Learning before matching

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Abstract

We study a problem of assigning objects to agents with endogenous information acquisition. Each object has a known common value and, for each agent, an unknown private value. Each agent has an opportunity to costlessly investigate up to one object and learn its private value which is drawn independently from a symmetric distribution common to all objects. After investigation, each agent reports a preference ranking over objects to a central authority. To isolate the role of learning, we assume that agents conduct their investigations simultaneously and that the central authority allocates objects according to a strategy-proof rule.

We consider two familiar families of rules: priority rules (parameterized by a priority order) and top trading cycles (TTC) rules (parameterized by an endowment).

Our main theorems compare equilibrium efficiency and social welfare of these families: 1) As expected, priority rules are ex-ante efficient while TTC rules are not. Nevertheless, no priority rule Pareto-dominates a TTC rule. 2) Adopting a max-min social welfare criterion, each priority rule is dominated by each TTC rule. 3) According to a utilitarian criterion, all rules in both families achieve the same total ex-ante utility.

Keywords: object allocation, information acquisition, priority rule, top trading cycles, Nash equilibrium, ex-ante efficiency, max-min social welfare.

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1 Introduction

The study of resource allocation takes agents' preferences as primitive and searches for rules with desirable properties. But how do agents form preferences? Agents often go to great lengths to investigate the value of items they may receive: Firms interview candidates, students visit schools, consumers read product reviews, and so on. Does the choice of allocation rule have an effect on this learning process? If so, does this effect have welfare implications? These are the questions that we ask.

Most rules used in practice are *strategy-proof*, meaning that agents have incentives to truthfully report their preferences. This is important for analysis of rules: a rule that is not *strategy-proof* may appear to have many desirable properties when agents are are assumed to truthfully report their preferences that no longer apply when agents report their preferences strategically. But the choice of rule may influence agents' incentives at an earlier stage as well. If agents' decisions to acquire information depend on which rule they expect to be applied, then the same caveats apply; evaluation of rules must account for the incentives they create at this stage as well.

For concreteness, consider a student applying to schools. By visiting, the student may learn whether the school is well-suited to him. However, if the student has limited time to travel or schools schedule visitation days simultaneously, he may be unable to acquire all information relevant to his preferences. Choosing which school to visit is now a strategic decision. Should the student visit his "safety school" where he is essentially guaranteed a seat or a "reach school" with a competitive admissions process? The anticipated allocation rule determines the likelihood that a school is an option for the student and therefore bears on his decision.

We study an object allocation problem in which each agent receives one object.[1] The objects may represent tasks, school seats, dorm rooms, offices, and so on. Agents evaluate objects according to their expected utility. Initially, every agent assigns the same expected value to a given object: its *common value*. Common values may arise when agents have access to common information or consult the same source.[2] For each agent, the value of an object is the sum of its common value and an idiosyncratic component, or *private value*. We model private values as random variables which are identically distributed according to a symmetric distribution and are independent across agents and objects. Additionally, to avoid trivialities, we suppose that the

[1]So that there is a genuine allocation problem, we assume that there are at least two agents and two objects.

[2]For example, many college-bound seniors use the ranking of *US News and World Reports* as a baseline.

support of the distribution is sufficiently large to potentially reverse each pairwise comparison. We also make a technical assumption, weaker than continuity of the distribution, to ensure that there are almost never indifferences.

To extend the model, we propose a simple learning technology: Each agent has the opportunity to learn his private value for one object. That is, investigating one is costless and perfectly reveals its private value. This technology applies, for example, when agents may freely sample the objects but face a time constraint. After investigation, agents report their preference rankings over objects to a central authority which then applies a pre-determined rule to allocate the objects. This induces a two-stage game:

- Investigation: Each agent chooses an object to investigate.
- Reporting: Each agent reports a preference ranking to the central authority.

Choices at each stage are simultaneous. As we are interested in the investigation stage, we restrict attention to *strategy-proof* rules. With a *strategy-proof* rule, each agent has a dominant strategy to truthfully report his preferences and so the second stage is essentially non-strategic.

We highlight two features of our model. First, while we include cardinal utility information, the rules that we study are ordinal, thereby retaining comparability with the standard model. That we include cardinal information in the model allows us to consider notions of *ex-ante efficiency*. Second, while agents have limited opportunities to investigate, their decisions are not driven by cost. Therefore, each agent's strategic choice is *what* to investigate rather than *whether* or *how much*.

Since the choice of what to learn is strategic, we start by characterizing the equilibria induced by different allocation rules. To focus our analysis, we emphasize two familiar families of rules: 1) Priority rules, each of which is parameterized by a priority order. The final allocation under such a rule is calculated by simulating sequential choice according to the priority order. 2) Top trading cycles (TTC) rules, each of which is parameterized by a fixed endowment. The final allocation under such a rule is calculated by simulating trade from the endowment. Our first results characterize the equilibria under priority rules (Theorem 1) and TTC rules (Theorem 2). Under a priority rule, the first agent investigates one of the objects with the two highest common values. The second agent investigates the object with the third highest common value, and in general, the ith agent investigates the object with the $(i+1)$st highest common value. If there are an equal number of objects and agents, the final agent's investigation decision is irrelevant and undetermined. Although we find multiple equilibria, these equilibria are welfare equivalent. Under a TTC rule, there is always an equilibrium in which each agent investigates his endowment. With three or more objects, this equilibrium is unique, and with two agents and two objects, all equilibria are welfare equivalent to this one.

Our main contribution is to evaluate these two families of rules according to their equilibrium efficiency and social welfare. Priority rules are *ex-ante efficient* (Proposition 1) whereas TTC rules are not (Proposition 2). While it would be tempting to favor priority rules based on these results, our social welfare comparisons temper this conclusion. First, no pair consisting of a priority rule and a TTC rule can be ex-ante Pareto ranked.[3] Moreover, the agent who is worst off under a priority rule is better off under each TTC rule. That is, TTC rules are superior to priority rules according to the Rawlsian max-min criterion (Theorem 3). We also consider utilitarian social welfare. Surprisingly, all priority rules and TTC rules achieve the same utilitarian welfare in equilibrium (Theorem 4). See Figure 1 for a schematic illustration of these results. For further comparison, we identify the welfare obtained without investigation and when investigation is optimally directed by a social planner. The performance of our rules relative to these benchmarks depends on the distributions of common and private values.

1.1 Related literature

Closest to our paper is Bade (2014b), which also studies object allocation. In her model, agents choose to acquire information about their preferences through a costly learning technology. By comparison, we a consider a technology with which learning the precise value of one object is free, but additional learning is infinitely costly. Although our model is more structured, it is not a special case. In particular, Bade (2014b) assumes a finite state space whereas we allow private values for each agent and each object to lie on a continuum. More importantly, by fixing the learning technology, we are able to analyze and compare equilibrium strategies and social welfare under various rules. In this sense, the results of our papers are complementary.

Both priority rules and TTC rules feature prominently in the literature on object allocation which, like us, has generally insisted on *ex-post efficiency* and *strategy-proofness*. Shapley and Scarf (1974) introduced the algorithm underlying the TTC rules in a model with endowments and the procedure has been adapted to various settings (e.g., Abdulkadiroğlu and Sönmez (2003)). Like priority rules, TTC rules belong to a family of "hierarchical exchange" rules characterized by Pápai (2000a). Pycia and Ünver (2014) identify the even larger class comprising all *ex-post efficient*

[3]This is in contrast with Bade (2014b) where a priority rule may ex-ante Pareto dominate a TTC rule.

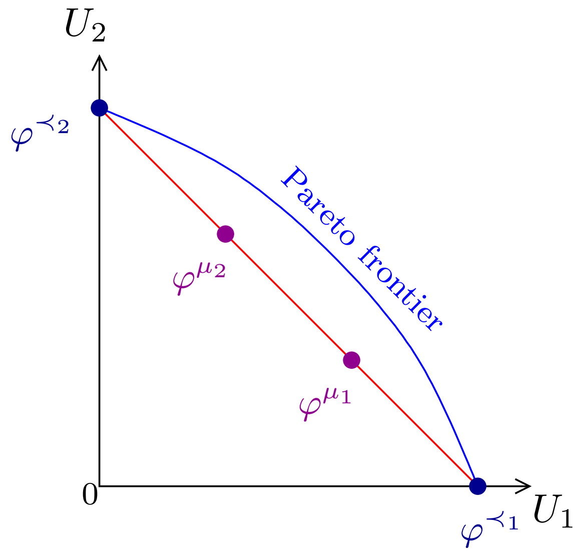

Figure 1: Welfare comparisons: When there are two agents and two objects, there are two priority rules: $\varphi^{\prec_1}$, which favors agent 1, and $\varphi^{\prec_2}$, which favors agent 2. Similarly, there are two TTC rules: φ^{μ_1}, which endows agent 1 with the ex-ante better object, and φ^{μ_2}, which endows agent 2 with that object. Representing the utility of agent 1 on the horizontal axis and agent 2 on the vertical axis, three of our results are illustrated: 1) While both priority rules reach the ex-ante Pareto frontier, neither dominates either of the TTC rules; 2) the TTC rules lead to outcomes that are more just in the Rawlsian sense; and 3) all four rules yield the same utilitarian welfare.

and *group strategy-proof* rules.

In recent work, Bu (2014) adds a characterization of priority rules to the list started by Svensson (1994). Morrill (2013) provides an axiomatic justification for TTC rules.[4] Abdulkadiroğlu and Sönmez (1998) note an important parallel between priority rules and TTC rules: each family traces the full set of *ex-post efficient* allocations when permuting the roles of the agents. Bade (2014a) extends this observation to all hierarchical exchange rules. While the standard model assumes strict preferences, indifferences are sometimes relevant and overturn some results (see, e.g., Erdil and Ergin (2008); Jaramillo and Manjunath (2012); Alcalde-Unzu and Molis (2011); Ehlers (2014)).

A separate body of research studies endogenous information. In an auction setting where agents may acquire information at a cost, Bergemann and Välimäki (2002) sin-

[4]Relatedly, Ma (1994) characterizes the "strict core."

gle out the Vickrey-Clark-Groves mechanism as optimal. Gerardi and Yariv (2008); Gershkov and Szentes (2009) consider voting in a jury setting and find that costly information drives a wedge between *ex-post efficiency* and incentives to acquire information so that both desiderata cannot be achieved simultaneously.[5]

Finally, our social welfare criteria address classical concerns (Rawls, 1972). In similar spirit to us, Hafalir and Miralles (2014) study max-min and utilitarian welfare. Their model, assigning agents to hierarchical positions, corresponds to our setting prior to investigation. Allowing for probabilistic assignment, Bhalgat et al. (2011) introduce additional measures of social welfare based on ordinal preferences.

The remainder of the paper is organized as follows: We formalize our model in Section 2. We characterize equilibria under priority rules and TTC rules in Section 3. In Section 4 we turn to the efficiency of these equilibria and in Section 5 we investigate their implications for social welfare. We conclude in Section 6. All proofs are in the appendix.

2 Model

A finite set of objects $A \equiv \{a_1, \ldots, a_m\}$ must be assigned to a finite set of agents $N \equiv \{1, \ldots, n\}$ where $2 \leq n \leq m$ so that each agent receives one object. Each object $a \in A$ has **common value** $v_a \in \mathbb{R}$ and no two objects have the same common value. Labeling objects in decreasing order of their common values, for each $a_k \in A$, let $v_k \equiv v_{a_k}$ so that $v_1 > v_2 > \cdots > v_m$. For each agent $i \in N$, each object $a \in A$ also has a **private value** ε_{ia} so the **value** of object a to agent i is

$$v_{ia} \equiv v_a + \varepsilon_{ia}.$$

The private values $(\varepsilon_{ia})_{i \in N, a \in A}$ are random variables which are initially unknown to the agents but potentially discoverable. We assume that $(\varepsilon_{ia})_{i \in N, a \in A}$ are independent and identically distributed with cdf F satisfying

1. Symmetry: for each $\alpha \in A$, $F(\alpha) = 1 - F(-\alpha)$;

2. Reversal: $0 < F(v_m - v_1) < F(v_1 - v_m) < 1$; and

3. Almost no indifference: for each pair $a, b \in A$, F is continuous at $v_a - v_b$.

[5] Bognar et al. (2015) model costly participation in voting and offer an interpretation in terms of costly information acquisition.

Conditions (1) and (2) ensure that F is symmetric about zero and that all pairwise comparisons can be reversed by sufficiently high (low) realizations of the private value; Condition (3) ensures that, having investigated an object, an agent is almost never indifferent between two objects. Using the ordering by common values, for each $i \in N$ and each $a_j \in A$, let $\varepsilon_{ij} \equiv \varepsilon_{ia_j}$.

2.1 Learning and equilibrium

Let $\mathcal{P}$ be the set of linear orders over A and $P_0 \equiv (a_1, \ldots, a_m)$. Initially, each agent ranks objects according to P_0, but has the opportunity to learn his private value for one object. Agent i's **investigation strategy**[6] is $\sigma_i \in A$ and an investigation strategy profile is $\sigma \equiv (\sigma_i)_{i \in N}$. Agent i's investigation reveals $\varepsilon_{i\sigma_i}$ and revises his ranking to $P_i(\varepsilon_{i\sigma_i})$. The updated ranking agrees[7] with P_0 on $A \backslash \{\sigma_i\}$, although the position of σ_i may differ. In particular, for each $a \in A \backslash \{\sigma_i\}$, $\sigma_i \ P_i(\varepsilon_{i\sigma_i}) \ a \Leftrightarrow \varepsilon_{i\sigma_i} > v_a - v_{i\sigma_i}$ so that $v_{ia} = v_a < v_{\sigma_i} + \varepsilon_{i\sigma_i}$. By Condition (3), $P_i(\varepsilon_{i\sigma_i})$ is a linear order with probability 1, so we assume $P_i(\varepsilon_{i\sigma_i}) \in \mathcal{P}$. Let $\varepsilon_\sigma \equiv (\varepsilon_{i\sigma_i})_{i \in N}$ and $P(\varepsilon_\sigma) \equiv (P_i(\varepsilon_{i\sigma_i}))_{i \in N}$.

After learning, each agent i reports $P_i \in \mathcal{P}$, (possibly different from $P_i(\varepsilon_{i\sigma_i})$) to a central authority which then applies a rule to determine the final allocation.[8] The set of (feasible) allocations is $X \equiv \{\nu \colon N \to A : \forall i, j \in N, \text{ if } i \neq j \text{ then } \nu_i \neq \nu_j\}$ so an **allocation** is a function that assigns a distinct object to each agent. A **rule** $\varphi \colon \mathcal{P}^N \to X$ assigns an allocation to each profile of reported preferences. A rule is **strategy-proof** if, conditional on investigation, it is a weakly dominant strategy for each agent i to report $P_i(\varepsilon_{i\sigma_i})$. As we are interested in understanding agents' investigation strategies, we restrict attention to *strategy-proof* rules and henceforth assume that agents report $P(\varepsilon_\sigma)$.

Let φ be a *strategy-proof* rule and $\sigma \in A^N$. For each ε_σ, let $\nu(\varepsilon_\sigma) \equiv \varphi(P(\varepsilon_\sigma))$. That is, at the realization ε_σ, $\nu(\varepsilon_\sigma)$ is the realized allocation. Agent i's **expected utility** at σ under φ is

$$U_i(\sigma, \varphi) \equiv \mathbf{E}_F[v_{i\nu(\varepsilon_\sigma)}] = \int \cdots \int v_{i\nu(\varepsilon_\sigma)} \, dF(\varepsilon_{1\sigma_1}) \cdots dF(\varepsilon_{n\sigma_n}).$$

[6] This specification restricts attention to pure strategies. Allowing mixed investigation strategies does not affect our results.

[7] By Condition (1), for each pair $a, b \in A \backslash \{\sigma_i\}$, $\mathbf{E}_F[\varepsilon_{ia}] = \mathbf{E}_F[\varepsilon_{ib}] = 0$ so $\mathbf{E}_F[v_a | \varepsilon_{i\sigma_i}] = v_a$ and $\mathbf{E}_F[v_b | \varepsilon_{i\sigma_i}] = v_b$. Thus, $a \ P_i(\varepsilon_{i\sigma_i}) \ b \Leftrightarrow v_a > v_b$.

[8] Equivalently, each agent reports a pair $(\sigma_i, \varepsilon_i) \in A \times \mathbb{R}$ representing $\varepsilon_{i\sigma_i}$. We define reports as preference rankings because our rules are traditionally defined over this domain.

The decomposition follows by independence. A (subgame perfect) **equilibrium** is $\sigma \in A^N$ such that for each $i \in N$ and each $\sigma'_i \in A^N$, $U_i(\sigma) \geq U_i(\sigma'_i, \sigma_{-i})$.

2.2 Rules

We focus on two classes of *strategy-proof* rules: priority rules and top trading cycles rules. Given an order $\prec$ over N, the **priority rule** associated with $\prec$, $\varphi^{\prec}$, begins with an order over agents and uses their reported preferences to simulate sequential choice: First, the agent with the highest priority is assigned his most preferred object, then the agent with the second highest priority is assigned his most preferred object among those that remain, and so on.[9] More precisely, for each $P \in \mathcal{P}^N$ and each $i \in N$,

$$\varphi_i^{\prec}(P) \equiv \underset{P_i}{\operatorname{argmax}}\ A \backslash \left(\bigcup_{j \prec i} \varphi_j^{\prec}(P) \right).$$

Given an allocation $\mu \in X$,[10] the **top trading cycles (TTC) rule** associated with μ, φ^{μ}, simulates trade as if μ were the endowment according to Gale's top trading cycles algorithm. Informally,[11] each agent points at his most preferred object and each object points at the agent endowed with it, which forms at least one cycle. Agents in a cycle are assigned the objects at which they are pointing and these agents and objects are removed. Each unassigned agent now points at his most preferred object among those that remain. This procedure continues until each agent has been assigned an object.

Henceforth, when considering $\varphi^{\prec}$, we label the agents so that $1 \prec 2 \prec \cdots \prec n$. Similarly, when considering φ^{μ}, we label the agents so that for each $i \in N$, $\mu_i = a_i$.

3 Equilibria

We now characterize the equilibria under priority rules (Theorem 1) and under TTC rules (Theorem 2).

[9]It is important to point out that it is merely *as if* the assignments are made sequentially. Each agent decides what to investigate *simultaneously* and *before* submitting preferences so that the allocation rule may be applied.

[10]We describe the rule under the assumption $n = m$. As our equilibrium analysis will show, additional objects play no role until the final round of the algorithm. In this round, we suppose that all remaining objects point at the final unassigned agent.

[11]See Shapley and Scarf (1974) for a complete definition.

3.1 Priority rules

In equilibrium, the agent with the highest priority investigates one of the two objects with the highest expected values and receives one of these objects. The agent with the second highest priority investigates the object with the third highest expected value. He receives one of the objects with the three highest expected values. The agent with the third highest priority investigates the object with the fourth highest expected value and so on. If $m = n$, then there is no object for the agent with the lowest priority to investigate. In this case, the agent has no strategic decision and may investigate any object. Although we identify multiple equilibria distinguished by the strategies of the agents with the highest and lowest priority, all equilibria are welfare-equivalent.

Theorem 1. *Let $\varphi^{\prec}$ be a priority rule. Then $\sigma \in A^N$ is an equilibrium under $\varphi^{\prec}$ if and only if (i) $\sigma_1 \in \{a_1, a_2\}$; and (ii) for each $i \in N$ with $1 < i < m$, $\sigma_i = a_{i+1}$.*

3.2 Top trading cycles rules

When $n = m \geq 3$, there is a unique equilibrium: each agent investigate his endowment.[12] When $n = m = 2$, each agent investigates one of the objects. All combinations constitute equilibria and are welfare-equivalent to the equilibrium in which both agents investigate their endowments. Recall that we are, for notational simplicity, supposing that agents are labeled so that for each $i \in N$, $\mu_i = a_i$.

Theorem 2. *Let φ^{μ} be a TTC rule. Then $\sigma \in A^N$ is an equilibrium under φ^{μ} if and only if (i) $n = m \geq 3$ and for each $i \in N$, $\sigma_i = a_i$; or (ii) $n = m = 2$ and $\sigma \in A^N$.*

In general, investigating either of two objects has the same probability of reversing an agent's ranking of those objects. Because the private values are independent and identically distributed, when both objects are in an agent's *option set*,[13] both strategies yield the same expected utility. This is why there are multiple equilibria under a TTC rule when there are two objects: whenever the agent has a choice, the object an agent investigated is in the agent's option set. This reasoning also explains why under a priority rule, the agent with the highest priority has two equilibrium strategies.

[12]It is worth emphasizing that this is not a true endowment since we are interested in the solution of a purely allocative problem. Instead, it is *as if* there were an endowment and agents could trade.

[13]We borrow the terminology of **?** of an "option set" to refer to all of the alternatives that an agent can obtain by varying his reported preference, holding fixed the reports of others.

We can sharpen the comparison between priority and TTC rules by considering inheritance structures (Pápai, 2000a). A priority rule can be described by an inheritance structure in which the first agent is initially endowed with all of the objects and agents inherit objects according to the priority order. In our model, the agent 1 always receives an object with one of the two highest common values, so agent 2 always inherits the remaining objects. Thus, it is equivalent to think of these objects as being part of agent 2's endowment. Continuing, the agent with second highest priority always receives an object with one of the three highest common values. In general, agent i always receives an object with one of the $i + 1$ highest common values.

Remark 1. A priority rule $\varphi^{\prec}$ can be represented as a priority structure in which each agent i as being endowed with a_{i+1}, allowing $a_{n+1} = \emptyset$. Then the initial endowment structure has the form $\hat{\mu} = \big(\{a_1, a_2\}, \{a_3\}, \{a_4\}, \ldots, \{a_n\}, \emptyset\big)$. Compared with φ^{μ}, the first agent's endowment has improved while the last agent's endowment has worsened. For the remaining agents, the comparison is ambiguous; the inheritance structure weakens their position with respect to preceding agents, but strengthens their position with respect to following agents.

4 Efficiency

With equilibria in hand, we next consider efficiency. Since agents have the opportunity to investigate objects, we may consider efficiency either before or after investigation.[14]

We first formalize the latter concept. Given $P \in \mathcal{P}^N$, an allocation $\nu \in A^N$ is **(Pareto) efficient** at P if it is not Pareto dominated: for each $\nu' \in A^N$ and $i \in N$, if $\nu'_i \; P_i \; \nu$, then there is $j \in N$ such that $\nu_j \; P_j \; \nu'_j$. A rule φ is **ex-post (Pareto) efficient** if for each $P \in \mathcal{P}^N$, $\varphi(P)$ is efficient.

Our next efficiency concept accounts for investigation strategies as well. An investigation profile $\sigma \in A^N$ and rule $\varphi \colon \mathcal{P}^N \to X$ are jointly **ex-ante (Pareto) efficient** if no alterative pair achieves an ex-ante Pareto improvement: for each $\sigma' \in A^N$ and $\varphi' \colon \mathcal{P}^N \to X$, if there is $i \in N$ such that $U_i(\sigma', \varphi') > U_i(\sigma, \varphi)$, then there is $j \in N$ such that $U_j(\sigma', \varphi') < U_j(\sigma, \varphi)$. A rule $\varphi \colon \mathcal{P}^n \to X$ is **ex-ante**

[14] Recall that, for each problem, there is uncertainty about ε_{ia} for each $i \in N$ and each $a \in A$. The allocation is chosen with the knowledge of at most one of these values for each agent. By *ex-post* versus *ex-ante*, we refer to the time at which this information is learned, not the time at which all uncertainty is resolved.

(Pareto) efficient if for each equilibrium $\sigma \in A^N$ under φ, the pair (σ, φ) is ex-ante efficient.

The inputs of all rules we consider are preference rankings. Accordingly, *ex-ante efficiency* involves this informational assumption. A further strengthening would allow the rule to condition on the realizations of private values rather than only the ordinal preferences they generate.[15] As our results are generally negative, we emphasize the weaker definition.

Remark 2. All priority rules and TTC rules are *ex-post efficient*, as are many other *strategy-proof* rules (Pápai, 2000a; Pycia and Ünver, 2014).

All priority rules are additionally *ex-ante efficient*.[16] Although our definitions differ slightly, Proposition 1 follows from the same logic as Theorem 2 of Bade (2014b) and we omit a formal proof.

Proposition 1. *Each priority rule is* ex-ante efficient.

With two agents and two objects, each TTC rule is also *ex-ante efficient*. In contrast, once there are three objects, no TTC rule is *ex-ante efficient*. This is similar to the inefficiency discovered by Bade (2014b), although qualitatively different. Bade (2014b) provides examples in which agents choose not to engage in socially beneficial investigation under TTC because the personal cost of investigation is too high. In our setting, however, investigation cost plays no role. Moreover, our weaker efficiency notions reveals a more robust inefficiency: each TTC rule may be improved upon by a rule that only considers ordinal preferences.

Proposition 2. *If $n = m = 2$, then each top trading cycles rule is ex-ante efficient. If $2 < m$, then no top trading cycles rule is* ex-ante efficient.

The third object plays an essential role in the example used to prove Proposition 2. The Pareto-dominating rule identifies additional beneficial trades between agents 1 and 2 by conditioning on the location of the third object in the agents' preferences. This explains why $n = m = 2$ is a special case.

We may ask whether this limited positive result applies to the stronger notion of *ex-ante efficiency* which allows the rule to condition on ε_σ directly. Unfortunately, it does not. Example 5 in the appendix shows that when $n = m = 2$, φ^μ is not *ex-ante efficient* under this stronger definition.

[15] This corresponds to the definition of ex-ante efficiency in terms of "outcome functions" in Bade (2014b).

[16] Priority rules continue to be *ex-ante efficient* under the more stringent definition.

5 Social welfare

We now turn to social welfare. We consider two leading criteria: max-min welfare and utilitarian welfare. According to the first criterion, TTC rules are unambiguously superior to priority rules (Theorem 3). In contrast, and surprisingly, the utilitarian criterion does not distinguish between these rules in our model: all TTC rules and priority rules achieve the same utilitarian welfare in equilibrium (Theorem 4). To better understand this common utilitarian welfare, we compare it to utilitarian welfare without investigation and utilitarian welfare when a social planner controls investigation.

In Appendix A.5 we provide explicit formulas for the equilibrium welfare of each agent under $\varphi^{\prec}$ and φ^{μ} using our previous analysis. Briefly comparing expressions, we see that under either $\varphi^{\prec}$ or φ^{μ}, the agents are ordered by ex-ante expected utility: agent 1 is better off than agent 2, and agent 2 is better off than agent 3, and so on. Also, comparing $n = m = 2$ and $n = m = 3$ under either rule, both agents 1 and 2 are at least as well off in the larger problem regardless of the rule we use. In fact, except for agent 1's ex-ante expected utility under $\varphi^{\prec}$, all utilities increase. Finally, comparing rules, when $n = m = 2$, agent 2's ex-ante expected utility is higher under φ^{μ} than under $\varphi^{\prec}$. Similarly, when $n = m = 3$, agent 3's ex-ante expected utility is higher under φ^{μ} than under $\varphi^{\prec}$. Our welfare analysis extends these observations.

5.1 Max-min welfare

Although priority rules are *ex ante efficient* whereas TTC rules are not, no priority rule ex-ante Pareto dominates a TTC rule. Moreover, for each priority rule and each TTC rule, the equilibrium welfare of the least well off agent is higher under the TTC rule than under the priority rule. Consequently, the max-min criterion favors TTC rules over priority rules.

Formally, for each vector $u \in \mathbb{R}^n$, let $\bar{u}$ be the vector formed from u by listing its coordinates in increasing order. For each pair $u, u' \in \mathbb{R}^n$, u **max-min dominates** u' if there is $i \in \{1, \ldots, n\}$ such that $\bar{u}_i > \bar{u}'_i$ and for each $j \in \{1, \ldots, i-1\}$, $\bar{u}_j \geq \bar{u}'_j$. Given $\sigma, \sigma' \in A^N$ and rules φ and φ', the pair (σ, φ) **max-min welfare dominates** (σ', φ') if $U(\sigma, \varphi)$ max-min dominates $U_i(\sigma', \varphi')$.

Theorem 3. *Each top trading cycles rule max-min welfare dominates each priority rule.*

While the worst-off agent is always better off under φ^{μ} than $\varphi^{\prec}$, the comparison is reversed for the best-off agent. In light of Remark 1, this has an intuitive interpretation: moving from φ^{μ} to $\varphi^{\prec}$ compresses the initial endowment structure from

$\mu \equiv (\{a_1\}, \{a_2\} \ldots, \{a_{n-1}\}, \{a_n\})$ to $\mu' \equiv (\{a_1, a_2\}, \{a_3\} \ldots, \{a_n\}, \emptyset)$. This benefits agent 1 and harms agent n.

When $n \geq 3$, we may also ask about the welfare of intermediate agents. For these agents, the comparison is ambiguous. Continuing to interpret the rules according to their endowment structures, we observe countervailing effects. Like agent n, each intermediate agent has a superior initial endowment under φ^{μ} than $\varphi^{\prec}$. However, under $\varphi^{\prec}$, each intermediate agent inherits an additional object and so the agent's option set includes at least two objects. Under φ^{μ} in contrast, the agent's option set sometimes consists only of his endowment. In Appendix B we provide examples (Examples 1 and 2) that show that either effect may dominate.

5.2 Utilitarian welfare

We now take the perspective of a social planner who aims to maximize utilitarian welfare. In contrast with the max-min approach, this necessarily involves interpersonal comparisons which are often suspect. Given the symmetry of our setting, however, agents' utilities have a common scale. Moreover, the agents have the same opportunities to investigate objects and, absent investigation, their utilities are identical. Therefore, we may interpret interpersonal comparisons as representing a hypothetical situation in which compensation is possible with respect to the common utility scale.

Let $\sigma \in A^N$ and φ be a rule. The **utilitarian welfare** of the pair (σ, φ) is then $U(\sigma, \varphi) \equiv \sum_N U_i(\sigma, \varphi)$. Implicit in our definition, φ applies to agents' truthful preferences $P(\varepsilon_\sigma)$.

5.2.1 Benchmark utilitarian social welfare

To begin our analysis of utilitarian welfare, we establish two benchmarks. At one extreme, a rule may choose a fixed allocation without regard for agents' preferences. In this case, the utilitarian social welfare is the sum of the common values of the objects. We denote this quantity by $U_0(n) \equiv \sum_{i=1}^{n} v_i$. While a malevolent social planner could potentially use investigation results to achieve lower welfare, a constant rule establishes a minimum that can be obtained through a *strategy-proof* rule.

We now search for an upper bound on welfare. To the extent that agents are able to take advantage of the information they learn about their private values, utilitarian social welfare may exceed U_0. Here we adopt the perspective of a social planner who selects both the investigation strategies and final allocation. Abstracting away from the reporting stage, we further suppose the planner observes all private values revealed by investigation and has all of the information available to the agents collectively. Since we ignore incentive constraints, this upper bound will be unattainable by *strategy-proof* rules.

First, we search for a policy that maximizes utilitarian welfare, beginning with optimal investigation strategies. Since investigation of one object is costless, utilitarian welfare is maximized when all agents investigate an object. In fact, we can say more: utilitarian welfare is maximized when all agents investigate *distinct* objects.

Lemma 1. *For each pair $n, m \in \mathbb{N}$, utilitarian welfare is maximized when the agents investigate distinct objects among those with the n highest common values.*

Although intuitive, the result is specific to our setting; the proof makes essential use of our assumptions that the private values are independent and identically distributed. Given agents' investigation strategies, an optimal allocation rule seeks to maximize the benefit from learning. Investigation reveals one private value for each agent which may be positive or negative. Ideally, each agent with a positive realization receives the object he investigated and each agent with a negative realization receives an object different from the one he investigated. This policy is almost feasible, but not quite: when exactly one agent has a negative realization, the recommendations are incompatible. In this case, an optimal policy computes the sum of the negative realization and the smallest positive realization. If the sum is negative, these agents exchange objects; if the sum is positive, they receive the objects that they investigated.

Having described an optimal policy, our next result bounds the utilitarian welfare it achieves. As the number of objects becomes large, these bounds converge. These bounds depend on the distribution of private value through $\eta_0 \equiv \mathbf{E}_F[x|x > 0]$.

Proposition 3. *For each pair $n, m \in \mathbb{N}$ with $2 \leq n \leq m$, the maximum utilitarian social welfare is bounded above by $U_0(n) + \frac{n}{2}\eta_0$ and bounded below by $U_0(n) + \frac{n}{2}\eta_0 - \frac{n}{2^n}\eta_0$. As $n \to \infty$, these bounds converge.*

5.2.2 Priority and top trading cycles rules

Our final theorem shows that, surprisingly, utilitarian welfare is the same under both $\varphi^{\prec}$ and φ^{μ}.

Theorem 4. *All priority rules and all top trading cycles rules achieve the same utilitarian welfare.*

The proof (Step 3 in Appendix A.5.4) has an interesting consequence for both types of rules: when the population increases from $m - 1$ to m, the difference in utilitarian welfare is entirely accounted for by the allocation of a previously discarded object. Although the new agent has an opportunity to investigate, neither rule makes use of it to increase utilitarian welfare.

We may also ask how the common utilitarian welfare under $\varphi^{\prec}$ and φ^{μ} compares to our benchmarks. This depends on the variability of common values and private values. Intuitively, these rules perform best when private values are relatively more important than common values. Learning is then likely to reverse several preference rankings and both families of rules approximate an optimal policy. On the other hand, when preference reversals are unlikely, the rules provide few opportunities to benefit from investigation. Thinking of the difference between utilitarian welfare under an optimal policy and no investigation as the available surplus, we provide examples in the Appendix to show that $\varphi^{\prec}$ and φ^{μ} may achieve arbitrarily large (Example 3) or arbitrarily small (Example 4) fractions of this surplus.

6 Discussion

Extending the object allocation problem to allow learning, we proposed a structured model with ex-ante common values and a limited learning technology which allows agents to investigate one object. This structure allows us to characterize equilibrium investigation strategies when priority rules or top trading cycles rules are applied to allocate objects. In equilibrium under a priority rule, the agent with the ith highest priority investigates the object with the $(i+1)$st highest common value. In equilibrium under a TTC rule, each agent investigates his endowment. Once there are three objects, this equilibrium is unique. For each rule, equilibrium welfare is unique and most agents have unique equilibrium strategies.

After characterizing equilibria, we turned to efficiency and social welfare. Unlike priority rules, which are *ex-ante efficient*, TTC rules may be ex-ante Pareto dominated by other ordinal rules. On the other hand, no priority rule ever ex-ante Pareto dominates a TTC rule. Moreover, the worst off agent is better off under each TTC rule than the worst off agent under each priority rule. Considering a utilitarian measure, all priority rules and all TTC rules achieve the same social welfare. While *ex-ante efficiency* favors priority rules, max-min welfare favors TTC rules, and the utilitarian criterion is silent.

Toward an understanding of endogenous information acquisition, our study is preliminary and leaves open many many questions. We briefly discuss avenues for future research. First, we may ask about the equilibria of other rules, particularly rules which are not themselves *strategy-proof*. To the extent that such rules are used in practice, we need to understand how they influence learning decisions.

Our model also admits numerous extensions. For example, agents may have access to a more powerful learning technology and be able to investigate multiple objects. Agents now face the problem of optimally allocating a fixed investigation budget. Alternatively, agents ex-ante preferences may be heterogeneous. This would allow us to account for students who prefer local schools. Relatedly, the distributions of private values may be correlated, either among object or among agents: if one student discovers that a school is a poor fit, similarly situated students are likely to draw the same conclusion. Pushing in a different direction, the distributions of private values may differ across objects: one school may have a special program that is a great success for some but a disaster for others while another school offers a standard curriculum with a relatively certain value.

Other generalizations would move away from the deterministic one-to-one assignment problem. Instead, agents may each receive a fixed number of objects.[17] If we are interested in equal treatment, we may turn to probabilistic assignments in the second stage.[18] The implications of our model for the school choice problem are also important to understand. This would entail adding capacities and priorities to each object. Similarly, in a two-sided matching model, agents on both sides may have opportunities to learn about potential partners. Pursing this extension may illuminate the mysterious process of dating.

A Proofs

A.1 Preliminaries

We start with some notation. For each pair $i, j \in \{1, \ldots, m\}$, let $p_{ij} \equiv 1 - F(|v_i - v_j|) = F(-|v_i - v_j|)$ and $\eta_{ij} \equiv \mathbf{E}_F[x|x > v_j - v_i]$. Then p_{ij} is the probability that learning the private value of either a_i or a_j will reverse their ranking in an agent's preferences. Similarly, η_{ij} is the expectation of an agent's private value for a_i given that it is large enough to keep or raise a_i above a_j in the agent's preference ranking. For each pair $i, j \in A$ with $i < j$, $p_{ij} = p_{ji} > 0$. However, $\eta_{ij} \neq \eta_{ji}$. Instead, $0 < \eta_{ij} < \eta_{ji}$ (see Lemma 2). Let $p_0 \equiv 1 - F(0) = F(0) = \frac{1}{2}$ and $\eta_0 \equiv \mathbf{E}_F[x|x > 0]$. Finally, for each $i \in N$ and each $P_{-i} \in \mathcal{P}^{N \setminus \{i\}}$, agent i's **option set** under φ is

$$O_i(P_{-i}, \varphi) \equiv \{a \in A : \exists P_i \in \mathcal{P} \text{ such that } \varphi_i(P_i, P_{-i}) = a\}.$$

Taking the preference reports of other agents as given, $O_i(P_{-i}, \varphi)$ represents those objects that agent i may receive by varying his report. Given an event E, agent i's

[17] See, for example, Pápai (2000b, 2001); Klaus and Miyagawa (2001).

[18] Bogomolnaia and Moulin (2001) first study this extension of the standard model.

expected utility conditional on E is $U_i(\sigma|E)$.

Next, we present a technical lemma that allows us to compare private values.

Lemma 2. *For each triple $i, j, k \in \{1, \ldots, m\}$ with $i < j < k$,*

1. $p_{ij} = p_{ji}$,
2. $p_{ij}\eta_{ij} = (1 - p_{ij})\eta_{ji}$, *and*
3. $p_{ik}(v_k - v_i + \eta_{ki}) < p_{ij}(v_j - v_i + \eta_{ji})$.

Proof. Let $i, j, k \in A$ with $i < j < k$.

(1) By definition, $p_{ij} = 1 - F(|v_i - v_j|) = 1 - F(|v_j - v_i|) = p_{ji}$.

(2) By symmetry of F, for each $\alpha \in \mathbb{R}_+$,

$$\begin{aligned}\int_{-\alpha}^{\alpha} x\, dF(x) &= \int_{-\alpha}^{0} x\, dF(x) + \int_{0}^{\alpha} x\, dF(x) \\ &= \int_{0}^{\alpha} -x\, dF(x) + \int_{0}^{\alpha} x\, dF(x) \\ &= 0.\end{aligned}$$

In particular,

$$\int_{v_j - v_i}^{\infty} x\, dF(x) = \int_{v_j - v_i}^{v_i - v_j} x\, dF(x) + \int_{v_i - v_j}^{\infty} x\, dF(x) = \int_{v_i - v_j}^{\infty} x\, dF(x).$$

Since $i < j$, $v_j < v_i$ and $p_{ij} = F(-|v_i - v_j|) = F(v_j - v_i) = 1 - F(v_i - v_j)$. By definition of conditional expectations,

$$\begin{aligned}\eta_{ij} &= \mathbf{E}_F[x|x > v_j - v_i] = \frac{\int_{v_j - v_i}^{\infty} x\, dF(x)}{1 - F(v_j - v_i)} = \frac{\int_{v_j - v_i}^{\infty} x\, dF(x)}{1 - p_{ij}} \text{ and} \\ \eta_{ji} &= \mathbf{E}_F[x|x > v_i - v_j] = \frac{\int_{v_i - v_j}^{\infty} x\, dF(x)}{1 - F(v_i - v_j)} = \frac{\int_{v_j - v_i}^{\infty} x\, dF(x)}{p_{ij}}.\end{aligned}$$

Therefore, $p_{ij}\eta_{ij} = (1 - p_{ij})\eta_{ji}$.

(3) Since $i < j < k$, $v_k < v_j < v_i$ and $p_{ij} < p_{ik}$. As computed in (2),

$$\begin{aligned}p_{ij}\eta_{ji} &= \int_{v_i - v_j}^{\infty} x\, dF(x) \text{ and} \\ p_{ik}\eta_{ki} &= \int_{v_i - v_k}^{\infty} x\, dF(x).\end{aligned}$$

Now comparing,

$$\begin{aligned}
p_{ij}\eta_{ji} - p_{ik}\eta_{ki} &= \int_{v_i-v_j}^{\infty} x\,dF(x) - \int_{v_i-v_k}^{\infty} x\,dF(x) \\
&= \int_{v_i-v_j}^{v_i-v_k} x\,dF(x) \\
&> \int_{v_i-v_j}^{v_i-v_k} v_i - v_j\,dF(x).
\end{aligned}$$

Additionally,

$$\begin{aligned}
&p_{ij}(v_j - v_i) - p_{ik}(v_k - v_i) \\
&\qquad = \int_{v_i-v_j}^{\infty} v_j - v_i\,dF(x) - \int_{v_i-v_k}^{\infty} v_k - v_i\,dF(x) \\
&\qquad = \int_{v_i-v_j}^{v_i-v_k} v_j - v_i\,dF(x) + \int_{v_i-v_k}^{\infty} (v_j - v_i) - (v_k - v_i)\,dF(x) \\
&\qquad = \int_{v_i-v_j}^{v_i-v_k} v_j - v_i\,dF(x) + \int_{v_i-v_k}^{\infty} v_j - v_k\,dF(x) \\
&\qquad > \int_{v_i-v_j}^{v_i-v_k} v_j - v_i\,dF(x).
\end{aligned}$$

Combining results,

$$\begin{aligned}
p_{ij}(v_j - v_i + \eta_{ji}) - p_{ik}(v_k - v_i + \eta_{ki}) &= \big(p_{ij}\eta_{ji} - p_{ik}\eta_{ki}\big) + \big(p_{ij}(v_j - v_i) - p_{ik}(v_k - v_i)\big) \\
&> \int_{v_i-v_j}^{v_i-v_k} v_i - v_j\,dF(x) + \int_{v_i-v_j}^{v_i-v_k} v_j - v_i\,dF(x) \\
&= 0.
\end{aligned}$$

Therefore, $p_{ik}(v_k - v_i + \eta_{ki}) < p_{ij}(v_j - v_i + \eta_{ji})$. □

Lemma 2(2) implies that for each pair $i, j \in \{1, \ldots, m\}$ with $i < j$, $0 < \eta_{ij} < \eta_{ji}$ and $v_i < v_j + \eta_{ji}$. Substituting according to definitions, Lemma 2(3) says that for each $h \in N$, $p_{ik}\mathbf{E}[v_{hk}|v_{hk} > v_i] < p_{ij}\mathbf{E}[v_{hj}|v_{hj} > v_i]$.

A.2 Proof of Theorem 1

Let $\varphi^{\prec}$ be a priority rule. If $n = m$, then for each $P \in \mathcal{P}^N$, agent n receives the object left over ater other' assignments are made, so $\varphi^{\prec}(P)$ is independent of P_n.

Since agent n's investigation strategy has no effect on the allocation, each $\sigma_n \in A$ may be part of an equilibrium. As this is the only difference between the cases $n = m$ and $n < m$, suppose now that $n < m$.

Step 1: Equilibrium strategies for agent 1. Let $\sigma_{-1} \in A^{N\setminus\{1\}}$. First suppose $\sigma_1 = a_1$. If $\varepsilon_{11} > v_2 - v_1$, then agent 1 reports P_0 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of $v_1 + \eta_{12}$. If instead $\varepsilon_{11} < v_2 - v_1$, then agent 1 reports preferences with a_2 at the top and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of v_2. Together,

$$U_1(a_1, \sigma_{-1}) = (1 - p_{12})(v_1 + \eta_{12}) + p_{12}v_2.$$

Next suppose $\sigma_1 = a_k \in A\setminus\{a_1\}$. If $\varepsilon_{1k} < v_1 - v_k$, then agent 1 reports P_0 and receives a_1. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{1k} > v_1 - v_k$, then agent 1 reports preferences with a_k at the top and receives a_k. This occurs with probability p_{1k} and yields a conditional expected utility of $v_k + \eta_{k1}$. Together,

$$U_1(a_k, \sigma_{-1}) = (1 - p_{1k})v_1 + p_{1k}(v_k + \eta_{k1}).$$

By Lemma 2(2), $p_{1k}\eta_{k1} = (1 - p_{1k})\eta_{1k}$. If $k = 2$, then

$$\begin{aligned} U_1(a_2, \sigma_{-1}) &= (1 - p_{12})v_1 + p_{12}v_2 + p_{12}\eta_{21} \\ &= (1 - p_{12})v_1 + p_{12}v_2 + (1 - p_{12})\eta_{12} \\ &= U_1(a_1, \sigma_{-1}). \end{aligned}$$

Therefore, agent 1 is indifferent between investigating a_1 and a_2. For $k > 2$,

$$p_{1k}\eta_{k1} = \int_{\alpha \geq v_1 - v_k} \alpha \, dF(\alpha) < \int_{\alpha \geq v_1 - v_2} \alpha \, dF(\alpha) = p_{12}\eta_{21}.$$

Also, $p_{1k} < p_{12}$ and $v_k - v_1 < v_2 - v_1$ so

$$\begin{aligned} U_1(a_k, \sigma_{-1}) &= (1 - p_{1k})v_1 + p_{1k}(v_k + \eta_{k1})) \\ &= v_1 + p_{1k}(v_k - v_1) + p_{1k}\eta_{k1} \\ &< v_1 + p_{12}(v_2 - v_1) + p_{12}\eta_{21} \\ &= (1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21}) \\ &= U_1(a_2, \sigma_{-1}). \end{aligned}$$

Therefore, for each $k > 2$, investigating a_k is strictly dominated for agent 1.

Step 2: Equilibrium strategy for agent 2. Let $\sigma_{-2} \in A^{N\setminus\{2\}}$ with $\sigma_1 \in \{a_1, a_2\}$. Given $\sigma_1 \in \{a_1, a_2\}$, agent 1 receives a_1 with probability $1 - p_{12}$ and receives a_2 with probability p_{12} independent of σ_2. By computing expected utilities, we show that $\sigma_2 = a_3$ is a unique best response for agent 2. Let $\sigma_2 = a_k$.

Case 1: $k = 1$. If agent 1 receives a_1, then agent 2 receives a_2 which yields a conditional expected utility of v_2. Suppose instead that agent 1 receives a_2. If $\varepsilon_{21} > v_3 - v_1$, then agent 2 reports preferences with a_1 at the top among $A\setminus\{a_2\}$ and receives a_1. This occurs with probability $1 - p_{13}$ and yields a conditional expected utility of $v_1 + \eta_{13}$. If instead $\varepsilon_{21} > v_3 - v_1$, then agent 2 reports preferences with a_3 at the top among $A\setminus\{a_2\}$ and receives a_3. This occurs with probability p_{13} and yields a conditional expected utility of v_3. Together,

$$U_2(a_1, \sigma_{-2}) = (1 - p_{12})v_2 + p_{12}\big[(1 - p_{13})(v_1 + \eta_{13}) + p_{13}v_3\big].$$

Case 2: $k = 2$. If agent 1 receives a_2, then agent 2 receives a_1 which yields a conditional expected utility of v_1. Suppose instead that agent 1 receives a_1. If $\varepsilon_{22} > v_3 - v_2$, then agent 2 reports preferences with a_2 at the top among $A\setminus\{a_1\}$ and receives a_2. This occurs with probability $1 - p_{23}$ and yields a conditional expected utility of $v_2 + \eta_{23}$. If instead $\varepsilon_{22} > v_3 - v_2$, then agent 2 reports preferences with a_3 at the top among $A\setminus\{a_1\}$ and receives a_3. This occurs with probability p_{23} and yields a conditional expected utility of v_3. Together,

$$U_2(a_2, \sigma_{-2}) = (1 - p_{12})\big[(1 - p_{23})(v_2 + \eta_{23}) + p_{23}v_3\big] + p_{12}v_1.$$

Case 3: $k \geq 3$. First, suppose agent 1 receives a_1. If $\varepsilon_{2k} < v_2 - v_k$, then agent 2 reports preferences with a_1 and a_2 at the top and receives a_2. This occurs with probability $1 - p_{2k}$ and yields a conditional expected utility of v_2. If instead $\varepsilon_{2k} > v_2 - v_k$, then agent 2 reports preferences with a_1 and a_k at the top and receives a_k. This occurs with probability p_{2k} and yields a conditional expected utility of $v_k + \eta_{k2}$. Second, suppose agent 1 receives a_2. If $\varepsilon_{2k} < v_1 - v_k$, then agent 2 reports preferences with a_1 at the top and receives a_1. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{2k} > v_1 - v_k$, then agent 2 reports preferences with a_k at the top and receives a_k. This occurs with probability p_{1k} and yields a conditional expected utility of $v_k + \eta_{k1}$. Together,

$$U_2(a_k, \sigma_{-2}) = (1 - p_{12})\big[(1 - p_{2k})v_2 + p_{2k}(v_k + \eta_{k2})\big] + p_{12}\big[(1 - p_{1k})v_1 + p_{1k}(v_k + \eta_{k1})\big].$$

We now compare these expected utilities. For $k > 3$, by Lemma 2(3), $p_{1k}\eta_{k1} < p_{13}\eta_{31}$ and $p_{2k}\eta_{k2} < p_{23}\eta_{32}$. Also, $p_{1k} < p_{13}$, $p_{2k} < p_{23}$, $v_k - v_1 < v_3 - v_1$, and

$v_k - v_2 < v_3 - v_2$. Therefore,

$$(1-p_{2k})v_2 + p_{2k}(v_k+\eta_{k2}) < (1-p_{23})v_2 + p_{23}(v_3+\eta_{32}) \text{ and}$$
$$(1-p_{1k})v_1 + p_{1k}(v_k+\eta_{k1}) < (1-p_{13})v_1 + p_{13}(v_3+\eta_{31})$$

so $U_2(a_k,\sigma_{-2}) < U_2(a_3,\sigma_{-2})$.

Next, by Lemma 2(2), $p_{13}\eta_{31} = (1-p_{13})\eta_{13}$ and $p_{23}\eta_{32} = (1-p_{23})\eta_{23}$. Also, by definition, $v_3+\eta_{32} > v_2$ and $v_3+\eta_{31} > v_1$. Therefore,

$$\begin{aligned}
U_2(a_3,\sigma_{-2}) &= (1-p_{12})\big[(1-p_{23})v_2 + p_{23}(v_3+\eta_{32})\big] + p_{12}\big[(1-p_{13})v_1 + p_{13}(v_3+\eta_{31})\big]\\
&> (1-p_{12})v_2 + p_{12}\big[(1-p_{13})v_1 + p_{13}v_3 + p_{13}\eta_{31}\big]\\
&= (1-p_{12})v_2 + p_{12}\big[(1-p_{13})v_1 + p_{13}v_3 + (1-p_{13})\eta_{13}\big]\\
&= U_2(a_1,\sigma_{-1}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
U_2(a_3,\sigma_{-2}) &= (1-p_{12})\big[(1-p_{23})v_2 + p_{23}(v_3+\eta_{32})\big] + p_{12}\big[(1-p_{13})v_1 + p_{13}(v_3+\eta_{31})\big]\\
&> (1-p_{12})\big[(1-p_{23})v_2 + p_{23}v_3 + p_{23}\eta_{32}\big] + p_{12}v_1\\
&> (1-p_{12})\big[(1-p_{23})v_2 + p_{23}v_3 + (1-p_{23})\eta_{23}\big] + p_{12}v_1\\
&= U_2(a_2,\sigma_{-1}).
\end{aligned}$$

Altogether, conditional on agent 1 choosing one of his dominant strategies, $\sigma_2 = a_3$ is a unique best response for agent 2.

Step 3: Equilibrium strategy for agent i, $i \geq 3$. The logic is similar to Step 2. Let $\sigma_{-i} \in A^{N\setminus\{i\}}$ with $\sigma_1 \in \{a_1,a_2\}$ and for each $j \in \{2,\ldots,i-1\}$, $\sigma_j = \sigma_{j+1}$. Then agents $\{1,\ldots,i-1\}$ collectively receive $i-1$ of the objects $\{a_1,\ldots,a_i\}$. Let $\sigma_i = a_k$. Suppose $a_l \in \{a_1,\ldots,a_i\}$ is the object none of the agents $\{1,\ldots,i-1\}$ receive.

Case 1: $k \leq i$. If $k \neq l$, then agent i receives a_l which yields a conditional expected utility of v_l. By comparison, investigating a_{i+1} yields expected utility $(1-p_{l(i+1)})v_l + p_{l(i+1)}(v_{i+1}+\eta_{(i+1)l}) > v_l$.

Suppose instead that $k = l$. If $\varepsilon_{il} > v_{i+1} - v_l$, then agent 2 reports preferences with a_l above a_{i+1} and receives a_l. This occurs with probability $1-p_{l(i+1)}$ and yields a conditional expected utility of $v_l + \eta_{l(i+1)}$. If instead $\varepsilon_{il} > v_{i+1} - v_l$, then agent 2 reports preferences with a_{i+1} above a_l and receives a_{i+1}. This occurs with probability $p_{l(i+1)}$ and yields a conditional expected utility of v_{i+1}. Agent i's expected utility in this case is then $(1-p_{l(i+1)})(v_l+\eta_{l(i+1)}) + p_{l(i+1)}v_{i+1}$. By comparison, investigating a_{i+1} again yields expected utility $(1-p_{l(i+1)})v_l + p_{l(i+1)}(v_{i+1}+\eta_{(i+1)l}) > v_l$. Since

$(1-p_{l(i+1)})\eta_{l(i+1)} = p_{l(i+1)}\eta_{(i+1)l}$, the expected utility is the same under either strategy. Since investigating a_{i+1} yields greater expected utility in the first case and equal expected utility in this case, σ_i is strictly dominated.

Case 2: $k \geq i+1$. If $\varepsilon_{ik} < v_l - v_k$, then agent 2 reports preferences with a_l above a_k and receives a_l. This occurs with probability $1 - p_{lk}$ and yields a conditional expected utility of v_l. If instead $\varepsilon_{ik} > v_l - v_k$, then agent 2 reports preferences with a_k above a_l and receives a_k. This occurs with probability p_{lk} and yields a conditional expected utility of $v_k + \eta_{kl}$. Therefore, agent i's expected utility conditional on agents $\{1, \dots, i-1\}$ receiving $\{a_1, \dots, a_i\}\backslash\{a_l\}$ is $(1-p_{lk})v_l + p_{lk}(v_k + \eta_{kl})$. For $k > i+1$,

$$(1-p_{lk})v_l + p_{lk}(v_k + \eta_{kl}) < (1-p_{l(i+1)})v_l + p_{l(i+1)}(v_{i+1} + \eta_{(i+1)l}).$$

Thus, investigating a_k yields strictly lower expected utility than investigating a_{i+1} in each case and the strategy is strictly dominated. Altogether, conditional on agents with higher priority best responding to the strategies of agents with even higher priority, $\sigma_i = a_{i+1}$ is a unique best response for agent i.

A.3 Proof of Theorem 2

Let φ^μ be a TTC rule and suppose $n = m \geq 3$. We consider the equilibrium strategies of the agents in order of the common values of their endowments. We argue that each agent has a unique best response to the equilibrium strategies of the preceding agents.

Step 1: Equilibrium strategies for agent 1. We show that $\sigma_1 = a_1$ is a strictly dominant strategy for agent 1 by analyzing his option sets. Let $P_{-1} \in \mathcal{P}^{N\backslash\{1\}}$ and $O_1 \equiv O_1(P_{-1}, \varphi^\mu)$. By definition of φ^μ, $a_1 \in O_1$. Let $a_k \in A\backslash\{a_1\}$. To compare $\sigma_1 = a_k$ and $\sigma_1 = a_1$, we consider two cases.

Case 1.1: $a_k \in O_1$. First consider $\sigma_1 = a_k$. If $\varepsilon_{1k} < v_1 - v_k$, then agent 1 reports preferences with a_1 at the top and receives a_1. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{1k} > v_1 - v_k$, then agent 1 reports preferences with a_k at the top and receives a_k. This occurs with probability p_{1k} and yields a conditional expected utility of $v_k + \eta_{k1}$. Then $U_1(a_k, \sigma_{-1}|a_k \in O_1) = (1-p_{1k})v_1 + p_{1k}(v_k + \eta_{k1})$.

Now consider $\sigma_1 = a_1$. If $\varepsilon_{11} > v_k - v_1$, then agent 1 reports preferences with a_1 above a_k and receives a_1 or a more preferred object. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of at least $v_1 + \eta_{1k}$. If instead $\varepsilon_{11} < v_k - v_1$, then agent 1 reports preferences with a_k above a_1 and receives a_k or better. This occurs with probability p_{1k} and yields a conditional expected utility of at least v_k. Agent 1's expected utility in this case is $U_1(a_1, \sigma_{-1}|a_k \in O_1) \geq (1 - p_{1k})(v_1 + \eta_{1k}) + p_{1k}v_k$.

By Lemma 2(2), $p_{1k}\eta_{k1} = (1 - p_{1k})\eta_{1k}$. Therefore, conditional on $a_k \in O_1$, agent 1's expected utility when investigating a_1 is at least as great as his expected utility when investigating a_k.

Case 1.2: $a_k \notin O_1$. First consider $\sigma_1 = a_k$. Then agent 1 reports preferences with a_1 at the top of O_1 and receives a_1. Then $U_1(a_k, \sigma_{-1}|a_k \notin O_1) = v_1$.

Now consider $\sigma_1 = a_1$. Since $a_1 \in O_1$, agent 1's conditional expected utility is at least v_1. To see that it is strictly greater, let $a_l \in A\backslash\{a_1, a_k\}$. For each σ_l, agent l reports preferences with a_1 at the top with positive probability. In this case, $a_l \in O_1$. Also, $\varepsilon_{11} < v_l - v_1$ with probability $p_{1l} > 0$. Then agent 1 reports preferences with a_l above a_1. Since these events are independent, there is positive probability that both $a_l \in O_1$ and $a_l \; P(\varepsilon_{11}) \; a_1$. In this case, agent 1 receives a_l or a more preferred object. Then $U_1(a_1, \sigma_{-1}|a_k \notin O_1) \geq v_l + \eta_{l1} > v_1$.

Altogether, $U_1(a_k, \sigma_{-1}) < U_1(a_1, \sigma_{-1})$. Since this is true for each $k \neq 1$, $\sigma_1 = a_1$ is a strictly dominant strategy for agent 1.

Step 2: Equilibrium strategy for agent 2. We show that $\sigma_2 = a_2$ is a strict best response to $\sigma_1 = a_1$. Let $P_{-2} \in \mathcal{P}^{N\backslash\{2\}}$ with P_1 determined by σ_1 and let $O_2 \equiv O_2(P_{-2}, \varphi^\mu)$. By definition of φ^μ, $a_2 \in O_2$. Moreover, $a_1 \in O_2$ with probability p_{12} and $a_1 \notin O_2$ with probability $1 - p_{12}$ independent of agent 2's strategy. Let $a_k \in A\backslash\{a_2\}$.

Case 2.1: $k = 1$. To compare $\sigma_2 = a_k$ and $\sigma_2 = a_2$, we distinguish two subcases.

Subcase 2.1.1: $a_1 \in O_2$. First consider $\sigma_2 = a_1$. If $\varepsilon_{21} > v_2 - v_1$, then agent 2 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of $v_1 + \eta_{12}$. If instead $\varepsilon_{21} < v_2 - v_1$, then agent 2 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of v_2. Then $U_2(a_1, \sigma_{-2}|a_1 \in O_2) = (1 - p_{12})(v_1 + \eta_{12}) + p_{12}v_2$.

Now consider $\sigma_2 = a_2$. If $\varepsilon_{22} < v_1 - v_2$, then agent 2 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{22} > v_1 - v_2$, then agent 2 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of $v_2 + \eta_{21}$. Then $U_2(a_2, \sigma_{-2}|a_1 \in O_2) = (1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21})$.

By Lemma 2(2), $p_{12}\eta_{21} = (1 - p_{12})\eta_{12}$. Therefore, $U_2(a_1, \sigma_{-2}|a_1 \in O_2) = U_2(a_2, \sigma_{-2}|a_1 \in O_2)$.

Subcase 2.1.2: $a_1 \notin O_2$. First consider $\sigma_2 = a_1$. Then agent 2 reports preferences with a_2 at the top of O_2 and receives a_2. Then $U_2(a_1, \sigma_{-2}|a_1 \notin O_2) = v_2$.

Now consider $\sigma_2 = a_2$. Since $a_2 \in O_2$, agent 2's conditional expected utility is at least v_2. To see that it is strictly greater, consider a_3. For each σ_3, agent 3 reports preferences with a_2 at the top of $A\backslash\{a_1\}$ with positive probability. Whenever $a_1 \notin O_2$, $a_1 \notin O_3$, so $a_3 \in O_2$. Also, $\varepsilon_{22} < v_3 - v_2$ with probability $p_{23} > 0$. In this event, agent 2 reports preferences with a_3 above a_2. Since these events are independent, they occur simultaneously with positive probability. In this joint event, agent 2 receives a_3 or a more preferred object. Then $U_2(a_2, \sigma_{-2}|a_1 \notin O_2) \geq v_3 + \eta_{32} > v_2$.

Altogether, $U_2(a_1, \sigma_{-2}) < U_2(a_2, \sigma_{-2})$. Therefore, $\sigma_2 = a_1$ is strictly dominated.

Case 2.2: $k > 2$. To compare $\sigma_2 = a_k$ and $\sigma_2 = a_2$, we distinguish four subcases.

Subcase 2.2.1: $a_1 \in O_2$ and $a_k \in O_2$. First consider $\sigma_2 = a_k$. If $\varepsilon_{2k} < v_1 - v_k$, then agent 2 reports preferences with a_1 at the top of O_2 and receives a_1. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{2k} > v_1 - v_k$, then agent 2 reports preferences with a_k at the top of O_2 and receives a_k. This occurs with probability p_{1k} and yields a conditional expected utility of $v_k + \eta_{k1}$. Then $U_2(a_k, \sigma_{-2}|a_1 \in O_2, a_k \in O_2) = (1-p_{1k})v_1 + p_{1k}(v_k + \eta_{k1}) = v_1 + p_{1k}(v_k - v_1 + \eta_{k1})$.

Now consider $\sigma_2 = a_2$. Then agent 2 reports preferences with either a_1 or a_2 at the top of O_2 and receives that object. Thus, as computed in Subcase 2.1.1, $U_2(a_2, \sigma_{-2}|a_1 \in O_2, a_k \in O_2) = (1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21}) = v_1 + p_{12}(v_2 - v_1 + \eta_{21})$.

By Lemma 2(3), since $2 < k$, $p_{1k}(v_k - v_1 + \eta_{k1}) < p_{12}(v_2 - v_1 + \eta_{21})$. Therefore, in this subcase agent 2's expected utility when investigating a_2 is higher than his expected utility when investigating a_k.

Subcase 2.2.2: $a_1 \in O_2$ and $a_k \notin O_2$. First consider $\sigma_2 = a_k$. Then agent 2 reports preferences with a_1 at the top of O_2 and receives a_1. This yields a conditional expected utility of v_1. Now consider $\sigma_2 = a_2$. As in Subcase 2.1.1, agent 2's expected utility is again $v_1 + p_{12}(v_2 - v_1 + \eta_{21}) > v_1$. Thus, in this subcase agent 2's expected utility when investigating a_2 is higher than his expected utility when investigating a_k.

Subcase 2.2.3: $a_1 \notin O_2$ and $a_k \in O_2$. First consider $\sigma_2 = a_k$. If $\varepsilon_{2k} < v_2 - v_k$, then agent 2 reports preferences with a_2 above a_k and receives a_2. This occurs with probability $1 - p_{2k}$ and yields a conditional expected utility of v_2. If instead $\varepsilon_{2k} > v_2 - v_k$, then agent 2 reports preferences with a_k above a_2 and receives a_k. This occurs with probability p_{2k} and yields a conditional expected utility of $v_k + \eta_{k2}$. Then $U_2(a_k, \sigma_{-2}|a_1 \notin O_2, a_k \in O_2) = (1 - p_{2k})v_2 + p_{2k}(v_k + \eta_{2k})$.

Now consider $\sigma_2 = a_2$. If $\varepsilon_{22} > v_k - v_2$, then agent 2 reports preferences with a_2 above a_k and receives a_2 or a more preferred object. This occurs with probability $1 - p_{2k}$ and yields a conditional expected utility of at least $v_2 + \eta_{2k}$. If instead $\varepsilon_{22} < v_k - v_2$, then agent 2 reports preferences with a_k above a_2 and receives a_2 or a more preferred object. This occurs with probability p_{2k} and yields a conditional expected utility of at least v_k. Then $U_2(a_2, \sigma_{-2} | a_1 \notin O_2, a_k \in O_2) \geq (1 - p_{2k})(v_2 + \eta_{2k}) + p_{12} v_k$.

By Lemma 2(2), $p_{2k}\eta_{k2} = (1-p_{2k})\eta_{2k}$. Therefore, $U_2(a_k, \sigma_{-2} | a_1 \notin O_2, a_k \in O_2) \leq U_2(a_2, \sigma_{-2} | a_1 \notin O_2, a_k \in O_2)$.

Subcase 2.2.4: $a_1 \notin O_2$ and $a_k \notin O_2$. First consider $\sigma_2 = a_k$. Then agent 2 reports preferences with a_2 at the top of O_2 and receives a_2. This yields a conditional expected utility of v_2. Now consider $\sigma_2 = a_2$. Since $a_2 \in O_2$, agent 2's conditional expected utility is at least v_2. Thus, $U_2(a_k, \sigma_{-2} | a_1 \notin O_2, a_k \notin O_2) \leq U_2(a_2, \sigma_{-2} | a_1 \notin O_2, a_k \notin O_2)$.

Now $a_1 \in O_2$ with positive probability, so at least one of Subcases 2.2.1 and 2.2.2 occurs with positive probability. Therefore, $U_2(a_k, \sigma_{-2}) < U_2(a_2, \sigma_{-2})$. Combining results, $\sigma_2 = a_2$ is a unique best response for agent 2.

Step 3: Equilibrium strategy for agent i, $i \geq 3$. We show that $\sigma_i = a_i$ is a best response to $(\sigma_1, \ldots, \sigma_{i-1}) = (a_1, \ldots, a_{i-1})$. Let $P_{-i} \in \mathcal{P}^{N \setminus \{i\}}$ where for each $j \in \{1, \ldots, i-1\}$, P_j is determined by σ_j and let $O_i \equiv O_i(P_{-i}, \varphi^\mu)$. By definition of φ^μ, $a_i \in O_i$. Moreover, for each $l \in \{1, \ldots, i-1\}$, there is positive probability that agent l reports preferences such that $a_i \; P_l \; a_l$ and for each $j \in \{1, \ldots, i-1\} \setminus \{l\}$, $a_j \; P_j \; a_l$. In this case, $a_l \in O_i$. Therefore, for each $a_l \in \{a_1, \ldots, a_{i-1}\}$, $a_l \in O_i$ with positive probability independent of agent i's strategy.

Let $a_k \in A \setminus \{a_i\}$, let $v_h \equiv \max\{v_l : a_l \in O_i \setminus \{a_k\}\}$, and let a_h be the object with common value v_h. Then $h \leq i$ since $a_i \in O_i$. Note that h is a random variable drawn from $\{1, \ldots, i\}$.

Case 3.1: $k < i$. We claim that ($\star$) for each pair $j, l \in \{1, \ldots, i-1\}$ with $j \neq l$, $\{a_j, a_l\} \not\subseteq O_i$. To see this, suppose by way of contradiction that there is such a pair. Then $a_i \; P_j \; a_j$ and $a_i \; P_l \; a_l$. Also, $v_i < v_j$ and $v_i < v_l$. Since P_j and P_l are determined by σ_j and σ_l, we have $a_l \; P_j \; a_i$ and $a_j \; P_l \; a_i$. But then agents j and l prefer trading with each other to trading with agent i, so $a_j \notin O_i$ and $a_l \notin O_i$. To compare $\sigma_i = a_k$ and $\sigma_i = a_i$, we distinguish two subcases.

Subcase 3.1.1: $a_k \in O_i$. Then by ($\star$), $a_h = a_i$ and agent i receives either a_k or a_i. First consider $\sigma_i = a_k$. If $\varepsilon_{ik} > v_k - v_i$, then agent i reports preferences with a_k at the top of O_i and receives a_k. This occurs with probability $1 - p_{ik}$ and yields a conditional expected utility of $v_k + \eta_{ki}$. If instead $\varepsilon_{ik} < v_k - v_i$, then agent i reports preferences with a_i at the top of O_i and receives a_i. This occurs with probability p_{ik} and yields a conditional expected utility of v_h. Then $U_i(a_k, \sigma_{-i} | a_k \in O_i) = (1 - p_{ik})(v_k + \eta_{ki}) + p_{ik} v_i$.

Now consider $\sigma_i = a_i$. If $\varepsilon_{ii} < v_k - v_i$, then agent i reports preferences with a_k at the top of O_i and receives a_k. This occurs with probability $1 - p_{ik}$ and yields a conditional expected utility of v_k. If instead $\varepsilon_{ii} > v_k - v_i$, then agent i reports preferences with a_i at the top of O_i and receives a_i. This occurs with probability p_{ik} and yields a conditional expected utility of $v_i + \eta_{ik}$. Then $U_i(a_i, \sigma_{-i}|a_k \in O_i) = (1 - p_{ik})v_k + p_{ik}(v_i + \eta_{ik})$.

By Lemma 2(2), $p_{ik}\eta_{ik} = (1 - p_{ik})\eta_{ki}$. Therefore, $U_i(a_k, \sigma_{-i}|a_k \in O_i) = U_i(a_i, \sigma_{-i}|a_k \in O_i)$.

Subcase 3.1.2: $a_k \notin O_i$. First consider $\sigma_i = a_k$. Then agent i reports preferences which rank O_2 according to their common values and receives a_h. Then for each $j \in \{1, \ldots, i\}, U_i(a_k, \sigma_{-i}|a_k \notin O_i, h = j) = v_h$.

For each $j \in \{1, \ldots, i-1\}$ there is positive probability that $h = j$ so we consider each of these events separately. If $\varepsilon_{ii} > v_h - v_i$, then i reports preferences with a_i at the top of O_i and receives a_i, This happens with probability p_{ih} and yields a conditional expected utility of $v_i + \eta_{ih}$. If instead $\varepsilon_{ii} < v_h - v_i$, then i reports preferences with a_h at the top of O_i and receives it. This happens with probability $1 - p_{ih}$ and yields a conditional expected utility of v_h. So $U_i(a_i, \sigma_{-i}|a_k \notin O_i, h = j) = (1 - p_{ih})v_h + p_{ih}(v_i + \eta_{ih})$. By Lemma 2(2), $p_{ih}\eta_{ih} = (1 - p_{ih})\eta_{hi}$. Therefore $U_i(a_i, \sigma_{-i}|a_k \notin O_i, h = j) > U_i(a_k, \sigma_{-i}|a_k \notin O_i, h = j) = v_h$. However, if $h = i$, then $U_i(a_i, \sigma_{-i}|a_k \notin O_i, h = i) \geq v_i = U_i(a_k, \sigma_{-i}|a_k \notin O_i, h = j)$.

Altogether, $U_i(a_k, \sigma_{-i}) \leq U_i(a_k, \sigma_{-i})$. Therefore, for $k < i$, $\sigma_i = a_k$ is not a best response to $(\sigma_1, \ldots, \sigma_{i-1})$.

Case 3.2: $k > i$. To compare $\sigma_i = a_k$ and $\sigma_i = a_i$, we distinguish four subcases.

Subcase 3.2.1: $h < i$ and $a_k \in O_i$. First consider $\sigma_i = a_k$. If $\varepsilon_{ik} < v_h - v_k$, then agent 2 reports preferences with a_h at the top of O_i and receives a_h. This occurs with probability $1 - p_{hk}$ and yields a conditional expected utility of v_h. If instead $\varepsilon_{ik} > v_h - v_k$, then agent i reports preferences with a_k at the top of O_i and receives a_k. This occurs with probability p_{hk} and yields a conditional expected utility of $v_k + \eta_{kh}$. Then $U_i(a_k, \sigma_{-i}|h < i, a_k \in O_i) = (1 - p_{hk})v_h + p_{hk}(v_k + \eta_{kh}) = v_h + p_{hk}(v_k - v_h + \eta_{kh})$.

Now consider $\sigma_i = a_i$. Then agent i reports preferences with either a_h or a_i at the top of O_i and receives that object. Thus, as computed in Subcase 3.1.1, $U_i(a_i, \sigma_{-i}|h < i, a_k \in O_i) = (1 - p_{hi})v_h + p_{hi}(v_i + \eta_{ih}) = v_h + p_{hi}(v_i - v_h + \eta_{ih})$.

By Lemma 2(3), since $h < i < k$, $p_{hk}(v_k - v_h + \eta_{kh}) < p_{hi}(v_i - v_h + \eta_{ih})$. Therefore, $U_i(a_k, \sigma_{-i}|h < i, a_k \in O_i) < U_i(a_i, \sigma_{-i}|h < i, a_k \in O_i) =$.

Subcase 3.2.2: $h < i$ and $a_k \notin O_i$. An argument identical to that in case 3.1.2 shows that $U_i(a_k, \sigma_{-i}|h < i, a_k \notin O_i) < U_i(a_i, \sigma_{-i}|h < i, a_k \notin O_i)$.

Subcase 3.2.3: $h = i$ and $a_k \in O_i$. First consider $\sigma_i = a_k$. If $\varepsilon_{ik} < v_i - v_k$, then agent i reports preferences with a_i above a_k and receives a_i. This occurs with probability $1 - p_{ik}$ and yields a conditional expected utility of v_i. If instead $\varepsilon_{ik} >$

$v_i - v_k$, then agent i reports preferences with a_k above a_i and receives a_k. This occurs with probability p_{ik} and yields a conditional expected utility of $v_k + \eta_{ki}$. Then $U_i(a_k, \sigma_{-i}|h = i, a_k \in O_i) = (1 - p_{ik})v_i + p_{ik}(v_k + \eta_{ki})$.

Now consider $\sigma_i = a_i$. If $\varepsilon_{ii} > v_k - v_i$, then agent i reports preferences with a_i above a_k and receives a_i or a more preferred object. This occurs with probability $1 - p_{ik}$ and yields a conditional expected utility of at least $v_i + \eta_{ik}$. If instead $\varepsilon_{ii} < v_i - v_k$, then agent i reports preferences with a_k above a_i and receives a_k or a more preferred object. This occurs with probability p_{ik} and yields a conditional expected utility of at least v_k. Then $U_i(a_i, \sigma_{-i}|h = i, a_k \in O_i) \geq (1 - p_{ik})(v_i + \eta_{ik}) + p_{ik}v_k$.

By Lemma 2(2), $p_{ik}\eta_{ki} = (1 - p_{ik})\eta_{ik}$. Therefore, $U_i(a_k, \sigma_{-i}|h = i, a_k \in O_i) \leq U_i(a_i, \sigma_{-i}|h = i, a_k \in O_i)$.

Subcase 3.2.4: $h = i$ and $a_k \notin O_i$. First consider $\sigma_i = a_k$. Then agent i reports preferences with a_i at the top of O_2 and receives a_i. Then $U_i(a_k, \sigma_{-i}|h = i, a_k \notin O_i) = v_i$. Now consider $\sigma_i = a_i$. Since $a_i \in O_i$, $U_i(a_i, \sigma_{-i}|h = i, a_k \notin O_i) \geq v_i = U_i(a_k, \sigma_{-i}|h = i, a_k \notin O_i)$.

Now $a_h \neq a_i$ with positive probability, so at least one of Subcases 3.2.1 and 3.2.2 occurs with positive probability. Therefore, $U_i(a_k, \sigma_{-i}) < U_i(a_i, \sigma_{-i})$. Thus, for $k > i$, $\sigma_i = a_k$ is not a best response to $(\sigma_1, \ldots, \sigma_{i-1})$. Instead, $\sigma_i = a_i$ is the unique best response. Summarizing, if $n = m \geq 3$ and $\sigma \in A^N$ is an equilibrium profile, then for each $i \in N$, $\sigma_i = a_i$.

We conclude by analyzing the special case $n = m = 2$. Let $\sigma \in A^N$.

Agent 1: First suppose $a_2 \in O_1$. Under $\sigma_1 = a_1$, if $\varepsilon_{11} > v_2 - v_1$, then agent 1 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of $v_1 + \eta_{12}$. If instead $\varepsilon_{11} < v_2 - v_1$, then agent 1 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of v_2. Then $U_1(a_1, \sigma_2|a_2 \in O_1) = (1 - p_{12})(v_1 + \eta_{12}) + p_{12}v_2$.

Under $\sigma_1 = a_2$, if $\varepsilon_{12} < v_1 - v_2$, then agent 1 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{12} > v_1 - v_2$, then agent 1 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of $v_2 + \eta_{21}$. Then $U_1(a_2, \sigma_2|a_2 \in O_1) = (1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21})$. By Lemma 2(2), $p_{12}\eta_{21} = (1 - p_{12})\eta_{12}$. Therefore, $U_1(a_1, \sigma_2|a_2 \in O_1) = U_1(a_2, \sigma_2|a_2 \in O_1)$.

Now suppose $a_2 \notin O_1$. Then agent 1 receives a_1. As this is independent of σ_1, $U_1(a_1, \sigma_2|a_2 \notin O_1) = U_1(a_2, \sigma_2|a_2 \notin O_1) = v_1$. Altogether, $U_1(a_1, \sigma_2) = U_1(a_2, \sigma_2)$.

Agent 2: First suppose $a_1 \in O_2$. Under $\sigma_2 = a_1$, if $\varepsilon_{21} > v_2 - v_1$, then agent 2 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of $v_1 + \eta_{12}$. If instead $\varepsilon_{11} < v_2 - v_1$, then agent 2 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of v_2. Then $U_2(a_1, \sigma_1 | a_1 \in O_2) = (1 - p_{12})(v_1 + \eta_{12}) + p_{12} v_2$.

Under $\sigma_2 = a_2$, if $\varepsilon_{22} < v_1 - v_2$, then agent 2 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{22} > v_1 - v_2$, then agent 2 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of $v_2 + \eta_{21}$. Then $U_2(a_2, \sigma_1 | a_1 \in O_2) = (1 - p_{12}) v_1 + p_{12}(v_2 + \eta_{21})$. By Lemma 2(2), $p_{12}\eta_{21} = (1 - p_{12})\eta_{12}$. Therefore, $U_2(a_1, \sigma_1 | a_1 \in O_2) = U_2(a_2, \sigma_1 | a_1 \in O_2)$.

Now suppose $a_1 \notin O_2$. Then agent 2 receives a_2. As this is independent of σ_2, $U_2(a_1, \sigma_1 | a_1 \notin O_2) = U_2(a_2, \sigma_1 | a_1 \notin O_2) = v_2$. Altogether, $U_2(a_1, \sigma_1) = U_2(a_2, \sigma_1)$. Combining results, if $n = m = 2$, then each profile $\sigma \in A^N$ constitutes an equilibrium.

A.4 Proof of Proposition 2

Case 1: $\boldsymbol{n = m = 2}$. Let $\sigma^\mu \in A^N$ be an equilibrium under φ^μ. There are two feasible allocations and two preference relations: $X \equiv \{(a_1, a_2), (a_2, a_1)\}$ and $\mathcal{P} \equiv \{P^1, P^2\}$ where $a_1 \; P^1 \; a_2$ and $a_2 \; P^2 \; a_2$. Let $\sigma \in A^N$ and $\varphi \colon \mathcal{P}^N \to X$ and suppose $\varphi \neq \varphi^\mu$.

For each $\varepsilon_\sigma \in \mathbb{R}^N$ such that $P_1(\varepsilon_\sigma) \neq P_2(\varepsilon_\sigma)$, there is a unique ex-post efficient allocation, so suppose $\varphi(P^1, P^2) = (a_1, a_2)$ and $\varphi(P^2, P^1) = (a_2, a_1)$. Two profiles remain. First suppose that $\varphi(P^1, P^1) \neq \varphi^\mu(P^1, P^1)$, so $\varphi(P^1, P^1) = (a_2, a_1)$. Conditional on $P_1(\varepsilon_\sigma) = P_2(\varepsilon_\sigma) = P^1$, agent 1 is better off under $(\sigma^\mu, \varphi^\mu)$ than under (σ, φ). If a Pareto improvement is possible, agent 1 must be better off under φ than under φ^μ in the final case, so $\varphi^\mu(P^2, P^2) = (a_2, a_1)$. Then φ is simply φ^μ with the roles of the agents reversed. In particular, $U_1(\sigma, \varphi) = U_2(\sigma^\mu, \varphi^\mu)$ and $U_2(\sigma, \varphi) = U_1(\sigma^\mu, \varphi^\mu)$, so (σ, φ) is not an ex-ante Pareto improvement over $(\sigma^\mu, \varphi^\mu)$.

Suppose instead that $\varphi(P^2, P^2) \neq \varphi^\mu(P^2, P^2)$, so $\varphi(P^2, P^2) = (a_2, a_1)$. Conditional on $P_1(\varepsilon_\sigma) = P_2(\varepsilon_\sigma) = P^2$, agent 2 is better off under $(\sigma^\mu, \varphi^\mu)$ than under (σ, φ). If a Pareto improvement, agent 2 must be better off under φ than under φ^μ in the final case, so $\varphi^\mu(P^1, P^1) = (a_2, a_1)$. Once again, φ is simply φ^μ with the roles of the agents reversed and (σ, φ) is not an ex-ante Pareto improvement over $(\sigma^\mu, \varphi^\mu)$. Instead, $(\sigma^\mu, \varphi^\mu)$ is *ex-ante efficient.*

Case 2: $\boldsymbol{2 = n}$ and $\boldsymbol{m = 3}$. We show by example that φ^μ is not *ex-ante efficient.*

Let $(v_1, v_2, v_3) \equiv (50, 49, 30)$ and $F \sim Unif[-30, 30]$. Then $p_{12} = \frac{29}{60}$, $p_{13} = \frac{10}{60}$, and $p_{23} = \frac{11}{60}$. Let $\sigma \equiv (a_1, a_2)$, which is the unique equilibrium under φ^μ. We construct a rule that yields an ex-ante Pareto improvement with σ. Let $\bar{P}, \bar{P}' \in \mathcal{P}^N$ be such that

$\bar{P}_1$	$\bar{P}_2$	$\bar{P}'_1$	$\bar{P}'_2$
a_1	a_1	a_2	a_2
a_2	a_3	a_3	a_1
a_3	a_2	a_1	a_3

Now define $\varphi \colon \mathcal{P}^N \to X$ by

$$\varphi(P) \equiv \begin{cases} (a_2, a_1) & \text{if } P = \bar{P} \text{ or } P = \bar{P}' \\ \varphi^\mu(P) & \text{otherwise} \end{cases}.$$

To compare ex-ante expected utilities, let $\varepsilon_\sigma \in \mathbb{R}^N$. If $P(\varepsilon_\sigma) \notin \{\bar{P}, \bar{P}'\}$, then (σ, φ) and (σ, φ^μ) coincide, so suppose $P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}$. The allocations are $\varphi(\bar{P}) = \varphi(\bar{P}') = (a_2, a_1)$, $\varphi^\mu(\bar{P}) = (a_1, a_3)$, and $\varphi^\mu(\bar{P}') = (a_3, a_2)$. Under (σ, φ), agent 1's conditional expected utility is $v_2 = 49$ and agent 2's conditional expected utility is $v_1 = 50$.

We now compute the conditional expected utilities under (σ, φ^μ). First, $Pr(P_1(\varepsilon_\sigma) = \bar{P}_1) = 1 - p_{12} = \frac{31}{60}$, $Pr(P_2(\varepsilon_\sigma) = \bar{P}_2) = p_{23} = \frac{11}{60}$, $Pr(P_1(\varepsilon_\sigma) = \bar{P}'_1) = p_{13} = \frac{10}{60}$, and $Pr(P_2(\varepsilon_\sigma) = \bar{P}'_2) = p_{12} = \frac{29}{60}$. Therefore, by independence,

$$\begin{aligned} Pr\big(P(\varepsilon_\sigma) = \bar{P}\big) &= \tfrac{31}{60} \cdot \tfrac{11}{60} = \tfrac{341}{3600}, \\ Pr\big(P(\varepsilon_\sigma) = \bar{P}'\big) &= \tfrac{10}{60} \cdot \tfrac{29}{60} = \tfrac{290}{3600}, \\ Pr\big(P(\varepsilon_\sigma) = \bar{P} \big| P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}\big) &= \tfrac{341}{631}, \text{ and} \\ Pr\big(P(\varepsilon_\sigma) = \bar{P}' \big| P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}\big) &= \tfrac{290}{631}. \end{aligned}$$

Next, computing expectations,

$$\begin{aligned} \mathbf{E}[v_1 + \varepsilon_{11} | P(\varepsilon_\sigma) = \bar{P}] &= 50 + \tfrac{1}{2}(-1 + 30) = 64.5 \text{ and} \\ \mathbf{E}[v_2 + \varepsilon_{22} | P(\varepsilon_\sigma) = \bar{P}'] &= 49 + \tfrac{1}{2}(1 + 30) = 64.5. \end{aligned}$$

Combining results,

$$\begin{aligned} U_1\big(\sigma, \varphi^\mu \big| P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}\big) &= \tfrac{341}{631} \cdot 64.5 + \tfrac{290}{631} \cdot 30 = \tfrac{61389}{1262} \approx 48.64 < 49 \text{ and} \\ U_2\big(\sigma, \varphi^\mu \big| P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}\big) &= \tfrac{341}{631} \cdot 30 + \tfrac{290}{631} \cdot 64.5 = \tfrac{57870}{1262} \approx 45.86 < 50. \end{aligned}$$

Therefore, each agent's ex-ante expected utility is higher under φ than under φ^μ.

Case 3: $3 < m$. We show how to embed the example in Case 1 as a subproblem. First, for $2 = n < m$, only the objects with the three highest common values are relevant and so the computations carry over unchanged.

Now suppose $2 < n \leq m$. Again let $F \sim Unif[-30, 30]$ and let $v \in \mathbb{R}^m$ be such that $(v_1, v_2, v_3) \equiv (50, 49, 30)$ and $30 > v_4 > \cdots > v_m > 20$. Let $\bar{P}, \bar{P}' \in \mathcal{P}^N$ be such that for each $i \in N \backslash \{1, 2\}$,

$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_i$	$\bar{P}'_1$	$\bar{P}'_2$	$\bar{P}'_i$
a_1	a_1	a_i	a_2	a_2	a_i
a_2	a_3	a_1	a_3	a_1	a_1
a_3	a_2	a_2	a_1	a_3	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Then $\bar{P}$ and $\bar{P}'$ each occur with positive probability under σ^μ. Moreover, for each $i \in N \backslash \{1, 2\}$ and each $\varepsilon_\sigma \in \mathbb{R}^N$ such that $P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}$, $\varphi_i^\mu(P(\varepsilon_\sigma)) = a_i = \mu_i$. Therefore, we may extend the definition of φ in Case 1 so that for each $i \in N \backslash \{1, 2\}$ and each $\varepsilon_\sigma \in \mathbb{R}^N$, $\varphi_i(P(\varepsilon_\sigma)) = \varphi_i^\mu(P(\varepsilon_\sigma))$. Then agent i's expected utility is the same under φ and under φ^μ. The conditional expected utilities for agents 1 and 2 are the same as before, so φ once again represents an ex-ante Pareto improvement over φ^μ.

A.5 Proofs of Theorem 3, Lemma 1, Proposition 3, and Theorem 4

We first provide explicit formulas for the equilibrium welfare of each agent under $\varphi^\prec$ and φ^μ. To do so, we introduce some additional notation. For each $k \in N$, let $N^k \equiv \{1, \ldots, k-1\}$. Then under $\varphi^\prec$, N^k is the set agents with higher priority than agent k. Similarly, under φ^μ, N^k is the set of agents whose endowments have common values higher than the common value of agent k's endowment. For each pair $l, k \in \{1, \ldots, m\}$ with $l \leq k$, let

$$P(l,k) \equiv \begin{cases} 1 \text{ if } l = k \\ p_{l(l+1)} \cdot p_{l(l+2)} \cdots p_{lk} & \text{if } l < k \end{cases}$$

and recursively define

$$Q(l,k) \equiv \begin{cases} P(l,k) & \text{if } l = 1 \\ 1 - \sum_{h=1}^{k-1} Q(h,k) & \text{if } 1 < l = k \\ \big(1 - Q(l,l)\big) \cdot P(l,k) & \text{if } 1 < l < k \end{cases} .$$

Lemma 3 below allows us to interpret these products in terms of agents' options sets: under either $\varphi^{\prec}$ or φ^{μ}, $Q(l,k)$ represents the probability that a_l is the object with the highest common value in agent k's option set.

Lemma 3. *For each pair $l,k \in \{1,\dots,n\}$ with $l \leq k$, $Q(l,k)$ is the probability that (i) a_l has the highest common value in agent k's option set under $\varphi^{\prec}$ and (ii) a_l has the highest common valuein agent k's option set under φ^{μ}.*

Proof. Let $l,k \in \{1,\dots,n\}$ with $l < k$. After verifying the probabilities for $l < k$, the formula for $l = k$ follows by subtraction. Let $\sigma^{\prec}, \sigma^{\mu} \in A^N$ be equilibria under $\varphi^{\prec}$ and φ^{μ} respectively and let $P^{\prec}, P^{\mu} \in \mathcal{P}^N$ be preferences determined by $\sigma^{\prec}$ and σ^{μ}. That is, $P^{\prec}$ and P^{μ} are random variables that depend on the realizations of $\varepsilon_{\sigma^{\prec}}$ and $\varepsilon_{\sigma^{\mu}}$ respectively.

Without loss of generality, suppose that for each $i \in N$, $\sigma_i^{\prec} = a_{i+1}$ and $\sigma_i^{\mu} = a_i$ where $a_{m+1} = a_1$ if necessary.

(i) Priority rule. According to $\sigma^{\prec}$, agents in N^k receive at least $k-1$ of the objects $\{a_1,\dots,a_k\}$. Thus, agent k's option set contains exactly one of these objects and the cases are mutually exclusive. Now suppose that $a_l \in O_k(P^{\prec}_{-k}, \varphi^{\prec})$. Then no agent in N^k receives a_l. We verify the probabilities by induction on l.

Case 1.1: $l = 1$. Then for each $i \in N^k$, agent i receives a_{i+1}. Therefore, $a_{i+1}\ P_i^{\prec}\ a_1$ which occurs with probability $p_{1(i+1)}$. Since these events are independent, the probability that $a_1 \in O_k(P^{\prec}_{-k}, \varphi^{\prec})$ is $p_{12} \cdot p_{13} \cdots p_{1k} = Q(1,k)$.

Case 1.2: $l = 2$. Then agent 1 receives a_1 and for each $i \in N^k \backslash \{1\}$, agent i receives a_{i+1}. Therefore, $a_1\ P_1^{\prec}\ a_2$ which occurs with probability $1 - p_{12} = 1 - Q(1,2)$. Also, for each $i \in N^k \backslash \{1\}$, $a_{i+1}\ P_i^{\prec}\ a_2$ which occurs with probability $p_{2(i+1)}$. Since these events are independent, the probability that $a_2 \in O_k(P^{\prec}_{-k}, \varphi^{\prec})$ is

$$(1 - p_{12}) p_{23} \cdot p_{24} \cdots p_{2k} = (1 - Q(1,2)) P(2,k) = Q(2,k).$$

Case 1.3: $l \geq 3$. Suppose that the claim is true for each $i < l$. Then the agents in N^k collectively receive $\{a_1,\dots,a_{l-1},a_{l+1},\dots,a_k\}$. Moreover, since each agent i always receives an object among $\{a_1,\dots,a_{i+1}\}$, this implies that the agents N^l collectively receive $\{a_1,\dots,a_{l-1}\}$. Therefore, $\{a_1,\dots,a_{l-1}\} \cap O_l(P^{\prec}_{-l}, \varphi^{\prec}) = \emptyset$. By hypothesis, for each $i < l$, $Q(i,l)$ is the probability that $a_i \in O_l(P^{\prec}_{-l}, \varphi^{\prec})$, so $1 - Q(i,l)$ is the probability $a_i \notin O_l(P^{\prec}_{-l}, \varphi^{\prec})$. By mutual exclusivity, $1 - \sum_{i=1}^{l-1} Q(i,l)$ is the probability that $\{a_1,\dots,a_{l-1}\} \cap O_l(P^{\prec}_{-l}, \varphi^{\prec}) = \emptyset$. Next, for each $i \in N^k \backslash (N^l \cup \{l\})$, agent i receives a_{i+1}. Therefore, $a_{i+1}\ P_i^{\prec}\ a_l$ which occurs with probability p_{1i}. Since these events are independent, the probability that $a_l \in O_k(P^{\prec}_{-k}, \varphi^{\prec})$ is

$$\left(1 - \sum_{i=1}^{l-1} Q(i,l)\right) p_{l(l+1)} \cdot p_{l(l+2)} \cdots p_{lk} = \big(1 - Q(l,l)\big) \cdot P(l,k) = Q(l,k).$$

Finally, since agent k's option set under $\varphi^{\prec}$ always includes exactly one of $\{a_1, \ldots, a_k\}$, the probability that $a_k \in O_k(P^{\prec}_{-k}, \varphi^{\prec})$ is $1 - \sum_{i=1}^{k-1} Q(i,k) = Q(k,k)$.

(ii) Top trading cycles rule. According to σ^{μ}, each agent i reports preferences that agree with P_0 on $A\backslash\{\mu_i\}$. Thus, for each pair $i, j \in N$ with $i < j$, $O_j(P^{\mu}_{-j}, \varphi^{\mu}) \subseteq O_i(P^{\mu}_{-i}, \varphi^{\mu})$. Moreover, if agent i receives a_j, then agent j receives a_i and for each $h \in N$ with $i < h < j$, agent h receives $\mu_h = a_h$. In particular, the events $a_i \in O_j(P^{\mu}_{-j}, \varphi^{\mu})$ and $a_h \in O_j(P^{\mu}_{-j}, \varphi^{\mu})$ are mutually exclusive. Now suppose that $a_l \in O_k(P^{\mu}_{-k}, \varphi^{\mu})$. Then no agent in N^k receives a_l. We verify the probabilities by induction on l.

Case 2.1: $l = 1$. Then $a_k \; P^{\mu}_1 \; a_1$. This occurs with probability p_{1k}. Furthermore, $\{a_1, \ldots, a_{k-1}\} \cap O_1(P^{\mu}_{-1}, \varphi^{\mu}) = \{a_1\}$. For each $i \in N^k\backslash\{1\}$, $a_i \notin \cap O_1(P^{\mu}_{-1}, \varphi^{\mu})$ implies $a_i \; P^{\mu}_i \; a_1$ which occurs with probability p_{1i}. Since these events are independent, the probability that $a_1 \in O_k(P^{\mu}_{-k}, \varphi^{\mu})$ is $p_{1k} \cdot p_{12} \cdot p_{13} \cdots p_{1(k-1)} = Q(1,k)$.

Case 2.2: $l = 2$. Then $a_k \; P^{\mu}_2 \; a_2$. This occurs with probability p_{2k}. Furthermore, $\{a_1, \ldots, a_{k-1}\} \cap O_2(P^{\mu}_{-2}, \varphi^{\mu}) = \{a_2\}$. For each $i \in N^k\backslash\{2\}$, $a_i \notin \cap O_2(P^{\mu}_{-2}, \varphi^{\mu})$ implies $a_i \; P^{\mu}_i \; a_2$. This occurs for $i = 1$ with probability $1 - p_{12} = 1 - Q(1,2)$ and for $i > 2$ with probability p_{2i}. Since these events are independent, the probability that $a_2 \in O_k(P^{\mu}_{-k}, \varphi^{\mu})$ is $p_{2k}(1 - p_{12}) \cdot p_{23} \cdots p_{2(k-1)} = Q(2,k)$.

Case 2.3: $l \geq 3$. Suppose that the claim is true for each $i < l$. Then $a_k \; P^{\mu}_l \; a_l$. This occurs with probability p_{lk}. Furthermore, $\{a_1, \ldots, a_{k-1}\} \cap O_l(P^{\mu}_{-l}, \varphi^{\mu}) = \{a_l\}$. By hypothesis, for each $i < l$, $Q(i,l)$ is the probability that $a_i \in O_l(P^{\mu}_{-l}, \varphi^{\mu})$, so $1 - Q(i,l)$ is the probability $a_i \notin O_l(P^{\mu}_{-l}, \varphi^{\mu})$. By mutual exclusivity, $1 - \sum_{i=1}^{l-1} Q(i,l)$ is the probability that $\{a_1, \ldots, a_{l-1}\} \cap O_l(P^{\mu}_{-l}, \varphi^{\mu}) = \emptyset$. Next, for each $i \in N^k\backslash(N^l \cup \{l\})$, $a_i \notin O_l(P^{\mu}_{-l}, \varphi^{\mu})$ implies $a_i \; P^{\mu}_i \; a_l$ which occurs with probability p_{il}. Since these events are independent, the probability that $a_l \in O_k(P^{\mu}_{-k}, \varphi^{\mu})$ is

$$p_{lk}\left(1 - \sum_{i=1}^{l-1} Q(i,k)\right) p_{l(l+1)} \cdot p_{l(l+2)} \cdots p_{l(k-1)} = \big(1 - Q(l,l)\big) \cdot P(l,k) = Q(l,k).$$

Finally, since agent k's option set under φ^{μ} always includes a_k, the probability that a_k is the object in $O_k(P^{\mu}_{-k}, \varphi^{\mu})$ with the highest common value is then $O_k(P^{\mu}_{-k}, \varphi^{\mu})$ is $1 - \sum_{i=1}^{k-1} Q(i,k) = Q(k,k)$. □

We now provide formulas for the each agent's equilibrium utility. For ease of notation, we adopt the conventions that $a_{m+1} \equiv a_1$ and for each $k \in N$, $p_{k(m+1)} \equiv 0$ and $\eta_{(m+1)k} \equiv 0$.

Proposition 4. *Let $\sigma \in A^N$ be an equilibrium under $\varphi^{\prec}$. Then for each $k \in N$,*

$$U_k(\sigma, \varphi^{\prec}) = \sum_{l=1}^{k} Q(l,k) \left[(1 - p_{l(k+1)})v_l + p_{l(k+1)}(v_{k+1} + \eta_{(k+1)l})\right].$$

Proof. Let $\sigma \in A^N$ be an equilibrium under $\varphi^{\prec}$, $P \in \mathcal{P}^N$ be preferences determined by σ, and $k \in N$. Without loss of generality, suppose that $\sigma_k = a_{k+1}$. We interpret the expressions in the formula as expected utilities conditional on the realization of agent k's option set.

According to $\varphi^{\prec}$, exactly one of $\{a_1, \ldots, a_k\}$ is in $O_k(P_{-k}, \varphi^{\prec})$ so these events are mutually exclusive and exhaustive. By Lemma 3, for each $l \in \{1, \ldots, k\}$, $a_l \in O_k(P_{-k}, \varphi^{\prec})$ with probability $Q(l,k)$. In this case, agent k reports preferences with either $a_l$or a_{k+1} at the top of $O_k(P_{-k}, \varphi^{\prec})$ and receives that object. If $\varepsilon_{k(k+1)} < v_l - v_{k+1}$, then $a_l \ P_k \ a_{k+1}$ and agent k receives a_l. This occurs with probability $1 - p_{l(k+1)}$ and yields a conditional expected utility of v_l. If instead $\varepsilon_{k(k+1)} > v_l - v_{k+1}$, then $a_{k+1} \ P_k \ a_l$ and agent k receives a_{k+1}. This occurs with probability $p_{l(k+1)}$ and yields a conditional expected utility of $v_{k+1} + \eta_{(k+1)l}$. Thus

$$U_k\big(\sigma, \varphi^{\prec} \big| a_l \in O_k(P_{-k}, \varphi^{\prec})\big) = (1 - p_{l(k+1)})v_l + p_{l(k+1)}(v_{k+1} + \eta_{(k+1)l}).$$

The stated formula follows by taking an expectation over the realization of the option set. □

Proposition 5. *Let $\sigma^\mu \in A^N$ be an equilibrium under φ^μ. Then for each $k \in N$,*

$$\begin{aligned} U_k(\sigma^\mu, \varphi^\mu) = &\sum_{l=1}^{k-1} Q(l,k) \left[(1 - p_{lk})v_l + p_{lk}(v_k + \eta_{kl})\right] \\ &+ \sum_{l=k+1}^{n} \big(Q(k, l-1) - Q(k,l)\big) \left[(1 - p_{kl})(v_k + \eta_{kl}) + p_{kl} v_l\right] \\ &+ Q(k,n) \left[(1 - p_{k(n+1)})(v_k + \eta_{k(n+1)}) + p_{k(n+1)} v_{n+1}\right]. \end{aligned}$$

Proof. Let $\sigma \in A^N$ be an equilibrium under φ^μ, $P \in \mathcal{P}^N$ be preferences determined by σ, and $k \in N$. Without loss of generality, suppose that $\sigma_k = a_k$. We interpret the expressions in the formula as expected utilities conditional on the realization of agent k's option set. Suppose that $O_k(P_{-k}, \varphi^\mu)\backslash\{a_k\} \neq \emptyset$ and let $a_l \in O_k(P_{-k}, \varphi^\mu)\backslash\{a_k\}$ be the object with the highest common value.

Case 1: $\boldsymbol{l < k}$. Then agent k reports preferences with either a_l or a_k at the top of $O_k(P_{-k}, \varphi^\mu)$ and receives that object. If $\varepsilon_{kk} < v_l - v_k$, then $a_l \; P_k \; a_k$ and agent k receives a_l. This occurs with probability $1 - p_{lk}$ and yields a conditional expected utility of v_l. If instead $\varepsilon_{kk} > v_l - v_k$, then $a_k \; P_k \; a_l$ and agent k receives a_k. This occurs with probability p_{lk} and yields a conditional expected utility of $v_k + \eta_{kl}$. Then

$$U_k\big(\sigma, \varphi^\mu \big| a_l \in O_k(P_{-k}, \varphi^\mu)\big) = (1 - p_{lk})v_l + p_{lk}(v_k + \eta_{kl}).$$

By Lemma 3, this occurs with probability $Q(l, k)$. Taking expectations yields the first summation in the stated formula.

Case 2: $\boldsymbol{k < l \leq n}$. Then agent k reports preferences with either a_k or a_l at the top of $O_k(P_{-k}, \varphi^\mu)$ and receives that object. If $\varepsilon_{kk} > v_l - v_k$, then $a_k \; P_k \; a_l$ and agent k receives a_k. This occurs with probability $1 - p_{kl}$ and yields a conditional expected utility of $v_k + \eta_{kl}$. If instead $\varepsilon_{kk} < v_l - v_k$, then $a_l \; P_k \; a_k$ and agent k receives a_l. This occurs with probability p_{kl} and yields a conditional expected utility of v_l. Then

$$U_k\big(\sigma, \varphi^\mu \big| O_k(P_{-k}, \varphi^\mu) \cap \{a_1, \ldots, a_l\} = \{a_k, a_l\}\big) = (1 - p_{kl})v_l + p_{kl}(v_k + \eta_{kl}).$$

We now compute the probability that $O_k(P_{-k}, \varphi^\mu) \cap \{a_1, \ldots, a_l\} = \{a_k, a_l\}$. This requires that, $\{a_1, \ldots, a_{k-1}\} \cap O_k(P_{-k}, \varphi^\mu) = \emptyset$. By Lemma 3, this occurs with probability $Q(k, k)$ and this depends only on the preferences of agents in N^k. Now consider $i \in N$ with $k < i$. Then also $\{a_1, \ldots, a_{k-1}\} \cap O_i(P_{-i}, \varphi^\mu) = \emptyset$. Since each such agent i reports preferences that agree with P_0 on $A\backslash\{\mu_i\}$, agent i reports preferences with either a_k or a_i at the top of $A\backslash\{a_1, \ldots, a_{k-1}\}$. If $\varepsilon_{ii} < v_k - v_i$, then $a_k \; P_i \; a_i$ and $a_i \in O_k(P_{-k}, \varphi^\mu)$. This occurs with probability $1 - p_{ki}$. If instead $\varepsilon_{ii} > v_k - v_i$, then $a_i \; P_i \; a_k$ and $a_i \notin O_k(P_{-k}, \varphi^\mu)$. This occurs with probability p_{ki}. Conditional on the preferences of agents in N^k, for each pair $i, j \in N$ with $k < i < j$, the events $a_i \in O_k(P_{-k}, \varphi^\mu)$ and $a_j \in O_k(P_{-k}, \varphi^\mu)$ are independent. Combining results,

$$\begin{aligned} Pr\big(O_k(P_{-k}, \varphi^\mu) \cap \{a_1, \ldots, a_l\} = \{a_k, a_l\}\big) &= Q(k, k) \cdot p_{k(k+1)} \cdot p_{k(k+2)} \cdots p_{k(l-1)} \cdot (1 - p_{kl}) \\ &= Q(k, l-1) - Q(k, l). \end{aligned}$$

Taking expectations yields the second summation in the stated formula.

Case 3: $\boldsymbol{n < l}$. Then $l = n + 1$. Agent k reports preferences with one of a_k and a_{n+1} at the top of $O_k(P_{-k}, \varphi^\mu)$ and receives that object. If $\varepsilon_{kk} > v_{n+1} - v_k$, then

$a_k \ P_k \ a_{n+1}$ and agent k receives a_k. This occurs with probability $1 - p_{k(n+1)}$ and yields a conditional expected utility of $v_k + \eta_{k(n+1)}$. If instead $\varepsilon_{kk} < v_{n+1} - v_k$, then $a_{n+1} \ P_k \ a_k$ and agent k receives a_l. This occurs with probability $p_{k(n+1)}$ and yields a conditional expected utility of v_{n+1}. Then

$$U_k\big(\sigma, \varphi^\mu \big| O_k(P_{-k}, \varphi^\mu) \cap \{a_1, \dots, a_l\} = \{a_k, a_{n+1}\}\big) = (1-p_{k(n+1)})v_{n+1} + p_{k(n+1)}(v_k + \eta_{k(n+1)}).$$

We now compute $Pr(O_k(P_{-k} \cap \{a_1, \dots, a_l\} = \{a_k, a_l\})$. If $n < m$, then Case 3 occurs with the remaining probability after considering the events of cases 1 and 2,

$$\begin{aligned} 1 - \sum_{h=1}^{k-1} Q(l,k) - \sum_{l=k+1}^{n} \big[Q(k,l-1) - Q(k,l)\big] &= Q(k,k) - \big[Q(k,k) + Q(k,n)\big] \\ &= Q(k,n). \end{aligned}$$

This yields final term in the stated formula.

If $n = m$, then Case 3 does not occur. Instead, still with probability $Q(k, n)$, $O_k(P_{-k}, \varphi^\mu) \backslash \{a_k\} = \emptyset$. Then $O_k(P_{-k}, \varphi^\mu) = \{a_k\}$ and agent k receives a_k regardless of his preferences. This yields a conditional expected utility of v_k. Given our conventions $\eta_{k(m+1)} = 0$ and $p_{k(m+1)} = 0$, the expression in Case 3 simplifies to v_k and the stated formula applies. □

For reference, Corollaries 1 and 2 simplify the utility formulas from Propositions 4 and 5 for special cases.

Corollary 1. *Let $\sigma^\prec \in A^N$ be an equilibrium under $\varphi^\prec$. If $n = m = 2$, then*

$$\begin{aligned} U_1(\sigma^\prec, \varphi^\prec) &= (1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21}) \text{ and} \\ U_2(\sigma^\prec, \varphi^\prec) &= (1 - p_{12})v_2 + p_{12}v_1. \end{aligned}$$

If $n = m = 3$, then

$$\begin{aligned} U_1(\sigma^\prec, \varphi^\prec) &= (1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21}), \\ U_2(\sigma^\prec, \varphi^\prec) &= (1 - p_{12})\big[(1 - p_{23})v_2 + p_{23}(v_3 + \eta_{32})\big] \\ &\quad + p_{12}\big[(1 - p_{13})v_1 + p_{13}(v_3 + \eta_{31})\big], \text{ and} \\ U_3(\sigma^\prec, \varphi^\prec) &= p_{12}p_{13}v_1 + (1 - p_{12})p_{23}v_2 + \big(1 - p_{12}p_{13} - (1 - p_{12})p_{23}\big)v_3. \end{aligned}$$

Corollary 2. *Let $\sigma^\mu \in A^N$ be an equilibrium under φ^μ. If $n = m = 2$, then*

$$\begin{aligned} U_1(\sigma^\mu, \varphi^\mu) &= (1 - p_{12})\big[(1 - p_{12})(v_1 + \eta_{12}) + p_{12}v_2\big] + p_{12}v_1 \text{ and} \\ U_2(\sigma^\mu, \varphi^\mu) &= p_{12}\big[p_{12}(v_2 + \eta_{21}) + (1 - p_{12})v_1\big] + (1 - p_{12})v_2. \end{aligned}$$

If $n = m = 3$, *then*

$$\begin{aligned}
U_1(\sigma^\mu, \varphi^\mu) &= (1 - p_{12})\big[(1 - p_{12})(v_1 + \eta_{12}) + p_{12}v_2\big] \\
&\quad + p_{12}(1 - p_{13})\big[(1 - p_{13})(v_1 + \eta_{13}) + p_{13}v_3\big] + p_{12}p_{13}v_1, \\
U_2(\sigma^\mu, \varphi^\mu) &= p_{12}\big[p_{12}(v_2 + \eta_{21}) + (1 - p_{12})v_1\big] \\
&\quad + (1 - p_{12})(1 - p_{23})\big[(1 - p_{23})(v_2 + \eta_{23}) + p_{23}v_3\big] \\
&\quad + (1 - p_{12})p_{23}v_2, \ \ \textit{and} \\
U_3(\sigma^\mu, \varphi^\mu) &= p_{13}p_{12}\big[p_{13}(v_3 + \eta_{31}) + (1 - p_{13})v_1\big] \\
&\quad + (1 - p_{12})p_{23}\big[p_{23}(v_3 + \eta_{32}) + (1 - p_{23})v_2\big] \\
&\quad + \big(1 - p_{13}p_{12} - (1 - p_{12})p_{23}\big)v_3.
\end{aligned}$$

A.5.1 Proof of Theorem 3

Let $\sigma^\prec, \sigma^\mu \in A^N$ be equilibria under $\varphi^\prec$ and φ^μ respectively and let $P^\prec, P^\mu \in \mathcal{P}^N$ denote preferences determined by $\sigma^\prec$ and σ^μ. Without loss of generality, suppose that for each $i \in N$, $\sigma_i^\prec = a_{i+1}$ and $\sigma_i^\mu = a_i$. We argue that agent n is better off under φ^μ than under $\varphi^\prec$.

By Lemma 3, for each $i \in \{1, \ldots, n-1\}$, $Q(i, n)$ is the probability that $a_i \in O_n(P^\prec_{-n}, \varphi^\prec)$ and the probability that $a_i \in O_n(P^\mu_{-n}, \varphi^\mu)$. Since these events occur with equal probability and are independent of $\sigma^\prec_n$ and σ^μ_n, we may compare ex-ante expected utilities by conditioning on these events.

Case 1: For some $\boldsymbol{i \in \{1, \ldots, n-1\}}$, $\boldsymbol{a_i \in O_n(P^\prec_{-n}, \varphi^\prec)}$ versus $\boldsymbol{a_i \in O_n(P^\mu_{-n}, \varphi^\mu)}$. By mutual exclusivity, $O_n(P^\prec_{-n}, \varphi^\prec) \cap \{a_1, \ldots, a_{n-1}\} = \{a_i\}$ and $O_n(P^\mu_{-n}, \varphi^\mu) \cap \{a_1, \ldots, a_{n-1}\} = \{a_i\}$.

Subcase 1.1: $n = m$. Priority: Since n has the lowest priority, all but one object is assigned to other agents. Since $a_i \in O_n(P^\prec_{-n}, \varphi^\prec)$, $O_n(P^\prec_{-n}, \varphi^\prec) = \{a_i\}$. So $U_n\big(\sigma^\prec, \varphi^\prec \big| a_i \in O_n(P^\prec_{-n}, \varphi^\prec)\big) = v_i$. TTC: Since $a_i \in O_n(P^\mu_{-n}, \varphi^\mu)$, $U_n\big(\sigma^\mu, \varphi^\mu \big| a_i \in O_n(P^\mu_{-n}, \varphi^\mu)\big) \geq v_i$. If $\varepsilon^\mu_{nn} > v_i - v_n$, then $a_n \ P^\mu_n \ a_i$ and agent n receives a_n. This occurs with probability $p_{in} > 0$ and yields a conditional expected utility of $v_n + \eta_{ni} > v_i$. Therefore, $U_n\big(\sigma^\prec, \varphi^\prec \big| a_i \in O_n(P^\prec_{-n}, \varphi^\prec)\big) < U_n\big(\sigma^\mu, \varphi^\mu \big| a_i \in O_n(P^\mu_{-n}, \varphi^\mu)\big)$.

Subcase 1.2: $n < m$. Priority: If $\varepsilon^\prec_{n(n+1)} < v_i - v_{n+1}$, then $a_i \ P^\prec_n \ a_{n+1}$ and agent n receives a_i. This occurs with probability $1 - p_{i(n+1)}$ and yields a conditional expected utility of v_i. If instead $\varepsilon^\prec_{n(n+1)} > v_i - v_{n+1}$, then $a_{n+1} \ P^\prec_n \ a_i$ and agent n receives a_{n+1}. This occurs with probability $p_{i(n+1)}$ and yields a conditional expected utility of $v_{n+1}+$

$\eta_{(n+1)i}$. Then

$$U_n\big(\sigma^{\prec}, \varphi^{\prec}\big|a_i \in O_n(P^{\prec}_{-n}, \varphi^{\prec})\big) = (1 - p_{i(n+1)})v_i + p_{i(n+1)}(v_{n+1} + \eta_{(n+1)i}) \\ = v_i + p_{i(n+1)}\big(v_{n+1} - v_i + \eta_{i(n+1)}\big).$$

TTC: If $\varepsilon^{\mu}_{nn} < v_i - v_n$, then $a_i\ P^{\mu}_n\ a_n$ and agent n receives a_i. This occurs with probability $1 - p_{in}$ and yields a conditional expected utility of v_i. If instead $\varepsilon^{\mu}_{nn} > v_i - v_n$, then $a_n\ P^{\mu}_n\ a_i$ and agent n receives a_n. This occurs with probability p_{in} and yields a conditional expected utility of $v_n + \eta_{ni}$. Then

$$U_n\big(\sigma^{\mu}, \varphi^{\mu}\big|a_i \in O_n(P^{\mu}_{-n}, \varphi^{\mu})\big) = (1 - p_{in})v_i + p_{in}(v_n + \eta_{ni}) = v_i + p_{in}(v_n - v_i + \eta_{ni}).$$

By Lemma 2(3), since $i < n < n+1$, $p_{i(n+1)}\big(v_{n+1} - v_i + \eta_{i(n+1)}\big) < p_{in}\big(v_n - v_i + \eta_{ni}\big)$. Therefore, $U_n\big(\sigma^{\prec}, \varphi^{\prec}\big|a_i \in O_n(P^{\prec}_{-n}, \varphi^{\prec})\big) < U_n\big(\sigma^{\mu}, \varphi^{\mu}\big|a_i \in O_n(P^{\mu}_{-n}, \varphi^{\mu})\big)$.

Case 2: $\boldsymbol{O_n(P^{\prec}_{-n}, \varphi^{\prec}) \cap \{a_1, \ldots, a_{n-1}\} = \emptyset}$ versus $\boldsymbol{O_n(P^{\mu}_{-n}, \varphi^{\mu}) \cap \{a_1, \ldots, a_{n-1}\} = \emptyset}$. Then $O_n(P^{\prec}_{-n}, \varphi^{\prec}) = O_n(P^{\mu}_{-n}, \varphi^{\mu}) = \{a_n, \ldots, a_m\}$.

Subcase 2.2: $n = m$. Then agent n receives a_n under each rule. This yields a conditional expected utility of v_n under either rule, so $U_n\big(\sigma^{\prec}, \varphi^{\prec}\big|O_n(P^{\prec}_{-n}, \varphi^{\prec}) = \{a_n\}\big) = U_n\big(\sigma^{\mu}, \varphi^{\mu}\big|O_n(P^{\mu}_{-n}, \varphi^{\mu}) = \{a_n\}\big)$.

Subcase 2.2: $n < m$. First consider $\varphi^{\prec}$. If $\varepsilon^{\prec}_{n(n+1)} < v_n - v_{n+1}$, then $a_n\ P^{\prec}_n\ a_{n+1}$ and agent n receives a_n. This occurs with probability $1 - p_{n(n+1)}$ and yields a conditional expected utility of v_n. If instead $\varepsilon^{\prec}_{n(n+1)} > v_n - v_{n+1}$, then $a_{n+1}\ P^{\prec}_n\ a_n$ and agent n receives a_{n+1}. This occurs with probability $p_{n(n+1)}$ and yields a conditional expected utility of $v_{n+1} + \eta_{(n+1)n}$. Then

$$U_n\big(\sigma^{\prec}, \varphi^{\prec}\big|a_i \in O_n(P^{\prec}_{-n}, \varphi^{\prec})\big) = (1 - p_{n(n+1)})v_n + p_{n(n+1)}(v_{n+1} + \eta_{(n+1)n}).$$

Now consider φ^{μ}. If $\varepsilon^{\mu}_{n(n+1)} > v_{n+1} - v_n$, then $a_n\ P^{\mu}_n\ a_{n+1}$ and agent n receives a_n. This occurs with probability $1 - p_{n(n+1)}$ and yields a conditional expected utility of $v_n + \eta_{n(n+1)}$. If instead $\varepsilon^{\mu}_{n(n+1)} < v_{n+1} - v_n$, then $a_{n+1}\ P^{\mu}_n\ a_n$ and agent n receives a_{n+1}. This occurs with probability $p_{n(n+1)}$ and yields a conditional expected utility of v_{n+1}. Then

$$U_n\big(\sigma^{\mu}, \varphi^{\mu}\big|a_i \in O_n(P^{\mu}_{-n}, \varphi^{\mu})\big) = (1 - p_{n(n+1)})(v_n + \eta_{n(n+1)}) + p_{n(n+1)}v_{n+1}.$$

By Lemma 2(2), $p_{n(n+1)}\eta_{(n+1)n} = (1 - p_{n(n+1)})\eta_{n(n_1)}$. Therefore, $U_n\big(\sigma^{\prec}, \varphi^{\prec}\big|a_i \in O_n(P^{\prec}_{-n}, \varphi^{\prec})\big) = U_n\big(\sigma^{\mu}, \varphi^{\mu}\big|a_i \in O_n(P^{\mu}_{-n}, \varphi^{\mu})\big)$.

Since Case 1 occurs with positive probability, $U_n(\sigma^{\prec}, \varphi^{\prec}) < U_n(\sigma^{\mu}, \varphi^{\mu})$. Therefore, max-min social welfare is higher under φ^{μ} than under φ^{μ}.

A.5.2 Proof of Lemma 1

Let $n, m \in \mathbb{N}$ and $\sigma \in A^N$. Since we wish to compare investigation strategies, we assume that, conditional on ε_σ, objects are allocated to maximize utilitarian welfare.

Step 1: Utilitarian welfare is maximized when $\{\sigma_1, \ldots, \sigma_n\} \subseteq \{a_1, \ldots, a_n\}$. Suppose instead that there are $i \in N$ and $a_l \in A$ with $n < l$ such that $\sigma_i = a_l$. Then there is $a_k \in A$ with $k < n$ such that for each $j \in N$, $\sigma_j \neq a_k$. Let $\hat{s} \in A^N$ modify σ so that $\hat{\sigma}_i = a_k$ and agent i now receives a_k whenever he originally received a_l. Since we have assumed that the allocation under σ maximizes utilitarian welfare conditional on ε_σ, agent i receives a_l with positive probability under σ, and in particular when $\varepsilon_{ik} = v_1 - v_k$. Because private values are independent and identically distributed, for each $j \in N \backslash \{i\}$, agent j's expected utility is the same under $\hat{\sigma}$ and under σ. Also, agent i's utility differs only in those cases where he now receives a_k. Since $k < l$, $v_k > v_l$ and in these cases agent i's conditional expected utility is higher under $\hat{\sigma}$. Therefore, $\hat{\sigma}$ achieves higher utilitarian welfare than does σ.

Step 2: Utilitarian welfare is maximized when $\{\sigma_1, \ldots, \sigma_n\} = \{a_1, \ldots, a_n\}$. By Step 1, we may suppose that $\{\sigma_1, \ldots, \sigma_n\} \subseteq \{a_1, \ldots, a_n\}$. It remains to show that agents optimally investigate distinct objects. Suppose instead that at least two agents investigate the same object under σ and let $B \equiv \{a_1, \ldots, a_n\} \backslash \{\sigma_1, \ldots, \sigma_n\}$ and $l \equiv |B|$. Let $c \in A$ be one of the objects investigated by the greatest number of agents, $K \equiv \{i \in N : \sigma_i = c\}$, and $k \equiv |K|$. By assumption, $B \neq \emptyset$ and $k \geq 2$.

Let $\hat{\sigma} \in A^N$ be such that $\{\hat{\sigma}_1, \ldots, \hat{\sigma}_n\} = \{a_1, \ldots, a_n\}$. Because private values are independent and identically distributed, their realizations are independent of the investigation strategy. Therefore, we may compare σ and $\hat{\sigma}$ by pairing realizations ε_σ and $\varepsilon_{\hat{\sigma}}$ such that for each $i \in N$, $\varepsilon_{i\sigma(i)} = \varepsilon_{i\hat{\sigma}(i)}$. With this identification implicit, for each $i \in N$, let $\hat{\epsilon}_i \equiv \varepsilon_{i\sigma(i)} = \varepsilon_{i\hat{\sigma}(i)}$ and $\hat{\epsilon} \equiv (\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n)$ so private value realizations are now distinguished only by agents. Then $\mathbf{E}_F[\hat{\epsilon}|\hat{\epsilon} > 0] = \eta_0$. For each $\hat{\epsilon}$, let $N^+(\hat{\epsilon}) \equiv \{i \in N : \hat{\epsilon}_i > 0\}$. Utilitarian welfare is bounded above by $U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i$. If $n < m$, then $\hat{\sigma}$ achieves this bound, so suppose $n = m$. If $|N^+(\hat{\epsilon})| \neq n - 1$, then $\hat{\sigma}$ also achieves this bound. Furthermore, if $|N^+(\hat{\epsilon})| = n - 1$, then the utilitarian welfare under $\hat{\sigma}$ is at least $U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i - \min_{N^+(\hat{\epsilon})} \hat{\epsilon}_i$. Taking expectations, the utilitarian welfare under $\hat{\sigma}$ is at least

$$U_0 + \tfrac{n}{2} \cdot \eta_0 - n\left(\tfrac{1}{2}\right)^n \mathbf{E}_F\left[\min_{N^+(\hat{\epsilon})} \hat{\epsilon}_i \middle| |N^+(\hat{\epsilon})| = n - 1\right].$$

To compare utilitarian welfare under σ, we distinguish several cases.

Case 1: $n \geq 3$ and $k \geq 3$. If $|N^+(\varepsilon)| = n$, then the utilitarian welfare under σ is $U_0 + \sum_{N^+(\hat{\epsilon})\setminus K} \hat{\epsilon}_i + \max_K \hat{\epsilon}_i < U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i$. If $|N^+(\hat{\epsilon})| = n-1$, then $|K \cap N^+(\hat{\epsilon})| \geq 2$ and the utilitarian welfare under σ is at most $U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i - \min_{K\cap N^+(\hat{\epsilon})} \hat{\epsilon}_i \leq U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i - \min_{N^+(\hat{\epsilon})} \hat{\epsilon}_i$. Finally, if $|N^+(\hat{\epsilon})| < n-1$, then the utilitarian welfare under σ is at most $U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i$. Altogether, the utilitarian welfare under σ is less than the utilitarian welfare under $\hat{\sigma}$.

Case 2: $n \geq 3$, $k = 2$, and $l \geq 2$. Since $l \geq 2$, there is $c' \in A\setminus\{c\}$ and $K' \subseteq N\setminus K$ with $|K'| = 2$ such that the agents in K' both investigate c'. If $|N^+(\hat{\epsilon})| = n$, then the utilitarian welfare under σ is $U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i - \min_K \hat{\epsilon}_i - \min_{K'} \hat{\epsilon}_i < U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i$. If $|N^+(\hat{\epsilon})| = n-1$, then both agents in at least one of K and K' obtain positive realizations and the utilitarian welfare under σ is at most $U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i - \min_{(K\cup K')\cap N^+(\hat{\epsilon})} \hat{\epsilon}_i \leq U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i - \min_{N^+(\hat{\epsilon})} \hat{\epsilon}_i$. Finally, if $|N^+(\hat{\epsilon})| < n-1$, then the utilitarian welfare under σ is at most $U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i$. Altogether, the utilitarian welfare under σ is less than the utilitarian welfare under $\hat{\sigma}$.

Case 3: $n \geq 4$, $k = 2$, and $l = 1$. If $K \subseteq N^+(\hat{\epsilon})$, then the utilitarian welfare under σ is at most $U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i - \min_K \hat{\epsilon}_i$. In all other cases, the utilitarian welfare under σ is at most $U_0 + \sum_{N^+(\hat{\epsilon})} \hat{\epsilon}_i$. By independence, $Pr(K \subseteq N^+(\hat{\epsilon})) = \left(\frac{1}{2}\right)^{|K|} = \frac{1}{4}$. Therefore, taking expectations, the utilitarian welfare under σ is at most

$$U_0 + \tfrac{n}{2} \cdot \eta_0 - \tfrac{1}{4}\, \mathbf{E}_F \left[\min_K \hat{\epsilon}_i \middle| K \subseteq N^+(\hat{\epsilon}) \right].$$

Since $n \geq 4$, $n\left(\frac{1}{2}\right)^n \leq \frac{1}{4}$ and

$$\mathbf{E}_F \left[\min_{N^+(\hat{\epsilon})} \hat{\epsilon}_i \middle| |N^+(\hat{\epsilon})| = n-1 \right] \leq \mathbf{E}_F \left[\min_K \hat{\epsilon}_i \middle| K \subseteq N^+(\hat{\epsilon}) \right].$$

Case 4: $n = 3$, $k = 2$, and $l = 1$. Then $N = \{1,2,3\}$ and there is $a \in A$ such that $A = \{a,b,c\}$. Label the agents so that $\sigma_1 = \sigma_2 = c$. Conditional on $N^+(\hat{\epsilon})$, the utilitarian welfare under each policy is:

$N^+(\hat{\epsilon})$	utility under $\hat{\sigma}$	utility under σ
$\{1,2,3\}$	$U_0 + \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3$	$U_0 + \max\{\hat{\epsilon}_1, \hat{\epsilon}_2\} + \hat{\epsilon}_3$
$\{1,2\}$	$U_0 + \max\{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3\}$	$U_0 + \max\{\hat{\epsilon}_1, \hat{\epsilon}_2\}$
$\{1,3\}$	$U_0 + \max\{\hat{\epsilon}_1, \hat{\epsilon}_3, \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3\}$	$U_0 + \hat{\epsilon}_1 + \hat{\epsilon}_3$
$\{2,3\}$	$U_0 + \max\{\hat{\epsilon}_2, \hat{\epsilon}_3, \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3\}$	$U_0 + \hat{\epsilon}_2 + \hat{\epsilon}_3$
$\{1\}$	$U_0 + \hat{\epsilon}_1$	$U_0 + \hat{\epsilon}_1$
$\{2\}$	$U_0 + \hat{\epsilon}_2$	$U_0 + \hat{\epsilon}_2$
$\{3\}$	$U_0 + \hat{\epsilon}_3$	$U_0 + \max\{\hat{\epsilon}_1 + \hat{\epsilon}_3, \hat{\epsilon}_2 + \hat{\epsilon}_3, 0\}$
$\emptyset$	U_0	U_0

By independence, each case occurs with probability $\frac{1}{8}$. In the cases $N^+(\hat{\epsilon}) = \{1\}$, $N^+(\hat{\epsilon}) = \{2\}$, and $N^+(\hat{\epsilon}) = \emptyset$, the utilities under $\hat{\sigma}$ and σ are equal. In the case $N^+(\hat{\epsilon}) = \{1, 2\}$, since $\max\{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3\} \geq \max\{\hat{\epsilon}_1, \hat{\epsilon}_2\}$, utility is at least as high under $\hat{\sigma}$ as under σ. For the remaining cases, we compute utility differences more precisely. Let $\bar{u} \equiv \mathbf{E}_F\big[\min\{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_3\}\big|\hat{\epsilon}_1 > 0,\ \hat{\epsilon}_2 > 0,\ \hat{\epsilon}_3 > 0\big]$.

Subcase 4.1: $N^+(\hat{\epsilon}) = \{1, 2, 3\}$. The difference in utility under $\hat{\sigma}$ and σ is

$$\big[U_0 + \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3\big] - \big[U_0 + \max\{\hat{\epsilon}_1, \hat{\epsilon}_2\} + \hat{\epsilon}_3\big] = \min\{\hat{\epsilon}_1, \hat{\epsilon}_2\}.$$

The difference in conditional expected utility is therefore

$$\begin{aligned}&\mathbf{E}_F\big[\min\{\hat{\epsilon}_1, \hat{\epsilon}_2\}\big|\hat{\epsilon}_1 > 0,\ \hat{\epsilon}_2 > 0,\ \hat{\epsilon}_3 > 0\big]\\ &\qquad > \mathbf{E}_F\big[\min\{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_3\}\big|\hat{\epsilon}_1 > 0,\ \hat{\epsilon}_2 > 0,\ \hat{\epsilon}_3 > 0\big] = \bar{u}.\end{aligned}$$

Subcase 4.2: $N^+(\hat{\epsilon}) = \{1, 3\}$. The difference in utility under $\hat{\sigma}$ and σ is

$$\big[U_0 + \max\{\hat{\epsilon}_1, \hat{\epsilon}_3, \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3\}\big] - \big[U_0 + \hat{\epsilon}_1 + \hat{\epsilon}_3\big] = \max\{-\hat{\epsilon}_3, -\hat{\epsilon}_1, \hat{\epsilon}_2\} = -\min\{\hat{\epsilon}_1, -\hat{\epsilon}_2, \hat{\epsilon}_3\}.$$

The difference in conditional expected utility is therefore

$$\begin{aligned}&\mathbf{E}_F\big[-\min\{\hat{\epsilon}_1, -\hat{\epsilon}_2, \hat{\epsilon}_3\}\big|\hat{\epsilon}_1 > 0,\ \hat{\epsilon}_2 < 0,\ \hat{\epsilon}_3 > 0\big]\\ &\qquad = -\mathbf{E}_F\big[\min\{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_3\}\big|\hat{\epsilon}_1 > 0,\ \hat{\epsilon}_2 > 0,\ \hat{\epsilon}_3 > 0\big] = -\bar{u}.\end{aligned}$$

Subcase 4.3: $N^+(\hat{\epsilon}) = \{2, 3\}$. By the same computations as in Subcase 4.2, the difference in conditional expected utility is also $-\bar{u}$.

Subcase 4.4: $N^+(\hat{\epsilon}) = \{3\}$. The difference in utility under $\hat{\sigma}$ and σ is

$$\big[U_0 + \hat{\epsilon}_3\big] - \big[U_0 + \max\{\hat{\epsilon}_1 + \hat{\epsilon}_3, \hat{\epsilon}_2 + \hat{\epsilon}_3, 0\}\big] = -\max\{\hat{\epsilon}_1, \hat{\epsilon}_2, -\hat{\epsilon}_3\} = \min\{-\hat{\epsilon}_1, -\hat{\epsilon}_2, \hat{\epsilon}_3\}.$$

The difference in conditional expected utility is therefore

$$\begin{aligned}&\mathbf{E}_F\big[\min\{-\hat{\epsilon}_1, -\hat{\epsilon}_2, \hat{\epsilon}_3\}\big|\hat{\epsilon}_1 < 0,\ \hat{\epsilon}_2 < 0,\ \hat{\epsilon}_3 > 0\big]\\ &\qquad = \mathbf{E}_F\big[\min\{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_3\}\big|\hat{\epsilon}_1 > 0,\ \hat{\epsilon}_2 > 0,\ \hat{\epsilon}_3 > 0\big] = \bar{u}.\end{aligned}$$

The cumulative difference in Subcases 4.1-4.4 is greater than $\bar{u} - \bar{u} - \bar{u} + \bar{u} = 0$. Therefore, the utilitarian welfare under σ is less than the utilitarian welfare under $\hat{\sigma}$.

Case 5: $n = 2$. Then $N = \{1, 2\}$ and $A = \{b, c\}$. Conditional on $N^+(\hat{\epsilon})$, the utilitarian welfare under each policy is:

$N^+(\hat{\epsilon})$	utility under $\hat{\sigma}$	utility under σ
$\{1,2\}$	$U_0 + \hat{\epsilon}_1 + \hat{\epsilon}_2$	$U_0 + \max\{\hat{\epsilon}_1, \hat{\epsilon}_2\}$
$\{1\}$	$U_0 + \max\{\hat{\epsilon}_1 + \hat{\epsilon}_2, 0\}$	$U_0 + \hat{\epsilon}_1$
$\{2\}$	$U_0 + \max\{\hat{\epsilon}_1 + \hat{\epsilon}_2, 0\}$	$U_0 + \hat{\epsilon}_2$
$\emptyset$	U_0	$U_0 + \max\{\hat{\epsilon}_1, \hat{\epsilon}_2\}$

By independence, each case occurs with probability $\frac{1}{4}$. Also,

$$\begin{gathered}
\mathbf{E}_F\big[\hat{\epsilon}_1 + \hat{\epsilon}_2 \big| \hat{\epsilon}_1 > 0,\, \hat{\epsilon}_2 > 0\big] = 2\eta_0, \\
\mathbf{E}_F\big[\max\{\hat{\epsilon}_1 + \hat{\epsilon}_2, 0\} \big| \hat{\epsilon}_1 > 0 > \hat{\epsilon}_2\big] = \mathbf{E}_F\big[\max\{\hat{\epsilon}_1 + \hat{\epsilon}_2, 0\} \big| \hat{\epsilon}_2 > 0 > \hat{\epsilon}_1\big], \text{ and} \\
Pr\big(\hat{\epsilon}_1 + \hat{\epsilon}_2 > 0 \big| \hat{\epsilon}_1 > 0 > \hat{\epsilon}_2\big) = Pr\big(\hat{\epsilon}_1 + \hat{\epsilon}_2 > 0 \big| \hat{\epsilon}_2 > 0 > \hat{\epsilon}_1\big) = \tfrac{1}{2}.
\end{gathered}$$

Combining these observations,

$$\begin{aligned}
\mathbf{E}_F\big[\max\{\hat{\epsilon}_1 + \hat{\epsilon}_2, 0\} \big| \hat{\epsilon}_1 > 0 > \hat{\epsilon}_2\big] &= \tfrac{1}{2}\,\mathbf{E}_F\big[\hat{\epsilon}_1 + \hat{\epsilon}_2 \big| \hat{\epsilon}_1 + \hat{\epsilon}_2 > 0,\, \hat{\epsilon}_1 > 0 > \hat{\epsilon}_2\big] \\
&= \tfrac{1}{2}\,\mathbf{E}_F\big[\max\{\hat{\epsilon}_1, \hat{\epsilon}_2\} - \min\{\hat{\epsilon}_1, \hat{\epsilon}_2\} \big| \hat{\epsilon}_1 > 0 > \hat{\epsilon}_2\big].
\end{aligned}$$

Now, taking expectations, the utilitarian welfare under $\hat{\sigma}$ is

$$\begin{aligned}
&U_0 + \tfrac{1}{4} \cdot 2\eta_0 + \tfrac{1}{4} \cdot 2\,\mathbf{E}_F\big[\max\{\hat{\epsilon}_1 + \hat{\epsilon}_2, 0\} \big| \hat{\epsilon}_1 > 0 > \hat{\epsilon}_2\big] \\
&\qquad = U_0 + \tfrac{1}{2}\,\eta_0 + \tfrac{1}{4}\,\mathbf{E}_F\big[\max\{\hat{\epsilon}_1, \hat{\epsilon}_2\} - \min\{\hat{\epsilon}_1, \hat{\epsilon}_2\} \big| \hat{\epsilon}_1 > 0 > \hat{\epsilon}_2\big].
\end{aligned}$$

Similarly, the utilitarian welfare under σ is

$$\begin{aligned}
&U_0 + \tfrac{1}{4} \cdot 2\eta_0 \\
&\quad + \tfrac{1}{4}\,\mathbf{E}_F\big[\max\{\hat{\epsilon}_1, \hat{\epsilon}_2\} \big| \hat{\epsilon}_1 > 0,\, \hat{\epsilon}_2 > 0\big] \\
&\quad + \tfrac{1}{4}\,\mathbf{E}_F\big[\max\{\hat{\epsilon}_1,\, \hat{\epsilon}_2\} \big| \hat{\epsilon}_1 < 0,\, \hat{\epsilon}_2 < 0\big] \\
&\qquad = U_0 + \tfrac{1}{2}\,\eta_0 + \tfrac{1}{4}\,\mathbf{E}_F\big[\max\{\hat{\epsilon}_1, \hat{\epsilon}_2\} - \min\{\hat{\epsilon}_1, \hat{\epsilon}_2\} \big| \hat{\epsilon}_1 > 0 > \hat{\epsilon}_2\big].
\end{aligned}$$

Therefore, the utilitarian welfare is the same under either policy.

A.5.3 Proof of Proposition 3

Let (σ, φ) be a policy. By Lemma 1, utilitarian social welfare is maximized when agents investigate distinct objects. Without loss of generality, suppose that for each $i \in N$, $\sigma_i = a_i$. Since the investigation strategies are fixed, we suppress the object in our notation and write ε_i for $\varepsilon_{i\sigma(i)}$.

Given ε, let $N^+(\varepsilon) \equiv \{i \in N : \varepsilon_i > 0\}$ and $N^-(\varepsilon) \equiv \{i \in N : \varepsilon_i < 0\}$. Utilitarian social welfare is maximized if (i) for each $i \in N^+$, $\varphi_i(\varepsilon) = a_i$ and (ii) for each $i \in N^-$,

$\varphi_i(\varepsilon) \neq a_i$ whenever feasible. That is, agents with positive realizations keep their objects and agents with negative realizations exchange objects. Conditional on ε, expected utilitarian social welfare is therefore bounded above by $U_0(n) + \eta_0|N^+|$. For each $i \in N$, $e_i > 0$ with probability $p_0 = \frac{1}{2}$. These events are independent, so in expectation $|N^+| = \frac{n}{2}$. Altogether, utilitarian social welfare is bounded above by $U_0(n) + \frac{n}{2}\eta_0$.

In general, (i) and (ii) are not feasible simultaneously. In particular, they are incompatible when $|N^-| = 1$: if exactly one agent receives a negative private value realization, then this agent must either keep the object investigated, exchange with an agent who received a positive private value realization, or receive a_{n+1} (if $n < m$). In all other cases, (i) and (ii) are compatible: if $|N^-| = 0$, then $N^+ = N$ and it is feasible for each agent to receive the object that he investigated; if $|N^-| \geq 2$, then there is a feasible allocation in which the agents in N^- exchange objects exclusively among themselves.

Now suppose $|N^-| = 1$. Since each policy that is feasible in the case $n = m$ is also feasible when $n < m$, we show that the lower bound is achievable when $n = m$. Let $i \in N^-$ and $j \in N^+$ be such that $\varepsilon_j = \min\{\varepsilon_k : k \in N^+\}$. The utilitarian optimal policy in this case has agents i and j exchange objects if $\varepsilon_i + \varepsilon_j < 0$ and keep their objects if $\varepsilon_i + \varepsilon_j < 0$. In either case, all agents $N \setminus \{i, j\}$ keep their objects. For a lower bound, suppose that agents i and j exchange objects in either case. Conditional on ε, expected utilitarian social welfare is at least $U_0(n) + \eta_0(|N^+| - 1) = U_0(n) + \eta_0(n-2)$. Now $Pr\big(|N^-| = 1\big) = \binom{n}{1} p_0^n = \frac{n}{2^n}$, so the utilitarian social welfare of this policy is at least $U_0(n) + \frac{n}{2}\eta_0 - \frac{n}{2^n}\eta_0$. Finally, as $n \to \infty$, $\frac{n}{2^n} \to 0$, so $U_0(n) + \frac{n}{2}\eta_0 - \frac{n}{2^n}\eta_0 \to U_0(n) + \frac{n}{2}\eta_0$.

A.5.4 Proof of Theorem 4

We argue by inspecting the formulas derived in Propositions 4 and 5. Let $n, m \in \mathbb{N}$ with $n \leq m$ and let $\sigma^{\prec}, \sigma^{\mu} \in A^N$ be equilibria under $\varphi^{\prec}$ and φ^{μ} respectively. We proceed by induction on the number of agents and objects. Let $U^{n,m}(\varphi)$ be the sum of the agents' ex-ante equilibrium utilities under φ.

Step 1: $\boldsymbol{n = m = 2}$. By Corollary 1,

$$\begin{aligned} U^{2,2}(\varphi^{\prec}) &= U_1^{2,2}(\sigma^{\prec}, \varphi^{\prec}) + U_2^{2,2}(\sigma^{\prec}, \varphi^{\prec}) \\ &= [(1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21})] + [(1 - p_{12})v_2 + p_{12}v_1] \\ &= U_0(n) + p_{12}\eta_{21}. \end{aligned}$$

By Lemma 2(2), $(1-p_{12})\eta_{12} = p_{12}\eta_{21}$. Now by Corollary 2,

$$\begin{aligned}
U^{2,2}(\varphi^{\mu}) &= U_1^{2,2}(\sigma^{\mu},\varphi^{\mu}) + U_2^{2,2}(\sigma^{\mu},\varphi^{\mu}) \\
&= (1-p_{12})\big[(1-p_{12})(v_1+\eta_{12}) + p_{12}v_2\big] + p_{12}v_1 \\
&\qquad + p_{12}\big[p_{12}(v_2+\eta_{21}) + (1-p_{12})v_1\big] + (1-p_{12})v_2 \\
&= U_0(n) + (1-p_{12})(1-p_{12})\eta_{12} + p_{12}p_{12}\eta_{21} \\
&= U_0(n) + (1-p_{12})p_{12}\eta_{21} + p_{12}p_{12}\eta_{21} \\
&= U_0(n) + p_{12}\eta_{21}.
\end{aligned}$$

Therefore, $U^{2,2}(\varphi^{\prec}) = U^{2,2}(\varphi^{\mu})$.

Step 2: Increasing the number of objects. Suppose that $U^{n,n}(\varphi^{\prec}) = U^{n,n}(\varphi^{\mu})$. Under either rule, objects $\{a_{n+2}, a_{n+3}, \ldots, a_m\}$ are never allocated and so all agents are indifferent to their presence. Thus, it suffices to consider $m = n+1$. Since our hypothesis applies to all problems with n agents and n objects, we may suppose that the new object has the lowest common value among all objects, namely v_{n+1}.

Under $\varphi^{\prec}$, only agent n ever receives a_{n+1}, and the difference in utilitarian welfare between $U^{n,n}(\varphi^{\prec})$ and $U^{n,n+1}(\varphi^{\prec})$ is the difference in agent n's ex-ante expected utility. Then by Proposition 4,

$$\begin{aligned}
U^{n,n+1}(\varphi^{\prec}) - U^{n,n}(\varphi^{\prec}) &= U_n^{n,n+1}(\sigma^{\prec},\varphi^{\prec}) - U_n^{n,n}(\sigma^{\prec},\varphi^{\prec}) \\
&= \sum_{l=1}^{n} Q(l,k)\big[(1-p_{l(n+1)})v_l + p_{l(n+1)}(v_{n+1}+\eta_{(n+1)l})\big] - \sum_{l=1}^{n} Q(l,k)v_l \\
&= \sum_{l=1}^{n} Q(l,n)p_{l(n+1)}(v_{n+1}+\eta_{(n+1)l} - v_l).
\end{aligned}$$

Under φ^{μ}, the arrival of the new object increases each agent's ex-ante expected utility. For each $k \in N$, the difference is the last term in the expression for U_k in Proposition 5:

$$\begin{aligned}
U_k^{n,n+1}(\sigma^{\mu},\varphi^{\mu}) - U_k^{n,n}(\sigma^{\mu},\varphi^{\mu}) &= Q(k,n)\big[(1-p_{k(n+1)}(v_k+\eta_{k(n+1)}) + p_{k(n+1)}v_{n+1}\big] - Q(k,n)v_k \\
&= Q(k,n)\big[(1-p_{k(n+1)})\eta_{k(n+1)} + p_{k(n+1)}(v_{n+1}-v_l)\big].
\end{aligned}$$

By Lemma 2(2), $(1-p_{k(n+1)})\eta_{k(n+1)l} = p_{k(n+1)}\eta_{(n+1)k}$. Substituting and summing over agents,

$$\begin{aligned}
U^{n,n+1}(\varphi^\mu) - U^{n,n}(\varphi^\mu) &= \sum_{k=1}^{n} Q(k,n)\big[(1-p_{k(n+1)})\eta_{k(n+1)} + p_{k(n+1)}(v_{n+1} - v_l)\big] \\
&= \sum_{k=1}^{n} Q(k,n)\big[p_{k(n+1)}\eta_{(n+1)k} + p_{k(n+1)}(v_{n+1} - v_l)\big] \\
&= \sum_{k=1}^{n} Q(k,n) p_{k(n+1)}(v_{n+1} + \eta_{(n+1)k} - v_l).
\end{aligned}$$

Then $U^{n,n+1}(\varphi^\prec) - U^{n,n}(\varphi^\prec) = U^{n,n+1}(\varphi^\mu) - U^{n,n}(\varphi^\mu)$. By hypothesis, $U^{n,n}(\varphi^\prec) = U^{n,n}(\varphi^\mu)$, so $U^{n,n+1}(\varphi^\prec) = U^{n,n+1}(\varphi^\mu)$ as well.

Step 3: Increasing the number of agents. Suppose that $U^{n-1,n}(\varphi^\prec) = U^{n-1,n}(\varphi^\mu)$. Since our hypothesis applies to all problems with $n-1$ agents and n objects, we may assume that the new agent has the lowest priority under $\varphi^\prec$ and is endowed with a_n under φ^μ.

Under $\varphi^\prec$, the arrival of agent n has no effect on the allocations or ex-ante expected utilities of the original agents. Therefore, the difference in utilitarian welfare is the ex-ante expected utility of agent $n+1$. Also, agent n has no meaningful investigation decision. This is reflected by our conventions $p_{l(m+1)} = 0$ and $\eta_{(m+1)l} = 0$. Then by Proposition 4,

$$\begin{aligned}
U^{n,n}(\varphi^\prec) - U^{n-1,n}(\varphi^\prec) &= U^{n,n}_n(\sigma^\prec, \varphi^\prec) \\
&= \sum_{l=1}^{n} Q(l,k)\big[(1-p_{l(n+1)})v_l + p_{l(n+1)}(v_{n+1} + \eta_{(n+1)l})\big] \\
&= \sum_{l=1}^{n} Q(l,k) v_l.
\end{aligned}$$

Under φ^μ, the arrival of the new agent decreases each original agent's ex-ante expected utility. The difference arises because a_n may now be unavailable. In terms of Proposition 5, for each $k \in N$, the comparison between a_k and a_n moves from the final term into the second summation. The difference is

$$\begin{aligned}
&U^{n,n}_k(\sigma^\mu, \varphi^\mu) - U^{n-1,n}_k(\sigma^\mu, \varphi^\mu) \\
&\quad = \big[\big(Q(k,n-1) - Q(k,n)\big)\big[(1-p_{kn})(v_k + \eta_{kn}) + p_{kn}v_n\big] + Q(k,n)v_k\big] \\
&\qquad\qquad - Q(k,n-1)\left[(1-p_{kn})(v_k + \eta_{kn}) + p_{kn}v_n\right] \\
&\quad = -Q(k,n)\big[(1-p_{kn})(v_k + \eta_{kn}) + p_{kn}v_n\big] + Q(k,n)v_k \\
&\quad = -Q(k,n)\big[(1-p_{kn})\eta_{kn} + p_{kn}(v_n - v_k)\big].
\end{aligned}$$

By Lemma 2(2), $(1-p_{kn})\eta_{kn} = p_{kn}\eta_{nk}$. Substituting and summing over the original agents,

$$\sum_{k=1}^{n-1} U_k^{n,n}(\sigma^\mu, \varphi^\mu) - U_k^{n-1,n}(\sigma^\mu, \varphi^\mu) = \sum_{k=1}^{n-1} -Q(k,n)\big[(1-p_{kn})\eta_{kn} + p_{kn}(v_n - v_k)\big]$$
$$= \sum_{k=1}^{n-1} -Q(k,n)p_{kn}(v_n + \eta_{nk} - v_k).$$

Now agent n's ex-ante expected utility is

$$U_n^{n,n}(\sigma^\mu, \varphi^\mu) = \sum_{l=1}^{n-1} Q(l,n)\big[(1-p_{ln})v_l + p_{ln}(v_n + \eta_{nl})\big] + Q(n,n)v_n.$$

Combining results,

$$\begin{aligned}
&U^{n,n}(\varphi^\mu) - U^{n-1,n}(\varphi^\mu) \\
&\quad = U_n^{n,n}(\sigma^\mu, \varphi^\mu) + \sum_{k=1}^{n-1} U_k^{n,n}(\sigma^\mu, \varphi^\mu) - U_k^{n-1,n}(\sigma^\mu, \varphi^\mu) \\
&\quad = Q(n,n)v_n + \sum_{l=1}^{n-1} Q(l,n)\big[(1-p_{ln})v_l + p_{ln}(v_n + \eta_{nl})\big] \\
&\qquad\qquad - \sum_{k=1}^{n-1} Q(k,n)p_{kn}(v_n + \eta_{nk} - v_k) \\
&\quad = Q(n,n)v_n + \sum_{l=1}^{n-1} Q(l,n)\big[(1-p_{ln})v_l + p_{ln}(v_n + \eta_{nl}) - p_{ln}(v_n + \eta_{nl} - v_l)\big] \\
&\quad = Q(n,n)v_n + \sum_{l=1}^{n-1} Q(l,n)v_l \\
&\quad = \sum_{l=1}^{n} Q(l,n)v_l.
\end{aligned}$$

Therefore, $U^{n,n}(\varphi^{\prec}) = U^{n,n}(\varphi^\mu)$.

B Examples

Example 1. An intermediate agent who is better off under a priority rule. Let $n = m = 3$, $(v_1, v_2, v_3) \equiv (8, 6, 5)$, and $F \sim Unif[-4, 4]$. Then $p_{12} = \frac{2}{8}$, $p_{13} = \frac{1}{8}$, and $p_{23} = \frac{3}{8}$. Also, $\eta_{23} = 1.5$, $\eta_{21} = 3$, $\eta_{31} = 3.5$, and $\eta_{32} = 2.5$. Let $\sigma^{\prec}, \sigma^{\mu} \in A^N$ be equilibria under $\varphi^{\prec}$ and φ^{μ} respectively. By Corollary 1,

$$U_2(\sigma^{\prec}, \varphi^{\prec}) = \tfrac{6}{8} \cdot \left[\tfrac{5}{8} \cdot 6 + \tfrac{3}{8}(5 + 2.5)\right] + \tfrac{2}{8} \cdot \left[\tfrac{7}{8} \cdot 8 + \tfrac{1}{8}(5 + 3.5)\right] = \tfrac{1776}{256}.$$

Similarly, by Corollary 2,

$$U_2(\sigma^{\mu}, \varphi^{\mu}) = \tfrac{2}{8} \cdot \left[\tfrac{2}{8}(6 + 3) + \tfrac{6}{8} \cdot 8\right] + \tfrac{6}{8} \cdot \tfrac{5}{8} \cdot \left[\tfrac{5}{8}(6 + 1.5) + \tfrac{3}{8} \cdot 5\right] + \tfrac{6}{8} \cdot \tfrac{3}{8} \cdot 6 = \tfrac{1748}{256}.$$

Therefore, $U_2(\sigma^{\prec}, \varphi^{\prec}) > U_2(\sigma^{\mu}, \varphi^{\mu})$.

Example 2. An intermediate agent who is better off under a top trading cycles rule. Let $n = m = 3$, $(v_1, v_2, v_3) \equiv (8, 7, 5)$, and $F \sim Unif[-4, 4]$. Then $p_{12} = \frac{3}{8}$, $p_{13} = \frac{1}{8}$, and $p_{23} = \frac{2}{8}$. Also, $\eta_{23} = 1$, $\eta_{21} = 2.5$, $\eta_{31} = 3.5$, and $\eta_{32} = 3$. Let $\sigma^{\prec}, \sigma^{\mu} \in A^N$ be equilibria under $\varphi^{\prec}$ and φ^{μ} respectively. By Corollary 1,

$$U_2(\sigma^{\prec}, \varphi^{\prec}) = \tfrac{5}{8} \cdot \left[\tfrac{6}{8} \cdot 7 + \tfrac{2}{8}(5 + 3)\right] + \tfrac{3}{8} \cdot \left[\tfrac{7}{8} \cdot 8 + \tfrac{1}{8}(5 + 3.5)\right] = \tfrac{1934}{256}.$$

Similarly, by Corollary 2,

$$U_2(\sigma^{\mu}, \varphi^{\mu}) = \tfrac{3}{8} \cdot \left[\tfrac{3}{8}(7 + 2.5) + \tfrac{5}{8} \cdot 8\right] + \tfrac{5}{8} \cdot \tfrac{6}{8} \cdot \left[\tfrac{6}{8}(7 + 1) + \tfrac{2}{8} \cdot 5\right] + \tfrac{5}{8} \cdot \tfrac{2}{8} \cdot 7 = \tfrac{1972}{256}.$$

Therefore, $U_2(\sigma^{\prec}, \varphi^{\prec}) < U_2(\sigma^{\mu}, \varphi^{\mu})$.

Example 3. Full investigation surplus is achieved under $\boldsymbol{\varphi^{\prec}}$ and $\boldsymbol{\varphi^{\mu}}$. Let $n = m = 2$, $(v_1, v_2) \in \mathbb{R}^2$ with $v_1 - v_2 = 1$, and $\delta \in \mathbb{R}_{++}$ with $\delta > \frac{1}{2}$. Let F be such that $Pr(x = 2\delta) = Pr(x = -2\delta) = \frac{1}{2}$. Then $p_{12} = \frac{1}{2}$ and $\eta_{12} = \eta_{21} = \eta_0 = 2\delta$. By Corollaries 1 and 2,

$$U(\varphi^{\prec}) = U(\varphi^{\mu}) = v_1 + v_2 + p_{12}\eta_{21} = U_0(n) + \delta.$$

We claim that this is the maximum utilitarian welfare. Let $\sigma \equiv (a_1, a_2)$ and $\varepsilon_\sigma \in \mathbb{R}^2$. Without loss of generality, suppose $\sigma_1 = a_1$ and $\sigma_2 = a_2$. There are four cases. If $\varepsilon_{11} > 0$ and $\varepsilon_{22} > 0$, then the allocation (a_1, a_2) maximizes utilitarian welfare and this welfare is $(v_1 + \eta_0) + (v_2 + \eta_0) = U_0(n) + 2\eta_0$. Next, if $\varepsilon_{11} > 0$ and $\varepsilon_{22} > 0$, then the allocation (a_2, a_1) maximizes utilitarian welfare and this welfare is $v_2 + v_1 = U_0(n)$. Third, if $\varepsilon_{11} > 0$ and $\varepsilon_{22} < 0$, then the utilitarian welfare under (a_1, a_2) is $(v_1 + \eta_0) + (v_2 - \eta_0) = U_0$ and the utilitarian welfare under (a_2, a_1) is $v_2 + v_1 = U_0$. Similarly, if $\varepsilon_{11} < 0$ and $\varepsilon_{22} > 0$, then the the utilitarian welfare under either allocation is $U_0(n)$. As these cases are equally likely, the maximum utilitarian welfare is

$$\tfrac{1}{4}(U_0(n) + 2\eta_0) + \tfrac{3}{4} \cdot U_0(n) = U_0(n) + \delta.$$

Therefore, $\varphi^{\prec}$ and φ^{μ} achieve the full investigation surplus.

Example 4. Negligible fraction of investigation surplus is achieved under $\boldsymbol{\varphi^{\prec}}$ and $\boldsymbol{\varphi^{\mu}}$. Let $n = m = 2$, $(v_1, v_2) \in \mathbb{R}^2$ with $v_1 - v_2 = 1$, $\delta \in \mathbb{R}_{++}$ with $\delta > 1$, and $F \sim Unif[-\delta, \delta]$. Then $p_{12} = \frac{\delta-1}{2\delta}$, $\eta_{21} = \frac{1+\delta}{2}$, and $\mathbf{E}_F[\varepsilon_{11} + \varepsilon_{22} | \varepsilon_{11} + \varepsilon_{22} > 0] = \frac{2\delta}{3}$. By Corollaries 1 and 2,

$$U(\varphi^{\prec}) = U(\varphi^{\mu}) = v_1 + v_2 + p_{12}\eta_{21} = U_0(n) + \tfrac{\delta^2-1}{4\delta}.$$

We now consider a policy that achieves higher utilitarian welfare. Let $\sigma \equiv (a_1, a_2)$. For each $\varepsilon_\sigma \in \mathbb{R}^2$, the allocation is (a_1, a_2) if $\varepsilon_{11} + \varepsilon_{22} > 0$ and (a_2, a_1) if $\varepsilon_{11} + \varepsilon_{22} < 0$. Since each case is equally likely, the utilitarian welfare is

$$\tfrac{1}{2}\big(v_1 + v_2 + \mathbf{E}_F[\varepsilon_{11} + \varepsilon_{22} | \varepsilon_{11} + \varepsilon_{22} > 0]\big) + \tfrac{1}{2}(v_1 + v_2) = U_0(n) + \tfrac{\delta}{3}.$$

As $\delta \to 1$, $\frac{\delta^2-1}{4\delta} \to 0$ and $\frac{\delta}{3} \to \frac{1}{3}$. Therefore, $U(\varphi^{\prec}) = U(\varphi^{\mu}) \to U_0(n)$ whereas the utilitarian maximum is bounded below by $U_0(n) + \frac{1}{3}$. For δ close to 1, $\varphi^{\prec}$ and φ^{μ} achieve a negligible fraction of the investigation surplus.

Example 5. If *ex-ante efficiency* is strengthened to allow conditioning on $\boldsymbol{\varepsilon_\sigma}$ directly, then $\boldsymbol{\varphi^{\mu}}$ is no longer *ex-ante efficient* even when $\mathbf{n = m = 2}$. Let $n = m = 2$, $(v_1, v_2) \equiv (3, 2)$, and $F \sim Unif[-2, 2]$. Then $p_{12} = \frac{1}{4}$, $\eta_{12} = 0.5$, and $\eta_{21} = 1.5$. Let $\sigma \in A^N$ be an equilibrium under φ^{μ} and, without loss of generality, suppose $\sigma = (a_1, a_2)$. We construct an outcome function that yields an ex-ante Pareto improvement with σ. Let

$$\begin{aligned} E_1 &\equiv \big\{(\varepsilon_{11}, \varepsilon_{22}) : \varepsilon_{11} + \varepsilon_{22} < 0,\ -2 < \varepsilon_{11} < 1,\ \text{and } 1 < e_{22} < 2\big\} \text{ and} \\ E_2 &\equiv \big\{(\varepsilon_{11}, \varepsilon_{22}) : \varepsilon_{11} + \varepsilon_{22} < 0,\ -1 < \varepsilon_{11} < 0,\ \text{and } 0 < e_{22} < 1\big\}. \end{aligned}$$

Now define $f\colon (A \times \mathbb{R})^N \to X$ by

$$f(\varepsilon_\sigma) \equiv \begin{cases} (a_2, a_1) & \text{if } \varepsilon_\sigma \in E_1 \cup E_2 \\ \varphi^{\mu}(P(\varepsilon_\sigma)) & \text{if } \varepsilon_\sigma \notin E_1 \cup E_2 \end{cases}.$$

Let $\varepsilon_\sigma \in \mathbb{R}^N$. If $\varepsilon_\sigma \notin E_1 \cup E_2$, then (σ, f) and (σ, φ^{μ}) coincide, so suppose $\varepsilon_\sigma \notin E_1 \cup E_2$. Under (σ, f), agent 1's conditional expected utility is $v_2 = 2$ and agent 2's conditional expected utility is $v_1 = 3$.

We now compute the conditional expected utilities under (σ, φ^{μ}). If $\varepsilon_\sigma \in E_1$, then $a_2\ P_2(\varepsilon_\sigma)\ a_1$. Similarly, if $\varepsilon_\sigma \in E_2$, $a_1\ P_1(\varepsilon_\sigma)\ a_2$. Therefore, for each $\varepsilon_\sigma \in E_1 \cup E_2$, $\varphi^{\mu}(P(\varepsilon)) = (a_1, a_2)$. By independence, $Pr(\varepsilon_\sigma \in E_1) = Pr(\varepsilon_\sigma \in E_2) = \frac{1}{2} \cdot p_{12} \cdot p_{12} = \frac{1}{32}$.

Also, the joint density of $(\varepsilon_{11}, \varepsilon_{22})$ on $[-2,2] \times [-2,2]$ is $f(x) = \frac{1}{16}$. Computing,

$$\begin{aligned}
\mathbf{E}_F[\varepsilon_{11}|\varepsilon_\sigma \in E_1] &= \frac{\int_{-2}^{-1}\int_1^{-x} \frac{x}{16}\,dy\,dx}{Pr(\varepsilon_\sigma \in E_1)} = \frac{-5/96}{1/32} = -\frac{5}{3}, \\
\mathbf{E}_F[\varepsilon_{22}|\varepsilon_\sigma \in E_1] &= \frac{\int_{-2}^{-1}\int_1^{-x} \frac{y}{16}\,dy\,dx}{Pr(\varepsilon_\sigma \in E_1)} = \frac{1/24}{1/32} = \frac{4}{3}, \\
\mathbf{E}_F[\varepsilon_{11}|\varepsilon_\sigma \in E_2] &= \frac{\int_{-1}^{0}\int_1^{-x} \frac{x}{16}\,dy\,dx}{Pr(\varepsilon_\sigma \in E_2)} = \frac{1/24}{1/32} = -\frac{2}{3}, \text{ and} \\
\mathbf{E}_F[\varepsilon_{22}|\varepsilon_\sigma \in E_2] &= \frac{\int_{-1}^{0}\int_1^{-x} \frac{y}{16}\,dy\,dx}{Pr(\varepsilon_\sigma \in E_2)} = \frac{-5/96}{1/32} = \frac{1}{3}.
\end{aligned}$$

Now $Pr(\varepsilon_\sigma \in E_1 | \varepsilon_\sigma \in E_1 \cup E_2) = Pr(\varepsilon_\sigma \in E_1 | \varepsilon_\sigma \in E_1 \cup E_2) = \frac{1}{2}$, so

$$\begin{aligned}
U_1(\sigma, \varphi^\mu | E_1 \cup E_2) &= \tfrac{1}{2}\,\mathbf{E}_F[v_1 + \varepsilon_{11}|\varepsilon_\sigma \in E_1] + \tfrac{1}{2}\,\mathbf{E}_F[v_1 + \varepsilon_{11}|\varepsilon_\sigma \in E_2] \\
&= \tfrac{1}{2}\big(3 - \tfrac{5}{3}\big) + \tfrac{1}{2}\big(3 - \tfrac{2}{3}\big) \\
&= \tfrac{11}{6} \qquad \text{and} \\
U_2(\sigma, \varphi^\mu | E_1 \cup E_2) &= \tfrac{1}{2}\,\mathbf{E}_F[v_2 + \varepsilon_{22}|\varepsilon_\sigma \in E_2] + \tfrac{1}{2}\,\mathbf{E}_F[v_2 + \varepsilon_{22}|\varepsilon_\sigma \in E_2] \\
&= \tfrac{1}{2}\big(2 + \tfrac{4}{3}\big) + \tfrac{1}{2}\big(2 + \tfrac{1}{3}\big) \\
&= \tfrac{17}{6}.
\end{aligned}$$

Since $\frac{11}{6} < 2$ and $\frac{17}{6} < 3$, both agents are better off under (σ, f).

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