x \in \underline{B}^t \cap \mathcal{L}$ such that there does not exist $x' \in \underline{B}^t \cap \mathcal{L}$ with $x < x'$.

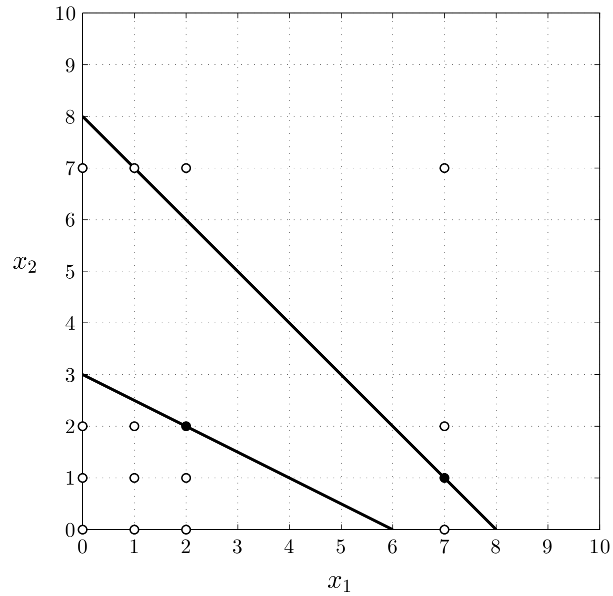


Figure 2: Violation of concave expected utility

and the inequalities (7) and (8) hold, where $\phi(\cdot, t)$ is defined by (4). This is a linear program and it is both *solvable* (in the sense that there is an algorithm that can decide within a known number of steps whether or not a solution exists) and computationally feasible.

Note that the Bernoulli utility function, whose existence is guaranteed by Theorem 1, need not be a concave function. Consider the example given in Figure 2 and suppose that $\pi_1 = \pi_2 = 1/2$. In this case, $\mathcal{X} = \{0, 1, 2, 7\}$, and one could check that (7) and (8) are satisfied (where $\phi(\cdot, t)$ is defined by (4)) if $\bar{u}(0) = 0$, $\bar{u}(1) = 2$, $\bar{u}(2) = 3$, and $\bar{u}(7) = 6$. Thus the data set is EU-rationalizable. However, any u that rationalizes the data cannot be concave. Indeed, since $(3, 1)$ is strictly within the budget set when $(2, 2)$ was chosen, $2u(2) > u(1) + u(3)$. By the concavity of u, $u(3) - u(2) \geqslant u(7) - u(6)$, and thus we obtain $u(6) + u(2) > u(7) + u(1)$, contradicting the optimality of $(7, 1)$.

We now turn to a setting in which no objective probabilities can be attached to each state. The data set $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ is said to be *rationalizable by subjective expected utility* (in short, SEU-rationalizable) if there exist beliefs $\pi = (\pi_1, \pi_2, \ldots, \pi_{\bar{s}}) \gg 0$ and an increasing and continuous function $u : \mathbb{R}_+ \to \mathbb{R}$ such that, for all $t = 1, 2, \ldots, T$,

$$\sum_{s=1}^{\bar{s}} \pi_s u(x_s^t) \geqslant \sum_{s=1}^{\bar{s}} \pi_s u(x_s) \text{ for all } x \in B^t.$$

In other words, at every observation t, the agent is acting as though he attributes a probability of π_s to state s and is maximizing expected utility. In this case, ϕ is independent of t, with $\phi(\mathbf{u}) = \sum_{s=1}^{\bar{s}} \pi_s u_s$. The conditions (7) and (8) can be written as

$$\sum_{s=1}^{\bar{s}} \pi_s \bar{u}(x_s^t) \geqslant \sum_{s=1}^{\bar{s}} \pi_s \bar{u}(x_s) \text{ for all } x \in \underline{B}^t \cap \mathcal{L} \text{ and} \tag{10}$$

$$\sum_{s=1}^{\bar{s}} \pi_s \bar{u}(x_s^t) > \sum_{s=1}^{\bar{s}} \pi_s \bar{u}(x_s) \text{ for all } x \in \left(\underline{B}^t \backslash \partial \underline{B}^t\right) \cap \mathcal{L}. \tag{11}$$

In other words, a necessary and sufficient condition for SEU-rationalizability is that we can find real numbers $\{\pi_s\}_{s=1}^{\bar{s}}$ and $\{\bar{u}(r)\}_{r \in \mathcal{X}}$ such that $\pi_s > 0$ for all s, $\sum_{s=1}^{\bar{s}} \pi_s = 1$, and (9), (10), and (11) are satisfied. Notice that we are simultaneously searching for u and (through $\pi_s > 0$) ϕ that rationalizes the data. This set of conditions forms a finite system of bilinear inequalities. The Tarski-Seidenberg Theorem tells us that such systems are decidable.

In the case when there are just two states, there is a straightforward way of implementing this test. Simply condition on the probability of state 1 (and hence the probability of state 2 as well), and then run a linear test to check if there is a solution to (9), (10), and (11). If not, choose another probability, implement the test, and repeat, if necessary. Even a search of up to two decimal places on the subjective probability of state 1 will lead to no more than one hundred linear tests, which can implemented with little difficulty.

3. Other applications of the lattice test

Theorem 1 can also be used to test models other than expected utility, with each model requiring different functional forms for $\phi(\cdot, t)$.

3.1 Rank dependent utility (RDU)

The RDU model (Quiggin, 1982) is a model of choice under risk where, for each state s, there is an objective probability $\pi_s > 0$ that is known to the agent (and which we assume is also known to the observer). Given a vector x, we can rank the entries of x from the smallest to the largest, with ties broken by the rank of the state. We denote by $r(x, s)$, the rank of x_s in x. For example, if there are five states and $x = (1, 4, 4, 3, 5)$, we have $r(x, 1) = 1$, $r(x, 2) = 3$, $r(x, 3) = 4$, $r(x, 4) = 2$, and $r(x, 5) = 5$. A rank dependent expected utility function gives to the bundle x the utility $V(x,\pi) = \sum_{s=1}^{\bar{s}} \rho(x,s,\pi)u(x_s)$ where $u : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and continuous function,

$$\rho(x,s,\pi) = g\left(\sum_{\{s':r(x,s')\leqslant r(x,s)\}} \pi_{s'}\right) - g\left(\sum_{\{s':r(x,s')<r(x,s)\}} \pi_{s'}\right), \tag{12}$$

and $g : [0,1] \to \mathbb{R}$ is an increasing and continuous function. (If $\{s' : r(x,s') < r(x,s)\}$ is empty, we let $g\left(\sum_{\{s':r(x,s')<r(x,s)\}} \pi_{s'}\right) = g(0)$.) If g is the identity function (or, more generally when g is affine), we simply recover the expected utility model. When it is nonlinear, the function g distorts the cumulative distribution of the lottery x, so that an agent maximizing RDU can behave as though the probability he attaches to a state depends on the relative attractiveness of the outcome in that state. Since u is increasing, $\rho(x,s,\pi) = \rho(\mathbf{u}(x),s,\pi)$. It follows that we can write $V(x,\pi) = \phi(\mathbf{u}(x),\pi)$, where for any vector $\mathbf{u} = (u_1, u_2, \ldots, u_{\bar{s}})$,

$$\phi(\mathbf{u},\pi) = \sum_{s=1}^{\bar{s}} \rho(\mathbf{u},s,\pi)u_s. \tag{13}$$

The function ϕ is a continuous and increasing in $\mathbf{u}$ (in all dimensions).

Suppose we wish to check whether $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ is RDU-rationalizable with probability weights $\{\pi^t\}_{t=1}^T$. RDU-rationalizability holds if and only if there are increasing functions $g : [0,1] \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$ such that, for each t, $V(x^t,\pi^t) \geqslant V(x,\pi^t)$ for all $x \in B^t$. To develop a necessary and sufficient test for this property, we first define the set

$$\Gamma = \left\{\gamma : \text{there is } t,\ s, \text{ and } x \in \mathcal{L} \text{ such that } \gamma = \sum_{\{s':r(x,s')\leqslant r(x,s)\}} \pi^t_{s'}\right\} \cup \{0\}.$$

Note that the set Γ is a finite subset of $[0,1]$ and includes both 0 and 1. We may denote the elements of Γ by γ^j, where $\gamma^{j-1} < \gamma^j$, with $\gamma^0 = 0$ and $\gamma^{\bar{m}} = 1$ (so Γ has $\bar{m}+1$ elements).

If $\mathcal{O}$ is RDU-rationalizable, there must be increasing functions $\bar{g} : \Gamma \to \mathbb{R}$ and $\bar{u} : \mathcal{X} \to \mathbb{R}$ such that

$$\sum_{s=1}^{\bar{s}} \bar{\rho}(x^t,s,\pi^t)\bar{u}(x^t_s) \geqslant \sum_{s=1}^{\bar{s}} \bar{\rho}(x,s,\pi^t)u(x_s) \ \text{ for all } x \in \underline{B}^t \cap \mathcal{L} \text{ and} \tag{14}$$

$$\sum_{s=1}^{\bar{s}} \bar{\rho}(x^t,s,\pi^t)\bar{u}(x^t_s) > \sum_{s=1}^{\bar{s}} \bar{\rho}(x,s,\pi^t)\bar{u}(x_s) \ \text{ for all } x \in \left(\underline{B}^t \backslash \partial \underline{B}^t\right) \cap \mathcal{L}, \tag{15}$$

where

$$\bar{\rho}(x,s,\pi) = \bar{g}\left(\sum_{\{s':r(x,s')\leqslant r(x,s)\}} \pi_{s'}\right) - \bar{g}\left(\sum_{\{s':r(x,s')<r(x,s)\}} \pi_{s'}\right). \tag{16}$$

This is clear since we can simply take $\bar{g}$ and $\bar{u}$ to be the restriction of g and u respectively. Conversely, suppose there are increasing functions $\bar{g} : \Gamma \to \mathbb{R}$ and $\bar{u} : \mathcal{X} \to \mathbb{R}$ such that (14), (15), and (16) are satisfied, and let $g : [0,1] \to \mathbb{R}$ be *any* continuous and increasing extension of $\bar{g}$. Defining $\phi(\mathbf{u}, \pi)$ by (13), the properties (14) and (15) may be re-written as

$$\phi(\mathbf{u}(x^t), \pi^t) \geqslant \phi(\mathbf{u}(x), \pi^t) \text{ for all } x \in \underline{B}^t \cap \mathcal{L} \text{ and}$$

$$\phi(\mathbf{u}(x^t), \pi^t) > \phi(\mathbf{u}(x), \pi^t) \text{ for all } x \in \left(\underline{B}^t \backslash \partial \underline{B}^t\right) \cap \mathcal{L}.$$

By Theorem 1, these properties guarantee that there exists $u : \mathbb{R}_+ \to \mathbb{R}$ that extends $\bar{u}$ such that the data set $\mathcal{O}$ is RDU-rationalizable.

To recap, we have shown that $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ *is RDU-rationalizable with probability weights* $\{\pi^t\}_{t=1}^T$ *if and only if there exist real numbers* $\{\bar{g}(\gamma)\}_{\gamma \in \Gamma}$ *and* $\{\bar{u}(r)\}_{r \in \mathcal{X}}$ *that satisfy*

$$\bar{g}(\gamma^{j-1}) < \bar{g}(\gamma^j) \textit{ for } j = 1, 2, \ldots, \bar{m}, \tag{17}$$

(9), (14), (15), and (16). As in the test for SEU-rationalizability, this test involves finding a solution to a finite set of bilinear inequalities (with unknowns $\{\bar{u}(r)\}_{r \in \mathcal{X}}$ and $\{\bar{g}(\gamma)\}_{\gamma \in \Gamma}$).[10]

In Section 4, we implement a test of Gul's (1991) model of disappoint aversion (DA). When there are just two states, the DA model is a special case of RDU. Given a bundle $x = (x_1, x_2)$, let H denote the state with the higher payoff and π_H its objective probability. In the DA model, the probability of the higher outcome is distorted and becomes

$$\gamma(\pi_H) = \frac{\pi_H}{1 + (1 - \pi_H)\beta}, \tag{18}$$

where $\beta \in (-1, \infty)$. If $\beta > 0$, then $\gamma(\pi_H) < \pi_H$ and the agent is said to be *disappointment averse*; if $\beta < 0$, then $\gamma(\pi_H) > \pi_H$ and the agent is *elation seeking*; lastly, if $\beta = 0$, then there is no distortion and the agent simply maximizes expected utility. For a bundle $(x_1, x_2) \in \mathbb{R}^2_+$,

$$\phi\left((u(x_1), u(x_2)), (\pi_1, \pi_2)\right) = (1 - \gamma(\pi_2))u(x_1) + \gamma(\pi_2)u(x_2) \text{ if } x_1 \leqslant x_2 \text{ and} \tag{19}$$

[10] It is also straightforward to modify the test to include restrictions on the shape of g. For example, to test that $\mathcal{O}$ is RDU-rationalizable with a convex function g we need to specify that $\bar{g}$ obeys

$$\frac{\bar{g}(\gamma^j) - \bar{g}(\gamma^{j-1})}{\gamma^j - \gamma^{j-1}} \leqslant \frac{\bar{g}(\gamma^{j+1}) - \bar{g}(\gamma^j)}{\gamma^{j+1} - \gamma^j} \text{ for } j = 1, \ldots, \bar{m} - 1.$$

It is clear that this condition is necessary for the convexity of g. It is also sufficient for the extension of $\bar{g}$ to a convex and increasing function $g : [0,1] \to \mathbb{R}$. Thus $\mathcal{O}$ is RDU-rationalizable with probability weights $\{\pi^t\}_{t=1}^T$ and a convex function g if and only if there exist real numbers $\{\bar{g}(\gamma)\}_{\gamma \in \Gamma}$ and $\{\bar{u}(r)\}_{r \in \mathcal{X}}$ that satisfy the above condition in addition to (9), (14), (15), (16), and (17).

$$\phi\left((u(x_1),u(x_2)),(\pi_1,\pi_2)\right) = \gamma(\pi_1)u(x_1) + (1-\gamma(\pi_1)))u(x_1) \text{ if } x_2 < x_1. \tag{20}$$

When the agent is elation seeking, ϕ is not concave in $\mathbf{u}$, so his preference over contingent consumption bundles need not be convex, even if u is concave. A data set is DA-rationalizable if and only if we can find $\beta \in (-1,\infty)$ and $\{\bar{u}(r)\}_{r\in\mathcal{X}}$ so that (14) and (15) are satisfied. Notice that, conditioning on the value of β, this test is linear in the remaining variables. We use this feature in our implementation of the DA model in Section 5.

3.2 Choice acclimating personal equilibrium (CPE)

The CPE model of Koszegi and Rabin (2007) (with a piecewise linear gain-loss function) specifies the agent's utility as $V(x) = \phi(\mathbf{u}(x),\pi)$, where

$$\phi((u_1,u_2,\ldots,u_{\bar{s}}),\pi) = \sum_{s=1}^{\bar{s}} \pi_s u_s + \frac{1}{2}(1-\lambda)\sum_{r,s=1}^{\bar{s}} \pi_r\pi_s|u_r - u_s|,$$

$\pi = \{\pi_s\}_{s=1}^{\bar{s}}$ are the objective probabilities, and $\lambda \in [1,2]$ is the coefficient of loss aversion.[11] A data set $\mathcal{O} = \{(x^t,B^t)\}_{t=1}^T$ can be rationalized by this model if and only if there is λ and $\bar{u}$ that solve (7), (8), and (9). Observe that, conditional on λ, this test is linear in the remaining variables, so it is feasible to implement it as a collection of linear tests (running over different values of $\lambda \in [1,2]$). Notably, this is true irrespective of the (finite) number of states.

3.3 Maxmin expected utility (MEU)

We again consider a setting where no objective probabilities can be attached to each state. An agent with maxmin expected utility behaves as though he evaluates each bundle $x \in \mathbb{R}_+^{\bar{s}}$ using the formula $V(x) = \phi(\mathbf{u}(x))$ where

$$\phi(\mathbf{u}) = \min_{\pi\in\Pi}\left\{\sum_{s=1}^{\bar{s}} \pi_s u_s\right\}, \tag{21}$$

where $\Pi \subset \Delta_{++} = \{\pi \in \mathbb{R}_{++}^{\bar{s}} : \sum_{s=1}^{\bar{s}} \pi_s = 1\}$ is nonempty, compact in $\mathbb{R}^{\bar{s}}$, and convex. (Π can be interpreted as a set of probability weights.) Given these restrictions on Π, the minimization problem in (21) always has a solution and ϕ is increasing in all dimensions.

[11] Our presentation of the CPE model follows Masatlioglu and Raymond (2014). The restriction of λ to $[1,2]$ guarantees that V is loss averse and respects first order stochastic dominance (see Masatlioglu and Raymond (2014)).

A data set $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ is MEU-rationalizable if and only if there exist Π and $\bar{u}$ that solve (7), (8), and (9). This requirement can be reformulated as the solvability of a set of bilinear inequalities and the two-state case is particularly straightforward. Indeed, we may assume without loss of generality that there is π_1^* and $\pi_1^{**} \in (0,1)$ such that $\Pi = \{(\pi_1, 1-\pi_1) : \pi_1^* \leqslant \pi_1 \leqslant \pi_1^{**}\}$. Then it is clear that $\phi(u_1, u_2) = \pi_1^* u_1 + (1-\pi_1^*)u_2$ if $u_1 \geqslant u_2$ and $\phi(u_1, u_2) = \pi_1^{**} u_1 + (1-\pi_1^{**})u_2$ if $u_1 < u_2$. Consequently, for any $(x_1, x_2) \in \mathcal{L}$, we have $V(x_1, x_2) = \pi_1^* \bar{u}(x_1) + (1-\pi_1^*)\bar{u}(x_2)$ if $x_1 \geqslant x_2$ and $V(x_1, x_2) = \pi_1^{**}\bar{u}(x_1) + (1-\pi_1^{**})\bar{u}(x_2)$ if $x_1 < x_2$ and this is independent of the precise choice of $\bar{u}$. Therefore, $\mathcal{O}$ is MEU-rationalizable if and only if we can find π_1^* and π_1^{**} in $(0,1)$, and an increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ that solve (7), (8), and (9). The requirement takes the form of a system of bilinear inequalities that is linear after conditioning on π_1^* and π_1^{**}.

The result below (which we prove in the Appendix) covers the case with multiple states. Note that the test involves solving a system of bilinear inequalities in the variables $\bar{\pi}_s(x)$ (for all s and $x \in \mathcal{L}$) and $\bar{u}(r)$ (for all $r \in \mathcal{X}$).

Proposition 1. *A data set $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ is MEU-rationalizable if and only if there is a function $\bar{\pi} : \mathcal{L} \to \Delta_{++}$ and an increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}$ such that*

$$\bar{\pi}(x^t) \cdot \bar{\mathbf{u}}(x^t) \geqslant \bar{\pi}(x) \cdot \bar{\mathbf{u}}(x) \text{ for all } x \in \mathcal{L} \cap \underline{B}^t, \tag{22}$$

$$\bar{\pi}(x^t) \cdot \bar{\mathbf{u}}(x^t) > \bar{\pi}(x) \cdot \bar{\mathbf{u}}(x) \text{ for all } x \in \mathcal{L} \cap (\underline{B}^t \backslash \partial \underline{B}^t), \text{ and} \tag{23}$$

$$\bar{\pi}(x) \cdot \bar{\mathbf{u}}(x) \leqslant \bar{\pi}(x') \cdot \bar{\mathbf{u}}(x) \text{ for all } (x, x') \in \mathcal{L} \times \mathcal{L}. \tag{24}$$

If these conditions hold, $\mathcal{O}$ admits an MEU-rationalization where Π (in (21)) is the convex hull of $\{\bar{\pi}(x)\}_{x \in \mathcal{L}}$ and $V(x) = \min_{\pi \in \Pi}\{\pi \cdot \bar{\mathbf{u}}(x)\} = \bar{\pi}(x) \cdot \bar{\mathbf{u}}(x)$ for all $x \in \mathcal{L}$.

3.4 Variational preferences

A popular model of decision making under uncertainty that generalizes maxmin expected utility is variational preferences (Maccheroni, Marinacci, and Rustichini, 2006). In this model, a bundle $x \in \mathbb{R}_+^{\bar{s}}$ has utility $V(x) = \phi(\mathbf{u}(x))$ where

$$\phi(\mathbf{u}) = \min_{\pi \in \Delta_{++}} \{\pi \cdot \mathbf{u} + c(\pi)\} \tag{25}$$

and $c : \Delta_{++} \to \mathbb{R}_+$ is a continuous and convex function with the following boundary condition: for any sequence $\pi^n \in \Delta_{++}$ tending to $\tilde{\pi}$, with $\tilde{\pi}_s = 0$ for some s, we obtain $c(\pi^n) \to \infty$. This boundary condition, together with the continuity of c, guarantee that there is $\pi^* \in \Delta_{++}$ that solves the minimization problem in (25).[12] Therefore, ϕ is well-defined and increasing in all its arguments. By Theorem 1, a data set $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ can be rationalized by variational preferences if and only if there exists a function $c : \Delta_{++} \to \mathbb{R}_+$ that is continuous, convex, and has the boundary property, and an increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ that together solve (7), (8), and (9), with ϕ defined by (25). The following result (proved in the Appendix) is a reformulation of this characterization that has a similar flavor to Proposition 1; note that, once again, the necessary and sufficient conditions on $\mathcal{O}$ are expressed as a set of bilinear inequalities.

Proposition 2. *A data set $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ can be rationalized by variational preferences if and only if there is a function $\bar{\pi} : \mathcal{L} \to \Delta_{++}$, a function $\bar{c} : \mathcal{L} \to \mathbb{R}_+$, and a strictly increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}$ such that*

$$\bar{\pi}(x^t) \cdot \bar{\mathbf{u}}(x^t) + \bar{c}(x^t) \geqslant \bar{\pi}(x) \cdot \bar{\mathbf{u}}(x) + \bar{c}(x) \text{ for all } x \in \mathcal{L} \cap \underline{B}^t, \tag{26}$$

$$\bar{\pi}(x^t) \cdot \bar{\mathbf{u}}(x^t) + \bar{c}(x^t) > \bar{\pi}(x) \cdot \bar{\mathbf{u}}(x) + \bar{c}(x) \text{ for all } x \in \mathcal{L} \cap (\underline{B}^t \backslash \partial \underline{B}^t), \text{ and} \tag{27}$$

$$\bar{\pi}(x) \cdot \bar{\mathbf{u}}(x) + \bar{c}(x) \leqslant \bar{\pi}(x') \cdot \bar{\mathbf{u}}(x) + \bar{c}(x') \text{ for all } (x, x') \in \mathcal{L} \times \mathcal{L}. \tag{28}$$

If these conditions hold, then $\mathcal{O}$ can be rationalized by a variational preference V such that $V(x) = \bar{\pi}(x) \cdot \bar{\mathbf{u}}(x) + \bar{c}(x)$ for all $x \in \mathcal{L}$, with c obeying $c(\bar{\pi}(x)) = \bar{c}(x)$ for all $x \in \mathcal{L}$.

3.5 Models with budget-dependent reference points

So far in our discussion we have assumed that the agent has a preference over different contingent outcomes, without being too specific as to what actually constitutes an outcome in the agent's mind. On the other hand, models such as prospect theory have often emphasized the impact of reference points, and *changing* reference points, on decision-making. Some of these phenomena can be easily accommodated within our framework.

[12] Indeed, pick any $\tilde{\pi} \in \Delta_{++}$ and define $S = \{\pi \in \Delta_{++} : \pi \cdot \mathbf{u} + c(\pi) \leqslant \tilde{\pi} \cdot \mathbf{u} + c(\tilde{\pi})\}$. The boundary condition and continuity of c guarantee that S is compact in $\mathbb{R}^{\bar{s}}$ and hence $\arg\min_{\pi \in S}\{\pi \cdot \mathbf{u} + c(\pi)\} = \arg\min_{\pi \in \Delta_{++}}\{\pi \cdot \mathbf{u} + c(\pi)\}$ is nonempty.

For example, imagine an experiment in which subjects are asked to choose from a constraint set of state contingent monetary prizes. Assuming that there are $\bar{s}$ states and that the subject never suffers a loss, we can represent each prize by a vector $x \in \mathbb{R}^{\bar{s}}_{+}$. The subject is observed to choose x^t from $B^t \subset \mathbb{R}^{\bar{s}}_{+}$, so the data set is $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^{T}$. The standard way of thinking about the subject's behavior is to assume his choice from B^t is governed by a preference defined on the prizes, which implies that the situation where he never receives a prize (formally the vector 0) is the subject's constant reference point. But a researcher may well be interested in whether the subject in fact has a different reference point or multiple reference points that vary with the budget (and perhaps manipulable by the researcher in some way). Most obviously, suppose that the subject has an endowment point $\omega^t \in \mathbb{R}^{\bar{s}}_{+}$ and a classical budget set $B^t = \{x \in \mathbb{R}^{\bar{s}}_{+} : p^t \cdot x \leqslant p^t \cdot \omega^t\}$. In this case, a possible hypothesis is that the subject will evaluate different bundles in B^t based on a utility function defined on the deviation from the endowment; in other words, the endowment is the subject's reference point. Another possible reference point is that bundle in B^t which gives the same payoff in every state.

Whatever it may be, suppose the researcher has a hypothesis about the possible reference point at observation t, which we shall denote by $\omega^t \in \mathbb{R}^{\bar{s}}_{+}$, and that the subject chooses according to some utility function $V : [-K, \infty)^{\bar{s}} \to \mathbb{R}_{+}$ where $K > 0$ is sufficiently large so that $[-K, \infty)^{\bar{s}} \subset \mathbb{R}^{\bar{s}}$ contains all the possible reference point-dependent outcomes in the data, i.e., the set $\bigcup_{t=1}^{T} \tilde{B}^t$, where

$$\tilde{B}^t = \{x' \in \mathbb{R}^{\bar{s}} : \; x' = x - e^t \text{ for some } x \in B^t\}.$$

Let $\{\phi(\cdot, t)\}_{t=1}^{T}$ be a collection of functions, where $\phi(\cdot, t) : [-K, \infty)^{\bar{s}} \to \mathbb{R}$ is increasing in all its arguments. We say that $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^{T}$ is *rationalizable by* $\{\phi(\cdot, t)\}_{t=1}^{T}$ *and the reference points* $\{e^t\}_{t=1}^{T}$ if there exists a continuous and increasing function $u : [-K, \infty) \to \mathbb{R}_{+}$ such that $\phi(\mathbf{u}(x^t), t) \geqslant \phi(\mathbf{u}(x), t)$ for all $x \in B^t$. This is formally equivalent to saying that the modified data set $\mathcal{O}' = \{(x^t - e^t, \tilde{B}^t)\}_{t=1}^{T}$ is rationalizable by $\{\phi(\cdot, t)\}_{t=1}^{T}$. Applying Theorem 1, rationalizability holds if and only if there is an increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}_{+}$ that obeys (7) and (8), where

$$\mathcal{X} = \{r \in \mathbb{R} : r = x^t_s - e^t_s \text{ for some } t, \, s\} \cup \{-K\}.$$

Therefore Theorem 1 allows us to test whether $\mathcal{O}$ is rationalizable by expected utility, or some other model, in conjunction with budget dependent reference points. Note that a test of rank dependent utility in this context is sufficiently flexible to accommodate phenomena emphasized by cumulative prospect theory (see Tversky and Kahneman (1992)), such as a function $u : [-K, \infty) \to \mathbb{R}$ that is S-shaped around 0 and probabilities distorted by a weighting function.

4. Goodness of fit

The revealed preference tests that we have presented in the previous two sections are 'sharp', in the sense that a data set either passes the test for a particular model or it fails. This either/or feature of the tests is not peculiar to our results but is true of all classical revealed preference tests, including Afriat's. It would, of course, be desirable to develop a way of measuring *the extent* to which a certain class of utility functions succeeds or fails in rationalizing a data set. We now give an account of the approach developed in the literature to address this issue (see, for example, Afriat (1972, 1973), Varian (1990), and Halevy, Persitz, and Zrill (2012)) and explain why implementing the same approach in our setting is possible (or at least that it is no more difficult than implementing the exact test).

Suppose that the observer collects a data set $\mathcal{O} = \{(x^t, B^t)\}_{t \in T}$; following the earlier papers, we focus attention on the case where B^t is a classical linear budget set, i.e., there is $p^t \in \mathbb{R}^{\bar{s}}_{++}$ such that $B^t = \{x \in \mathbb{R}^{\bar{s}}_{+} : p^t \cdot x \leqslant p^t \cdot x^t\}$. With no loss of generality, we normalize the price vector p^t so that $p^t \cdot x^t = 1$. Given a number $e^t \in [0, 1]$ we define

$$B^t(e^t) = \{x \in R^{\bar{s}}_{+} : x \leqslant x^t\} \cup \{x \in R^{\bar{s}}_{+} : p^t \cdot x \leqslant e^t\}.$$

Clearly $B^t(e^t)$ is smaller than B^t and shrinks with the value of e^t. Let $\mathcal{U}$ be some collection of continuous and strictly increasing utility functions. We define the set $E(\mathcal{U})$ in the following manner: a vector $\mathbf{e} = (e^1, e^2, \ldots, e^T)$ is in $E(\mathcal{U})$ if there is some function $U \in \mathcal{U}$ that rationalizes the modified data set $\mathcal{O}(\mathbf{e}) = \{(x^t, B^t(e^t))\}_{t=1}^T$, i.e., $U(x^t) \geqslant U(x)$ for all $x \in B^t(e^t)$. Clearly, the data set $\mathcal{O}$ is rationalizable by a utility function in $\mathcal{U}$ if and only if the unit vector $(1, 1, \ldots, 1)$ is in $E(\mathcal{U})$. We also know that $E(\mathcal{U})$ must be nonempty since it contains the vector 0 and it is clear that if $\mathbf{e} \in E(\mathcal{U})$ then $\mathbf{e}' \in E(\mathcal{U})$, where $\mathbf{e}' < \mathbf{e}$. The closeness of the set $E(\mathcal{U})$ to the unit vector is a measure of how well the utility functions in $\mathcal{U}$ can explain the data. Afriat (1972, 1973) suggests measuring this distance with the supnorm, so the distance between $\mathbf{e}$ and 1 is $D_A(\mathbf{e}) = 1 - \min_{1 \leqslant t \leqslant T}\{e^t\}$, while Varian (1990) suggests that we choose the square of the Euclidean distance, i.e., $D_V(\mathbf{e}) = \sum_{t=1}^{T}(1-e^t)^2$.

Measuring distance by the supnorm has the advantage that it is computationally more straightforward. Note that $D_A(\mathbf{e}) = D_A(\tilde{\mathbf{e}})$ where $\tilde{e}^t = \min\{e^1, e^2, \ldots, e^T\}$ for all t and, since $\tilde{\mathbf{e}} \leqslant \mathbf{e}$, we obtain $\tilde{\mathbf{e}} \in E(\mathcal{U})$ whenever $\mathbf{e} \in E(\mathcal{U})$. Therefore,

$$\min_{\mathbf{e} \in E(\mathcal{U})} D_A(\mathbf{e}) = \min_{\tilde{\mathbf{e}} \in \widetilde{E}(\mathcal{U})} D_A(\tilde{\mathbf{e}}),$$

where $\widetilde{E}(\mathcal{U}) = \{\mathbf{e} \in E(\mathcal{U}) : e^t = e^1 \ \forall\ t\}$. In other words, in searching for $\mathbf{e} \in E(\mathcal{U})$ that minimizes the supnorm distance from $(1, 1, .., 1)$, we can focus our attention on those vectors in $E(\mathcal{U})$ that shrink each observed budget set by the same proportion. Given a data set $\mathcal{O}$, Afriat refers to $\sup\{e : (e, e, \ldots, e) \in E(\mathcal{U})\}$ as the *critical cost efficiency index* (CCEI); we say that $\mathcal{O}$ is *rationalizable in* $\mathcal{U}$ *at the cost efficiency index/threshold* e' (or simply *efficiency index/threshold* e') if $(e', e', \ldots, e') \in E(\mathcal{U})$.

Calculating the CCEI (or an index based on the Euclidean metric or some other metric) will require checking whether a particular vector $\mathbf{e} = (e^1, e^2, \ldots, e^T)$ is in $E(\mathcal{U})$, i.e., whether $\{(x^t, B^t(e^t))\}_{t \in \mathcal{T}}$ is rationalizable by a member of $\mathcal{U}$. In the case where $\mathcal{U}$ is the family of increasing and continuous utility functions, it is known that a modified version of GARP (that excludes strict revealed preference cycles based on the modified budget sets $B^t(e^t)$) is both a necessary and sufficient condition for the rationalizability of $\{(x^t, B^t(e^t))\}_{t=1}^{T}$ (see Afriat (1972, 1973)).[13]

More generally, the calculation of CCEI will hinge on whether there is a suitable test for the rationalizability of $\{(x^t, B^t(e^t))\}_{t \in \mathcal{T}}$ by members of $\mathcal{U}$. Even if a test of the rationalizability of $\{(x^t, B^t)\}_{t \in T}$ by members of $\mathcal{U}$ is available, this test may rely on the convexity or linearity of the budget sets B^t; in this case, extending the test so as to check the rationalizability of $\mathcal{O}(e) = \{(x^t, B^t(e^t))\}_{t \in T}$ is not straightforward since the sets $B^t(e^t)$ are clearly non-convex. Crucially, this is *not* the case with the lattice test, which is applicable even for non-convex constraint sets. Thus extending our testing procedure to measure goodness of

[13] Alternatively, consult Forges and Minelli (2009) for a generalization of Afriat's Theorem to nonlinear budget sets; the test developed by Forges and Minelli can be applied to $\{(x^t, B^t(e^t))\}_{t=1}^{T}$.

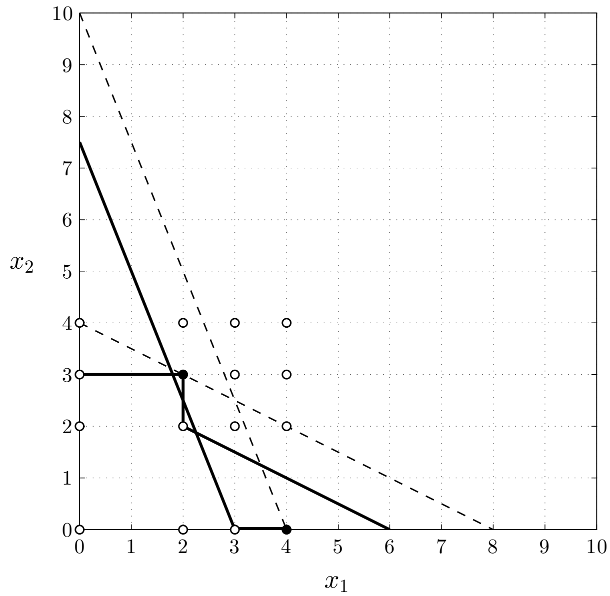


Figure 3: Establishing the CCEI

fit in the form of the efficiency index involves no additional difficulties.

To illustrate how CCEI is calculated, consider Figure 3 which depicts a data set with two observations. Since each chosen bundle is strictly within the budget set (as depicted by the dashed lines) in the *other* observation, the data violate GARP and cannot be rationalized by a non-satiated utility function. If we shrink both sets by some factor (as depicted by the solid lines), then eventually $(2, 3)$ is no longer contained in the shrunken budget set containing $(4, 0)$; at this efficiency threshold, the data set is rationalizable by some locally non-satiated utility function (see Forges and Minelli, 2009). Whether the data set also passes a more stringent requirement such as EU-rationalizability at this efficiency threshold can be checked via the lattice test, performed on the finite lattice indicated in the figure.

4.1 Approximate smooth rationalizability

While Theorem 1 guarantees that there is a continuous function u that extends $\bar{u} : \mathcal{X} \to \mathbb{R}_+$ and rationalizes the data when the required conditions are satisfied, this function is not necessarily smooth. Of course, the smoothness of u is commonly assumed in applications of expected utility and related models and its implications can appear to be stark. For example, suppose that it is commonly known that states 1 and 2 occur with equal probability and we observe the agent choosing $(1, 1)$ at a price vector (p_1, p_2), with $p_1 \neq p_2$. This observation is incompatible with a smooth EU model; indeed, given that the two states are equiprobable, the slope of the indifference curve at $(1, 1)$ must equal -1 and thus it will not be tangential to the budget line and will not be a local optimum. On the other hand, it is trivial to check that this observation is EU-rationalizable in our sense. In fact, one could even find a continuous and *concave* $u : \mathbb{R}_+ \to \mathbb{R}$ for which $(1, 1)$ maximizes expected utility. (Such a u will, of course, have a kink at 1.)

These two facts can be reconciled by noticing that, even though this observation cannot be exactly rationalized by a smooth Bernoulli utility function, it is in fact possible to find smooth functions that come arbitrarily close to rationalizing it. Given an increasing and continuous function u and a compact interval of $\mathbb{R}_+$, there is an increasing and smooth function $\tilde{u}$ that is uniformly and arbitrarily close to u on that interval. As such, if $u : \mathbb{R}_+ \to \mathbb{R}_+$ rationalizes $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^T$ by $\{\phi(\cdot, t)\}_{t=1}^T$, then, for any $e \in (0, 1)$, there is a smooth function $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}_+$ that rationalizes $\mathcal{O}' = \{(x^t, B^t(e))\}_{t=1}^T$ by $\{\phi(\cdot, t)\}_{t=1}^T$. In other words, if a data set is rationalizable by an increasing and continuous Bernoulli utility function, then it can be rationalized by an increasing and smooth utility function for any efficiency threshold arbitrarily close to 1. In this sense, imposing a strong requirement such as smoothness on the Bernoulli utility function does not radically alter a model's ability to explain a given data set.

5. Implementation

We implement our tests using data from the portfolio choice experiment in Choi *et al.* (2007), which was performed on 93 undergraduate subjects at the University of California, Berkeley. Every subject was asked to make consumption choices across 50 decision problems under risk. To be specific, he or she was asked to divide a budget between two Arrow-Debreu securities, with each security paying one token if the corresponding state was realized, and zero otherwise. In a symmetric treatment applied to 47 subjects, each state of the world occurred with probability 1/2, and in two asymmetric treatments applied to 17 and 29 subjects, the probability of the first state was 1/3 and 2/3, respectively. These probabilities were objectively known. Income was normalized to one, and state prices were chosen at random and varied across subjects. Choi *et al.* (2007) analyzed the data by first implementing

Treatment	GARP	DA	EU
$\pi_1 = 1/2$	12/47 (26%)	1/47 (2%)	1/47 (2%)
$\pi_1 \neq 1/2$	4/46 (9%)	1/46 (2%)	1/46 (2%)
Total	16/93 (17%)	2/93 (2%)	2/93 (2%)

Table 1: Pass rates for exact rationalizability

GARP tests on the observations for each subject; those subjects who passed, or came very close to passing (and were therefore consistent with utility maximization) were then fitted individually to a two-parameter version of the disappointment aversion model of Gul (1991).

We repeat the GARP tests of Choi *et al.* (2007) and then subject the data to further tests for DA- and EU-rationalizability, using the lattice procedure we have developed. EU-rationalizability was checked using the test described in Section 2, which simply involves ascertaining whether or not there is a solution to a set of linear inequalities. As we pointed out in Section 3.1, disappointment aversion is a special case of rank dependent utility when there are only two states. The test for DA-rationalizability is a linear test after controlling for β (and hence the distorted probability of the favorable state, $\gamma(\pi_H)$ (see (18))). We implement this test by letting $\gamma(\pi_H)$ take up to 99 different values in $(0, 1)$ and then performing the corresponding linear test. For example, in the symmetric case, $\gamma(\pi_H)$ took on the values 0.01, 0.02, . . . , 0.98, 0.99. Disappointment averse behavior is captured by $\gamma(\pi_H) < 1/2$ (so $\beta > 0$), while elation seeking behavior is captured by $\gamma(\pi_H) > 1/2$ (so $\beta < 0$).

The aggregated rationalizability results are displayed in Table 1. Across 50 decision problems, 16 out of 93 subjects obeyed GARP and were therefore rationalizable by a continuous and strongly monotone utility function; subjects with the symmetric treatment performed distinctly better than those with the asymmetric treatment. Hardly anyone passed the more stringent EU test and there was no improvement with the DA test either.

These results are unsurprising given that we have observed 50 decisions for every subject. Following Choi *et al.* (2007) we now investigate the efficiency thresholds at which subjects pass the tests. We first calculate the CCEI associated with utility maximization for each of the 93 subjects. These distributions are depicted in Figure 4. The black bars in Figure 4a and the solid black line in Figure 4b correspond to the CCEI results for utility maximization among the experimental subjects. (Note that Figure 4a is a replication of Figure 4 in Choi

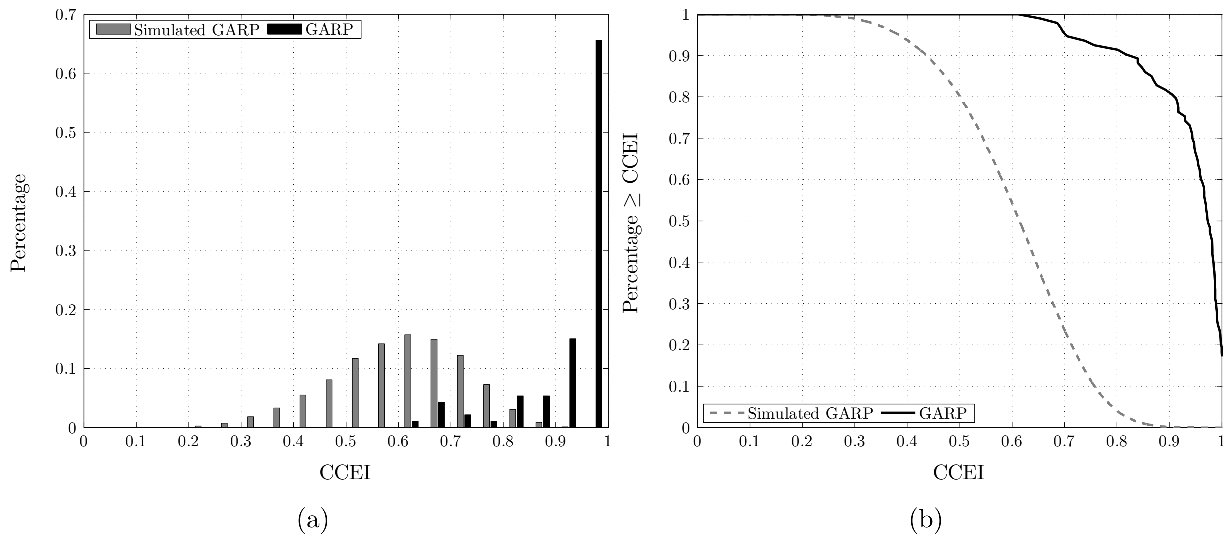
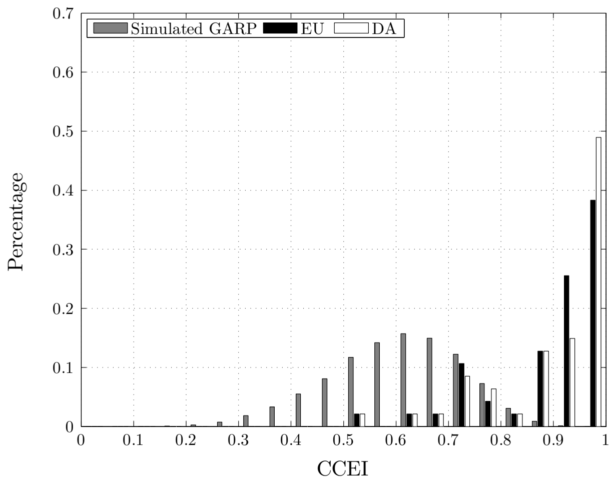


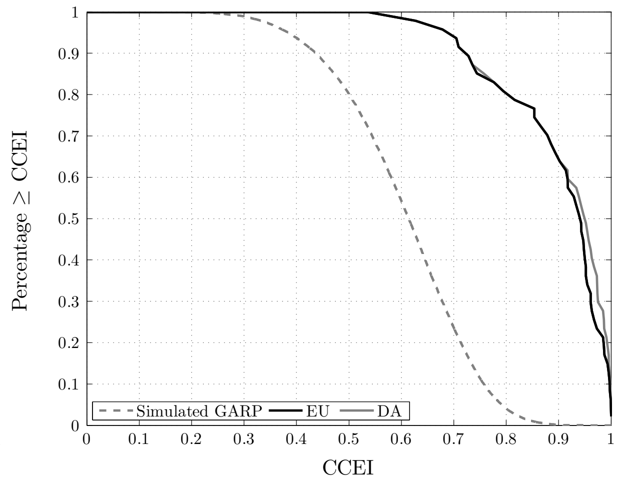
Figure 4: CCEI for utility maximization

et al. (2007).) We see from Figure 4b that more than 80% of subjects have a CCEI above 0.9, and more than 91% have a CCEI above 0.8. A first glance at these results suggest that the data are largely rationalizable by utility maximization.

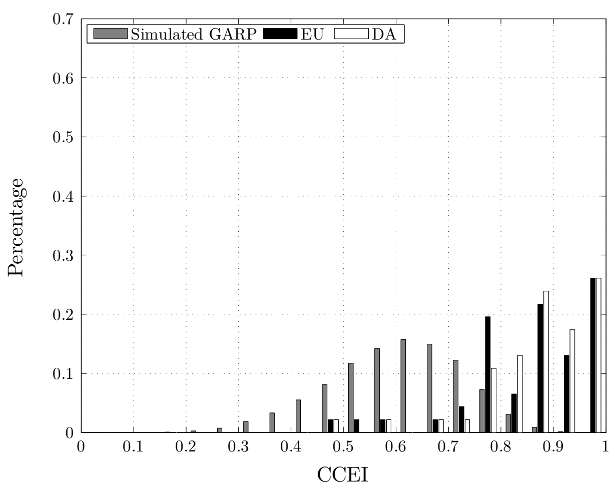
To better understand whether a CCEI of 0.9 implies the relative success or failure of a model to explain a given data set, it is useful to postulate an alternative hypothesis of some other form of behavior against which a comparison can be made. We adopt an approach first suggested by Bronars (1987) that simulates random uniform consumption, i.e., which posits that consumers are choosing randomly uniformly from their budget lines. The Bronars (1987) approach has become common practice in the revealed preference literature as a way of assessing the 'power' of revealed preference tests. We follow exactly the procedure of Choi *et al.* (2007) and generate a random sample of 25,000 simulated subjects, each of whom is choosing randomly uniformly from 50 budget lines that are selected in the same random fashion as in the experimental setting. The gray bars in Figure 4a and the dashed gray line in Figure 4b correspond to the CCEI results for our simulated subjects. The experimental and simulated distributions are starkly different. For example, while 80% of subjects have a CCEI of 0.9 or higher, the chance of a randomly drawn sample passing GARP at an efficiency threshold of 0.9 is negligible, which lends support to utility maximization as a model of choice among contingent consumption bundles.



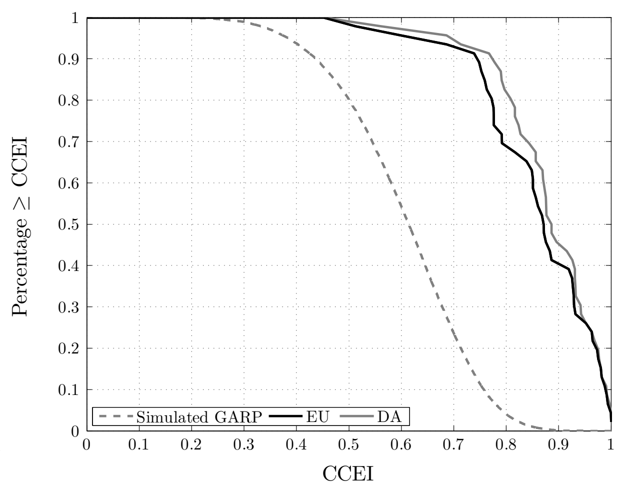
(a) $\pi_1 = 1/2$



(b) $\pi_1 = 1/2$



(c) $\pi_1 \neq 1/2$



(d) $\pi_1 \neq 1/2$

Figure 5: CCEI for EU and DA maximization

Going beyond Choi *et al.* (2007), we then calculate the CCEIs for EU maximization and DA maximization among the 93 subjects. These distributions are shown in Figure 5, where Figures 5a and 5b correspond to the symmetric treatment, and Figures 5c and 5d to the asymmetric treatment. Since these models are more stringent than utility maximization, one would expect the CCEIs to be lower, and they are. Nonetheless, around half of all subjects still obtain CCEIs of 0.9 or more. The DA model has a higher pass rate than the EU model (as it must since it is a more general model) with the improvement more significant for the asymmetric case and at lower CCEI levels. The CCEIs for the random subjects (under EU or DA) are not depicted, but plainly they will have to be even lower than for GARP and therefore very different from the CCEI distributions under EU or DA. We conclude that a large number of the subjects behave in a way that is nearly consistent with EU/DA, a group which is too sizable to be dismissed as occurring naturally in random behavior.

While these results are highly suggestive, we would like a more formal way of comparing across different candidate theories of behavior. The three models we have discussed so far are, in increasing order of stringency, utility maximization, disappointment aversion, and expected utility. What is needed in comparing these models is a way of trading off a model's frequency of correct predictions (which favors utility maximization) with the precision of its predictions (which favors expected utility). To do this, we make use of an axiomatic measure of predictive success proposed by Selten (1991). Selten's *index of predictive success* (which we shall refer to simply as the *Selten index*) is defined as the difference between the relative frequency of correct predictions (the 'hit rate') and the relative size of the set of predicted outcomes (the 'precision'). Our use of this index to evaluate different consumption models is not novel; see, in particular, Beatty and Crawford (2011).

To calculate the Selten index, we need the empirical frequency of correct predictions and the relative size of the set of predicted outcomes. To measure the latter, we use the frequency of hitting the set of predicted outcomes with uniform random draws. Specifically, for each subject, we generate 1,000 synthetic data sets containing consumption bundles chosen randomly uniformly from the actual budget sets facing that subject. (Recall that each subject in Choi *et al.* (2007) faces a different collection of 50 budget sets.) For a given efficiency threshold and for each model (whether GARP, DA, or EU), we calculate the Selten index for every subject, which is either 1 (pass) or 0 (fail) minus the fraction of the 1,000 randomly simulated subject-specific data sets that pass the test (for that model).[14] The index ranges from -1 to 1, where -1 corresponds to failing a lenient test and 1 to passing an stringent test. Lastly, we take the arithmetic average of these indices across

[14] Each of the 1000 synthetic data sets is treated in the following way. Firstly, we subject it to a GARP test; if it fails, we need not go further. If it passes, we test for EU-rationalizability and, if it fails that, we test for DA-rationalizability. We test for GARP using Warshall's algorithm and the EU tests are linear, so both are computationally undemanding. The test for DA-rationalizability is computationally intensive, since each test involves performing 99 linear tests (see earlier discussion in this section). To summarize, for each subject, the maximum number of tests performed on the synthetic data is 1,000 GARP tests plus $1,000 \times 99$ linear tests for DA-rationalizability.

		Efficiency Level				
		0.80	0.85	0.90	0.95	1.00
GARP	$\pi_1 = 1/2$	42/47 (89%) 0.83 [0.94]	40/47 (85%) 0.84 [0.99]	38/47 (81%) 0.81 [1.00]	32/47 (68%) 0.68 [1.00]	12/47 (26%) 0.26 [1.00]
	$\pi_1 \neq 1/2$	43/46 (93%) 0.88 [0.94]	40/46 (87%) 0.86 [0.99]	37/46 (80%) 0.80 [1.00]	29/46 (63%) 0.63 [1.00]	4/46 (9%) 0.09 [1.00]
	Total	85/93 (91%) **0.86** [0.94]	80/93 (86%) **0.85** [0.99]	75/93 (81%) **0.81** [1.00]	61/93 (66%) **0.66** [1.00]	16/93 (17%) **0.17** [1.00]
DA	$\pi_1 = 1/2$	37/47 (79%) 0.79 [1.00]	36/47 (77%) 0.77 [1.00]	30/47 (64%) 0.64 [1.00]	23/47 (49%) 0.49 [1.00]	1/47 (2%) 0.02 [1.00]
	$\pi_1 \neq 1/2$	37/46 (80%) 0.80 [1.00]	31/46 (67%) 0.67 [1.00]	20/46 (43%) 0.43 [1.00]	12/46 (26%) 0.26 [1.00]	1/46 (2%) 0.02 [1.00]
	Total	74/93 (80%) **0.80** [1.00]	67/93 (72%) **0.72** [1.00]	50/93 (54%) **0.54** [1.00]	35/93 (38%) **0.38** [1.00]	2/93 (2%) **0.02** [1.00]
EU	$\pi_1 = 1/2$	37/47 (79%) 0.79 [1.00]	36/47 (77%) 0.77 [1.00]	30/47 (64%) 0.64 [1.00]	18/47 (38%) 0.38 [1.00]	1/47 (2%) 0.02 [1.00]
	$\pi_1 \neq 1/2$	31/46 (67%) 0.67 [1.00]	28/46 (61%) 0.61 [1.00]	18/46 (39%) 0.39 [1.00]	12/46 (26%) 0.26 [1.00]	1/46 (2%) 0.02 [1.00]
	Total	68/93 (73%) **0.73** [1.00]	64/93 (69%) **0.69** [1.00]	48/93 (52%) **0.52** [1.00]	30/93 (32%) **0.32** [1.00]	2/93 (2%) **0.02** [1.00]

Table 2: Pass rates, power, and Selten indices

subjects in order to obtain an aggregate Selten index.[15] Equivalently, the Selten index is the difference between the empirical frequency of a correct prediction and the precision, which is the relative size of predicted outcomes (averaged across agents). Following Bronars (1987), we refer to 1 minus the average relative size of predicted outcomes as a model's *power*; a power index very close to 1 means that the model's predictions are highly specific.

The results are shown in Table 2. Within each cell in the table are four numbers, with the pass rate in the top row and, on the lower row, the Selten index and the power, with the latter in brackets. For example, with $\pi_1 = 1/2$ and at an efficiency threshold of 0.8, 42 out of 47 subjects (89%) pass GARP, the power is 0.94, and so the Selten index is $0.83 = 0.89-(1-0.94)$. What emerges immediately is that, at any efficiency level and for any model, the Selten indices of predictive success are almost completely determined by the pass rates. This is because all of the models have uniformly high power (in fact, very close to 1). It turns

[15] This aggregation is supported by the axiomatization in Selten (1991). Note that our procedure for calculating the Selten index is essentially the same as that in Beatty and Crawford (2011).

out that with 50 observations for each subject, even utility maximization has very high power. Given this, the Selten index tells us that *utility maximization outperforms disappointment aversion, which in turn outperforms expected utility*, and this ordering holds at any efficiency level and across both treatments. All three models have indices well within the positive range, indicating that they are clearly superior to the hypothesis of uniform random choice (which has a Selten index of 0). While academic discussion is often focussed on comparing different models that have been tailor-made for decisions under risk and uncertainty, these findings suggest that we should *not* take it for granted that such models are necessarily better than the standard utility maximization model. At least in the data analyzed here, one could argue that this simple model does a better job in explaining the data, even after accounting for its relative lack of specificity.

Our next objective is to investigate the success of EU and DA in explaining agent behavior, *conditional on the agent maximizing some utility function*. To do this, we focus on those subjects who pass GARP. For evert subject and at a given efficiency threshold, we generate 1,000 synthetic data sets that obey GARP (or modified GARP at the relevant efficiency threshold (see Section 4)) and are thus rationalizable by some increasing utility function.[16] (Note that since we focus on five different efficiency thresholds, this implies a total of 5,000 synthetic data sets for each subject.) We then subject each of these data sets to tests for EU- and DA-rationalizability (at the corresponding efficiency threshold). This gives us the precision or predictive power of EU or DA *within the context of behavior that is consistent with utility maximization*. We then calculate the Selten index; in this case, it is the difference between the empirical frequency of passing the model (the number who pass EU/DA amongst those who pass GARP) and the precision, i.e., the relative size of predicted outcomes (averaged across subjects).

[16] The procedure for creating a synthetic data set is as follows. We randomly pick a budget line (out of the 50 budget lines) and then randomly pick a bundle on that line. We next randomly pick a second budget line, and pick randomly from that part of the line which guarantees that this observation, along with the first, obeys GARP (or, in the case where the efficiency index is lower than 1, a modified version of GARP). We then randomly pick a third budget line, and choose a random bundle on that part of the line which guarantees that the three observations together obey GARP (or modified GARP). Note that there must exist such a bundle on the third budget line; indeed it can be chosen to be the demand (on the third budget line) of any utility function rationalizing the first two observations. We now choose a fourth budget line, and so on.

		Efficiency Level				
		0.80	0.85	0.90	0.95	1.00
DA	$\pi_1 = 1/2$	37/42 (88%) 0.87 [0.99]	36/40 (90%) 0.90 [1.00]	30/38 (79%) 0.79 [1.00]	23/32 (72%) 0.72 [1.00]	1/12 (8%) 0.08 [1.00]
	$\pi_1 \neq 1/2$	37/43 (86%) 0.85 [0.99]	31/40 (78%) 0.78 [1.00]	20/37 (54%) 0.54 [1.00]	12/29 (41%) 0.41 [1.00]	1/4 (25%) 0.25 [1.00]
	Total	74/85 (87%) **0.86** [0.99]	67/80 (84%) **0.84** [1.00]	50/75 (67%) **0.67** [1.00]	35/61 (57%) **0.57** [1.00]	2/16 (12%) **0.12** [1.00]
EU	$\pi_1 = 1/2$	37/42 (88%) 0.87 [0.99]	36/40 (90%) 0.90 [1.00]	30/38 (79%) 0.79 [1.00]	18/32 (56%) 0.56 [1.00]	1/12 (8%) 0.08 [1.00]
	$\pi_1 \neq 1/2$	31/43 (72%) 0.71 [0.99]	28/40 (70%) 0.70 [1.00]	18/37 (49%) 0.49 [1.00]	12/29 (41%) 0.41 [1.00]	1/4 (25%) 0.25 [1.00]
	Total	68/85 (80%) **0.79** [0.99]	64/80 (80%) **0.80** [1.00]	48/75 (64%) **0.64** [1.00]	30/61 (49%) **0.49** [1.00]	2/16 (12%) **0.12** [1.00]

Table 3: Pass Rates, power, and Selten indices (conditional on passing GARP)

The results are displayed in Table 3, where the numbers in each box have interpretations analogous to those in Table 2. For example, with $\pi_1 = 1/2$ and an efficiency threshold of 0.8, 42 subjects pass GARP, of which 37 (88%) pass DA. The power of the DA model (amongst data sets that pass GARP at the 0.8 threshold), is 0.99 and so the Selten index is 0.87. (Notice that the number of subjects with the symmetric treatment passing GARP shrinks from 42 to 12 as the efficiency threshold is raised.) The EU and DA models remain very sharp, with power indices that are very close to 1. In other words, even if we are choosing randomly amongst data sets that obey GARP, the probability of choosing a data set obeying EU or DA is close to zero. On the other hand, the Selten index tells us that, amongst the actual subjects, more than half of those who obey GARP obey EU or DA, with DA having a higher Selten index. So the DA and EU models *do* in fact capture a large proportion of those subjects who display utility-maximizing behavior.

Appendix

The proof of Theorem 1 uses the following lemma.

Lemma 1. *Let $\{C^t\}_{t=1}^T$ be a finite collection of constraint sets in $R_+^{\bar{s}}$ that are compact and downward closed (i.e., if $x \in C^t$ then so is $y \in R_+^{\bar{s}}$ such that $y < x$) and let the functions $\{\phi(\cdot, t)\}_{t=1}^T$ be continuous and increasing in all dimensions. Suppose that there is a finite set*

$\mathcal{X}$ *of* R_+*, an increasing function* $\bar{u} : \mathcal{X} \to R_+$*, and* $\{M^t\}_{t\in T}$ *such that the following holds:*

$$M^t \geqslant \phi(\bar{\mathbf{u}}(x), t) \;\; \text{for all } x \in C^t \cap \mathcal{L} \text{ and} \tag{29}$$

$$M^t > \phi(\bar{\mathbf{u}}(x), t) \;\; \text{for all } x \in (C^t \backslash \partial C^t) \cap \mathcal{L}, \tag{30}$$

where $\mathcal{L} = \mathcal{X}^{\bar{s}}$ *and* $\bar{\mathbf{u}}(x) = (\bar{u}(x_1), \bar{u}(x_2), \ldots, \bar{u}(x_{\bar{s}}))$*. Then there is a continuous and increasing function* $u : R_+ \to R_+$ *that extends* $\bar{u}$ *such that*

$$M^t \geqslant \phi(\mathbf{u}(x), t) \;\; \text{for all } x \in C^t \text{ and} \tag{31}$$

$$\text{if } x \in C^t \text{ and } M^t = \phi(\mathbf{u}(x), t), \text{ then } x \in \partial C^t \cap \mathcal{L} \text{ and } M^t = \phi(\bar{\mathbf{u}}(x), t). \tag{32}$$

Remark: The property (32) needs some explanation. Conditions (29) and (30) allow for the possibility that $M^t = \phi(\bar{\mathbf{u}}(x'), t)$ for some $x' \in \partial C^t \cap \mathcal{L}$; we denote the set of points in $\partial C^t \cap \mathcal{L}$ with this property by X'. Clearly any extension u will preserve this property, i.e., $M^t = \phi(\mathbf{u}(x'), t)$ for all $x' \in X'$. Property (32) says that we can choose u such that for all $x \in C^t \backslash X'$, we have $M^t > \phi(\bar{\mathbf{u}}(x), t)$.

Proof: We shall prove this result by induction on the dimension of the space containing the constraint sets. It is trivial to check that the claim is true if $\bar{s} = 1$. In this case, $\mathcal{L}$ consists of a finite set of points on R_+ and each C^t is a closed interval with 0 as its minimum. Now let us suppose that the claim holds for $\bar{s} = m$ and we shall prove it for $\bar{s} = m + 1$. If, for each t, there is an increasing and continuous utility function $u^t : R_+ \to R_+$ extending $\bar{u}$ such that (31) and (32) hold, then the the same conditions will hold for the increasing and continuous function $u = \min_{t\in T} u^t$. So we can focus our attention on constructing u^t for a single constraint set C^t.

Suppose $\mathcal{X} = \{0, r^1, r^2, r^3, \ldots, r^I\}$, with $r^0 = 0 < r^i < r^{i+1}$, for $i = 1, 2, \ldots, I - 1$. Let $\bar{r} = \max\{r \in R_+ : (r, 0, 0, \ldots, 0) \in C^t\}$ and suppose that $(r^i, 0, 0, \ldots, 0) \in C^t$ if and only if $i \leqslant N$ (for some $N \leqslant I$). Consider the collection of sets of the form $D^i = \{y \in R^m_+ : (r^i, y) \in C^t\}$ (for $i = 1, 2, \ldots, N$); this is a finite collection of compact and downward closed sets in R^m_+. By the induction hypothesis applied to $\{D^i\}_{i=1}^N$, with $\{\phi(\bar{u}(r^i), \cdot, t)\}_{i=1}^N$ as the collection of functions, there is an increasing function $u^* : R_+ \to R_+$ that extends $\bar{u}$ such that

$$M^t \geqslant \phi(\bar{u}(r^i), \mathbf{u}^*(y), t) \;\; \text{for all } (r^i, y) \in C^t \text{ and} \tag{33}$$

$$\text{if } (r^i, y) \in C^t \text{ and } M^t = \phi(\bar{u}(r^i), \mathbf{u}^*(y), t), \text{ then } (r^i, y) \in \partial C^t \cap \mathcal{L} \text{ and } M^t = \phi(\bar{\mathbf{u}}(r^i, y), t). \tag{34}$$

For each $r \in [0, \bar{r}]$, define

$$U(r) = \{u \leqslant u^*(r) : \max\{\phi(u, \mathbf{u}^*(y), t) : (r, y) \in C^t\} \leqslant M^t\}.$$

This set is nonempty; indeed $\bar{u}(r^k) = u^*(r^k) \in U(r)$, where r^k is the largest element in $\mathcal{X}$ that is weakly smaller than r. This is because, if $(r, y) \in C^t$ then so is (r^k, y), and (33) guarantees that $\phi(\bar{u}(r^k), \mathbf{u}^*(y), t) \leqslant M^t$. The downward closedness of C^t and the fact that u^* is increasing also guarantees that $U(r) \subseteq U(r')$ whenever $r < r'$. Now define $\tilde{u}(r) = \sup U(r)$; the function $\tilde{u}$ has a number of significant properties. (i) *For* $r \in \mathcal{X}$, $\tilde{u}(r) = u^*(r) = \bar{u}(r)$ (by the induction hypothesis). (ii) $\tilde{u}$ *is a nondecreasing function* since U is nondecreasing. (iii) $\tilde{u}(r) > \bar{u}(r^k)$ *if* $r > r^k$*, where* r^k *is largest element in* $\mathcal{X}$ *smaller than* r. Indeed, because C^t is compact and ϕ continuous, $\phi(\tilde{u}(r), \mathbf{u}^*(y), t) \leqslant M^t$ for all $(r, y) \in C^t$. By way of contradiction, suppose $\tilde{u}(r) = \bar{u}(r^k)$ and hence $\tilde{u}(r) < u^*(r)$. It follows from the definition of $\tilde{u}(r)$ that, for any sequence u_n, with $\tilde{u}(r) < u_n < u^*(r)$ and $\lim_{n\to\infty} u_n = \tilde{u}(r)$, there is $(r, y_n) \in C^t$ such that $\phi(u_n, \mathbf{u}^*(y_n), t) > M^t$. Since C^t is compact, we may assume with no loss of generality that $y_n \to \hat{y}$ and $(r, \hat{y}) \in C^t$, from which we obtain $\phi(\tilde{u}(r), \mathbf{u}^*(\hat{y}), t) = M^t$. Since C^t is downward closed, $(r^k, \hat{y}) \in C^t$ and, since $\bar{u}(r^k) = u^*(r^k)$, we have $\phi(\mathbf{u}^*(r^k, \hat{y}), t) = M^t$. This can only occur if $(r^k, \hat{y}) \in \partial C^t \cap \mathcal{L}$ (because of (34)), but it is clear that $(r^k, \hat{y}) \notin \partial C^t$ since $(r^k, \hat{y}) < (r, \hat{y})$. (iv) *If* $r_n < r^i$ *for all* n *and* $r_n \to r^i \in \mathcal{X}$*, then* $\tilde{u}(r_n) \to u^*(r^i)$. Suppose to the contrary, that the limit is $\hat{u} < u^*(r^i) = \bar{u}(r^i)$. Since u^* is continuous, we can assume, without loss of generality, that $\tilde{u}(r_n) < u^*(r_n)$. By the compactness of C^t, the continuity of ϕ, and the definition of $\tilde{u}$, there is $(r_n, y_n) \in C^t$ such that $\phi(\tilde{u}(r_n), \mathbf{u}^*(y_n), t) = M^t$. This leads to $\phi(\hat{u}, \mathbf{u}^*(y'), t) = M^t$, where y' is an accumulation point of y_n and $(r^i, y') \in C^t$. But since ϕ is increasing, we obtain $\phi(u^*(r^i), \mathbf{u}^*(y'), t) > M^t$, which contradicts (33).

Given the properties of $\tilde{u}$, we can find a continuous and increasing function u^t such that u^t extends $\bar{u}$, i.e., $u^t(r) = \bar{u}(r)$ for $r \in \mathcal{X}$, $u^t(r) < u^*(r)$ for all $r \in R_+ \backslash \mathcal{X}$ and $u^t(r) < \tilde{u}(r) \leqslant u^*(r)$ for all $r \in [0, \bar{r}] \backslash \mathcal{X}$. (In fact we can choose u^t to be smooth everywhere except possibly on $\mathcal{X}$.) We claim that the conditions (31) and (32) are satisfied for C^t. To see this, note that for $r \in \mathcal{X}$ and $(r, y) \in C^t$, the induction hypothesis guarantees that (33) and (34) hold and they will continue to hold if u^* is replaced by u^t. In the case where $r \notin \mathcal{X}$ and $(r, y) \in C^t$, since $u^t(r) < \tilde{u}(r)$ and ϕ is increasing, we obtain $M^t > \phi(\mathbf{u}^t(r, y), t)$. *QED*

Proof of Theorem 1: This follows immediately from Lemma 1 if we set $C^t = \underline{B}^t$, and $M^t = \phi(\bar{\mathbf{u}}(x^t), t)$. If $\bar{u}$ obeys conditions (7) and (8) then it obeys conditions (29) and (30). The rationalizability of $\mathcal{O}$ by $\{\phi(\cdot, t)\}_{t \in T}$ then follows from (31). *QED*

Proof of Proposition 1: Suppose that $\mathcal{O}$ is rationalizable by ϕ as defined by (21). For any x in the finite lattice $\mathcal{L}$, let $\bar{\pi}(x)$ be an element in $\arg\min_{\pi \in \Pi} \pi \cdot \mathbf{u}(x)$ and let $\bar{u}$ be the restriction of u to $\mathcal{X}$. Then it is clear that the conditions (22) – (24) hold.

Conversely, suppose that there is a function $\bar{\pi}$ and an increasing function $\bar{u}$ obeying the conditions (22) – (24). Define Π as the convex hull of $\{\bar{\pi}(x) : x \in \mathcal{L}\}$; Π is a non-empty and convex subset of Δ_{++} and it is compact in $\mathbb{R}^{\bar{s}}$ since $\mathcal{L}$ is finite. Suppose that there exists $x \in \mathcal{L}$ and $\pi \in \Pi$ such that $\pi \cdot \bar{\mathbf{u}}(x) < \bar{\pi}(x) \cdot \bar{\mathbf{u}}(x)$. Since π is a convex combination of elements in $\{\bar{\pi}(x) : x \in \mathcal{L}\}$, there must exist $x' \in \mathcal{L}$ such that $\bar{\pi}(x') \cdot \bar{\mathbf{u}}(x) < \bar{\pi}(x) \cdot \bar{\mathbf{u}}(x)$, which contradicts (24). We conclude that $\bar{\pi}(x) \cdot \bar{\mathbf{u}}(x) = \min_{\pi \in \Pi} \pi \cdot \bar{\mathbf{u}}(x)$ for all $x \in \mathcal{L}$. We define $\phi : \mathbb{R}^{\bar{s}}_+ \to \mathbb{R}$ by $\phi(\mathbf{u}) = \min_{\pi \in \Pi} \pi \cdot \mathbf{u}$. Then the conditions (22) and (23) are just versions of (7) and (8) and so Theorem 1 guarantees that there is $u : \mathbb{R}_+ \to \mathbb{R}_+$ extending $\bar{u}$ such that $\mathcal{O}$ is rationalizable by $V(x) = \phi(\mathbf{u}(x))$. *QED*

Proof of Proposition 2: Suppose $\mathcal{O}$ is rationalizable by ϕ as defined by (25). Let $\bar{u}$ be the restriction of u to $\mathcal{X}$. For any x in $\mathcal{L}$, let $\bar{\pi}(x)$ be an element in $\arg\min_{\pi \in \Delta_{++}} \{\pi \cdot \mathbf{u}(x) + c(\pi)\}$, and let $\bar{c}(x) = c(\bar{\pi}(x))$. Then it is clear that the conditions (26) – (28) hold.

Conversely, suppose that there is an increasing function $\bar{u}$ and functions $\bar{\pi}$ and $\bar{c}$ obeying conditions (26) – (28). For every $\pi \in \Delta_{++}$, define $\tilde{c}(\pi) = \max_{x \in \mathcal{L}} \{\bar{c}(x) - (\pi - \bar{\pi}(x)) \cdot \bar{\mathbf{u}}(x)\}$. It follows from (28) that $\bar{c}(x') \geqslant \bar{c}(x) - (\bar{\pi}(x') - \bar{\pi}(x)) \cdot \bar{\mathbf{u}}(x)$ for all $x \in \mathcal{L}$. Therefore, $\tilde{c}(\bar{\pi}(x')) = \bar{c}(x')$ for any $x' \in \mathcal{L}$. The function $\tilde{c}$ is convex and continuous but it need not obey the boundary condition. However, we know there is a function c defined on Δ_{++} that is convex, continuous, obeys the boundary condition, with $c(\pi) \geqslant \tilde{c}(\pi)$ for all $\pi \in \Delta_{++}$ and $c(\pi) = \tilde{c}(\pi)$ for $\pi \in \{\bar{\pi}(x) : x \in \mathcal{L}\}$. We claim that, with c so defined, $\min_{\pi \in \Delta_{++}} \{\pi \cdot \bar{\mathbf{u}}(x) + c(\pi)\} = \pi(x) \cdot \bar{\mathbf{u}}(x) + \bar{c}(x)$ for all $x \in \mathcal{L}$. Indeed, for any $\pi \in \Delta_{++}$,

$$\pi \cdot \bar{\mathbf{u}}(x) + c(\pi) \geqslant \pi \cdot \bar{\mathbf{u}}(x) + \tilde{c}(\pi) \geqslant \pi \cdot \bar{\mathbf{u}}(x) + \bar{c}(x) - (\pi - \bar{\pi}(x)) \cdot \bar{\mathbf{u}}(x) = \bar{\mathbf{u}}(x) + \bar{c}(x).$$

On the other hand, $\bar{\pi}(x) \cdot \mathbf{u}(x) + c(\bar{\pi}(x)) = \bar{\pi}(x) \cdot \mathbf{u}(x) + \bar{c}(x)$, which establishes the claim. We define $\phi : \mathbb{R}^{\bar{s}}_+ \to \mathbb{R}$ by (25); then (26) and (27) are just versions of (7) and (8) and so Theorem 1 guarantees that there is $u : \mathbb{R}_+ \to \mathbb{R}_+$ extending $\bar{u}$ such that $\mathcal{O}$ is rationalizable by $V(x) = \phi(\mathbf{u}(x))$. *QED*

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