Pure Strategy Markov Equilibrium in some classes of Stochastic Games and Applications

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Abstract: We examine classes of stochastic games that are generated by economics models and investigate conditions under which a Stationary Markov perfect equilibrium in pure strategies can be found. We study two broad classes of stochastic games, both with a finite state space. One class of stochastic games has a low payoff state and the transition probability satisfies a convex-concave condition, with a convexity condition holding for the low-payoff state. The other class of stochastic games discussed here is one in which the single-period payoff function is increasing in the state variable with transition probabilities that satisfy a first-order stochastic dominance condition.

Key Words: Stochastic Games, Markov Perfect Equilibrium, Stationary Markov Equilibrium, Subgame Perfect Equilibrium, R& D games, Sales competition, Resource extraction games.

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1 Introduction

Many stochastic dynamic economic models are stochastic games in which the actions of the agents affect the state variable as well as the current outcome and the payoff of the agents. Examples of such stochastic games are numerous, for instance R & D games as well as resource extraction games are important examples of economic models that are stochastic games. Also almost all dynamic models of imperfect competition and oligopolies fit into the general framework of stochastic games. There is now a growing literature that deals with the question of how to solve these games. An attractive line of work has focused on conditions under which one would be able to find stationary Markov-perfect equilibrium in pure strategies. Markov-perfect strategies have drawn a lot of attention as these strategies depend only on the current state, and therefore, are relatively simple to use and compute. Hence, results that provide conditions on the existence of stationary Markov-perfect strategies would be very useful in understanding models of dynamic competition in markets, resource extraction games, R & D games, competition in sales, dynamic provision of public goods, games of capital accumulation and many others.

The question of the existence of a Markov-perfect equilibrium in pure strategies has been raised in many different contexts. In Doralzeski and Satherthwaite [6], Amir [1], Sundaram ([15] and Majumdar and Sundaram [9] conditions are given under which one can find stationary Markov Perfect equilibrium in pure strategies for specific economic models. The models, however, differ from each other in important ways and thus the results obtained for one set of models do not necessarily provide answers for the other models. In [6] Markov-perfect equilibrium strategies ar proposed to solve the dynamic games of market competition and conditions are investigated under which a stationary Markov-perfect equilibrium in pure strategies can be found. This is also the issue raised in Ericson and Pakes [8], Pakes and McGuire [12], [13], Maskin and Tirole [10]. In [1] the focus is on games of capital accumulation and conditions are provided on the transition probabilities under which a Markov-perfect equilibrium exists, The work of Sundaram [15] and Majundar and Sundaram [9] look at games of resource extraction which are dynamic games, and give conditions for the existence of stationary Markov-perfect strategies. They show that if the players have the same payoff function in addition to conditions that hold in the case of resource extraction games, then there is a symmetric equilibrium in pure strategies. In the results presented here the focus is on finding conditions under which a stationary Markov-perfect equilibrium in pure strategies can be found for classes of stochastic that are generated by economic models like dynamic competition in markets, dynamic games of product innovation and R & D games as well as resource extraction games.

There is also a growing literature that deals with the existence of equilibrium under general conditions on stochastic games in mixed strategies. Thus most recently Duggan [7] establishes the existence of stationary Markov-perfect equilibrium when the state space can be split into the standard part and a non-atomic payoff irrelevant part. Mertens and Parthasarathy [11] look for stationary equilibrium in strategies that depend on both the last period's state and the current period's state. The result in Chakrabarti [5] shows that if the transition probabilities are absolutely continuous with respect to a fixed non-atomic probability measure then there are stationary equilibrium strategies that depend on the last period and the current period's state; that is, there exists stationary semi-Markov strategies. This literature, however, focuses on mixed strategies and looks for general conditions when the state space is uncountable. This literature thus does not provide results for the existence of pure strategy equilibrium

Here we focus on the question of the existence of stationary Markov-perfect equilibrium in pure strategies when the state space is finite. Thus the results here are most closely connected to questions that have been raised in the strand of the literature that deals with the existence of pure strategies. Since the strategies are no longer mixed, in order to obtain the convexity of the best responses of the players, one has to look for conditions on the structure of the payoff functions. In static or one-shot games the condition on the payoff function that leads to convex best responses is quasi-concavity. In the case of stochastic games, even if the single-period payoff functions are concave, the continuation payoff which gives the expected future payoff of a player may not be. Additional conditions on the transition probabilities are required, for instance conditions on the transition probabilities that would make the relevant best responses convex-valued. We examine two classes of stochastic games that have single-period utility functions that are concave in the actions of the player. The first class of stochastic games that we analyze has a low payoff state for each player and the transition probability satisfies a convexity condition for the low payoff state and a concavity condition for the other states. The second class of stochastic games that we discuss here have single-period payoffs that are increasing in the state variable, and the transition probability satisfies a first-order stochastic dominance condition as well as a convexity-concavity condition. We show that for both classes of stochastic games there is a stationary Markov Perfect equilibrium in pure strategies.

We provide several examples of interesting economic models to which these apply. The first class of stochastic games can be applied to dynamic market competition as in Ericsson and Pakes [8] and Doralzeski and Satherthwaite [6] and other models of dynamic competition as well as to games of resource extraction. The second class of models also have connection to games of dynamic competition in which the payoffs of a firm can be increasing in the state variable and show that the results can be applied to certain R & D games and to models of sales competition. We describe the overall framework in section 2, in section 3 we describe the class of stochastic games with the low payoff state, in section 4 we describe the class of stochastic games with payoffs that are nondecreasing in the state variable. In section 5 we provide the main existence result that applies to both classes of stochastic games and in section 6 we conclude.

2 Description of the Basic Model

A discrete-time stochastic dynamic game with n players and a finite state space is given by $[S, (A_i, u_i)_{i=1}^n, q]$, where

(i) $S = \{s_0, s_1, \cdots, s_K\}$ is the *State Space*. S is finite.

(ii) A_i is the action space of player i. It is a *compact* and a *convex* subset of an Euclidean Space which is invariant over time. $A = A_1 \times A_2 \times \cdots \times A_n$ i.e. A is the space of all action n-tuples.

(iii) $u_i : S \times A \to \mathbb{R}$ is the single period payoff function of player i. It is *continuous* on A, and *concave* on A_i.

(iv) The transition probability $q : S \times A \to \mathcal{P}(S)$ is continuous on A, that is the probability density $q(s_j|s, a)$ of the state s_j for $j = 0, \cdots, K$ is a continuous function of a.

The infinite-horizon stochastic game is one in which each player i chooses an action in the action set A_i of player i and the action chosen can depend on the entire past history of realized states and actions. A *strategy* of a player is a *plan* consisting of action choices in each period as a function of the past history. A *Markov Strategy* is one in which the choice of action in any period depends only on the current state and not on the entire past history.

Definition 1 *A* **Markov** *strategy of a player i is a sequence of functions $\{f_{it}\}_{t=1}^{\infty}$ such that $f_{it} : S \to A_i$. A* **stationary Markov** *strategy is a Markov strategy such that $f_{it} = f_{i(t+1)}$ for all $t \geq 1$.*

We should note here that the strategy of a player is a *pure* strategy as it selects an action in A_i for each player i and does not involve any randomization. The payoff of a player in the infinite-horizon stochastic game is the discounted sum of the single-period payoffs. Thus, the payoff of player i is given by

$$u_i^{\infty}(s) = \sum_{t=1}^{\infty} \delta^{t-1} u_i(s_t, a_t)$$

when the history of the game is $\{s_t, a_t\}_{t=1}^{\infty}$ and the initial state is s. Therefore, the payoff a player i when the Markov Strategy combination $f = (f_1, \cdots, f_n)$ is played is

$$u_i^{\infty}(f)(s) = \sum_{t=1}^{\infty} \delta^{t-1} u_i(s_t, f_t(s_t)) \tag{1}$$

where $f_t(s_t)$ is the action n-tuple in period t and the initial state is s.

Let f_{-i} denote the Markov strategy combination of the players other than player i. A Markov perfect equilibrium[1] is then defined as follows.

Definition 2 *A Markov strategy combination $f^{\star}$ is a* **Markov Perfect Equilibrium** *if for all $t \geq 1$ and for all $s_j \in S$ we have*

$$u_i^{\infty}(f^{\star})(s_j) \geq u_i^{\infty}(f_i, f_{-i}^{\star})(s_j). \tag{2}$$

A stationary Markov strategy combination $f^{\star}$ is a **stationary Markov Perfect Equilibrium** *if (2) holds for the stationary strategy combination $f^{\star}$.*

Here we show that the stochastic game has a stationary Markov Perfect equilibrium. We prove this by first observing that given a stationary strategy combination of

[1]For a more detailed discussion of Markov Perfect equilibrium one can refer to Chakrabarti [5].

the other players, a player i has to solve a stochastic dynamic programming problem to determine the best response to the stationary strategies of the other players. The best responses of a player are thus found from solving the stochastic dynamic programming problem for the player. One can then use a fixed point argument to find an equilibrium strategy. The result is then obtained by showing that this gives the required stationary equilibrium strategy. Given a stationary Markov Strategy combination f_{-i} of the players other than i, the discounted sum of payoffs of player i is given by

$$\sum_{t=1}^{\infty} \delta^{t-1} u_i(s_t, a_{it}, f_{-i}(s_t)).$$

Hence, given a stationary strategy combination f_{-i} of the players other than i, the payoff of player i can be written as

$$\sum_{t=1}^{\infty} \delta^{t-1} u_i^{f_{-i}}(s_t, a_{it}), \tag{3}$$

where the single-period payoffs are indexed by f_{-i}. The following result then follows from standard stochastic dynamic programming results. see for instance Stokey and Lucas [16] and Bhattacharya and Majumdar [3].

Lemma 1 *For every stationary Markov Strategy f_{-i} of the players other than i, there is a stationary Markov strategy of player i that maximizes the payoff of player i.*

Proof: Given a stationary Markov strategy f_{-i}, player $i's$ problem is to find a strategy that maximizes

$$\sum_{t=1}^{\infty} \delta^{t-1} u_i^{f_{-i}}(s_t, a_{it})(s_j)$$

for each $s_j \in S$. This as we know from stochastic dynamic programming implies that in each period, a_{it} solves

$$\max_{a_{it} \in A_i} [u_i^{f_{-i}}(s_j, a_{it}) + \delta \int_S g_i(s') q(ds' | s_j, a_{it}, f_{-i}(s_j))]$$

where $g_i(s')$ is the expected future payoff of player i conditional on some future choice of strategy by player i given the strategy f_{-i} of the players other than i.

A stationary solution to the above problem can be found if the following problem has a solution.

$$g_i(s_j) = \max_{a_{it} \in A_i} [u_i^{f_{-i}}(s_j, a_{it}) + \delta \int_S g_i(s') q^{f_{-i}}(ds' | s_j, a_{it})]. \tag{4}$$

As S is finite, $A_i's$ are compact and the transition probability $q(.|s_j, a_{it}, f_{-i}(s_j)) = q^{f_{-i}}(ds'|s_j, a_{it}) = q(.|s_j, a_{it}, f_{-i}(s_j))$ is continuous in a_i, results from stochastic dynamic programming show that the above problem has a solution. This then shows that there is a stationary Markov strategy that solves (4) above. ■

In fact, the result indicates that for a stationary Markov strategy of player i, there is an optimal value function $g_i^\star(f_{-i}) : S \to \mathbb{R}$ that solves (4). The optimal stationary strategy of player i, namely $f_i^\star(f_{-i})$, is then the one that solves

$$\max_{a_i \in A_i}[u_i^{f_{-i}}(s_j, a_i) + \delta \int_S g_i^\star(f_{-i})(s')q(ds'|s_j, a_i)]$$

for each $s_j \in S$.

3 Stochastic Games with a Low-payoff State

In this section we discuss the class of stochastic games in which every player has a low payoff state. The low payoff state for a player is the state in which the player's payoff is lower than in any other state. We show that if the stochastic game has such a low-payoff state for each player then under some fairly reasonable conditions the stochastic game will have a Markov Perfect equilibrium in Pure Strategies. The probability density of the low-payoff state is convex while the probability densities of the other states are concave in the actions of the player. This together with a stochastic dominance condition on the transition probabilities then imply that the relevant payoff function of a player at each period is concave in the actions of the player.

(V) There is a state s_0 such that for any pair of actions a_{-i} and a'_{-i} in A_{-i} of players other than i, for any a_i

$$(i)\ u_i(s_0, a_i, a_{-i}) \leq u_i(s_j, a_i, a'_{-i}) \text{ for all } s_j \neq s_0, \text{ and}$$
$$(ii)\ q(s_k|s_j, a_i, a_{-i}) \geq q(s_k|s_0, a_i, a'_{-i}) \text{ for } s_k, s_j \neq s_0.$$

(VI) For each i, for $j = 1, \cdots, K$ the probability densities of the state $s_k \neq s_0$

$$q(s_k|s, a_i, a_{-i}) \text{ is a concave function of } a_i \text{ for all } s,$$

and

$$q(s_0|s, a_i, a_{-i}) \text{ is a convex function of } a_i \text{ for all } s.$$

Condition (v) states that the payoff of a player in state s_0 is lower than the payoff in any other state irrespective of the actions taken by the player. The state s_0 is also such that the probability to transition to any other higher payoff state is lower if the current state is s_0, than if the current state was some other state. Condition (VI) is a concavity-convexity condition consistent with the fact that the densities across states sum to 1.

Remark: It should be pointed out that the low-payoff state could be different for different players. For the results that follow it is enough that each player has a *low-payoff* state and that the transition probabilities satisfy condition (VI) for each player's low-payoff state.

We show in the following examples that such a "low-payoff" state occur quite naturally in many economic models.

Example 1 Dynamic Market Competition

Consider a market in which a number of firms produce differentiated goods. The products of the firms are substitutes. There are $K+1$ possible pay-off relevant states for each firm and in each period a firm can choose an action in its action set $A_i = [0, \bar{I}_i]$ where $I_i \in [0, \bar{I}_i]$ indicates the level of investment undertaken by the firm.[2] The single-period profit function of firm i is given by

$$\pi_i(s_k, I_i, a_{-i}) = r_i(s_k) - c_i(I_i)$$

where $r_i(s_k)$ is the profit of the firm in state s_k and $c_i(I_i)$ is the cost of investing in demand generation for the next period. The transition probability $q(.|s_k, I_i, a_{-i})$ is a function of the actions chosen in the current period and the current state, and determines the probability densities over the states in the next period. $q(s|s_k, I_i, a_{-i})$ is then the probability of state s in the next period when s_k, I_i, a_{-i} are the actions and the state in the current period. For each firm there is a state s_{0i} in which the state of the demand for the firm's product is so low that

$$r_i(s_{0i}) \leq 0.$$

Therefore

$$\begin{aligned}\pi_i(s_{0i}, I_i, a_{-i}) &= r_i(s_{0i} - c_i(I_i) \leq r_i(s_k) - c_i(I_i) \\ &\leq \pi_i(s_k, I_i, a'_{-i}).\end{aligned}$$

[2]This has many of the features of the models analyzed by Pakes and McGuire, see for example [12] and Doraszelski and Satherthwaite [6].

for all I_i and a_{-i}, a'_{-i} and the firm has to invest substantial amounts in demand generating strategies to increase the likelihood of moving to a higher payoff state. In this case one has

$$
\begin{aligned}
&(i) \qquad \pi_i(s_{0i}, 0, a_{-i}) \leq 0 = \pi_i(s_k, \bar{I}_i, a'_{-i}) \leq \pi_i(s_k, I_i, a'_{-i}) \text{ for any pair } a_{-i}, a'_{-i}, \\
&(ii) \qquad q(s_j|s_k, 0, a_{-i}) \geq q(s_j|s_{0i}, 0, a'_{-i}) \text{ for all } s_j, s_k \neq s_{0i}.
\end{aligned}
$$

Condition (i) is not hard to see, it states that the payoff from the low-payoff state is lower than the payoff from one of the higher payoff states even if the cost of the investment is kept at zero. Condition (ii) states that if the current state is the *low payoff state*, then the transition probability of reaching a higher payoff state is lower than if the current state is one of the higher payoff states when the level of investment is low. We will also impose the following condition.

$$
(iii) \frac{\partial q(s_j|s_k, I_i, a_{-i})}{\partial I_i} \geq \frac{\partial q(s_j|s_{0i}, I_i, a'_{-i})}{\partial I_i} \geq 0 \text{ for all } s_j, s_k \neq s_{0i}.
$$

This fairly natural condition states that the probability of a higher payoff state increases with current investment, and if the current state is a higher payoff state, then this probability increases faster than if the current state is a *low payoff* state. Note that it is not the case that a firm's chances of moving to a higher demand state from the low demand state is zero, in fact it could be fairly high, especially if the firm invests in demand generating strategies.[3]

Proposition 1 *If (i), (ii) and (iii) hold then conditions (V) and (VI) are satisfied.*

Proof: Condition (i) gives (V).

Conditions (ii) and (iii) together imply that

$$
q(s_j|s_k, I_i, a_{-i}) \geq q(s_j|s_{0i}, I_i, a'_{-i}) \text{ for all } s_j, s_k \neq s_{0i}
$$

for any $I_i \in [0, \bar{I}_i]$. This gives condition (VI). ■

Example 2 *Dynamic Market Competition with random marginal costs*

[3]Firms can be in such low payoff states. For instance Blackberry posted huge losses as a result of a 54 percent drop in revenue in 2013, see [14].

This example also fits into the general category of dynamic market competition but is different in many aspects from the model of the previous example. This is a market in which firms compete in producing and selling a good over time, but now the product sold by the firms are identical. The demand in each period is subject to a random external shock and in each period is drawn from an independent and identical distribution that is not influenced by the actions of the firms in any period. The marginal costs of the firms are randomly drawn and can take the values $c_0 > c_1 > c_2 > \cdots > c_L$. The high marginal cost c_0 is such that when the price is equal to c_0 the quantity demanded is zero. As the products are identical, in each period, given the demand and the marginal costs of the firms, the firms play a Cournot quantity setting game. Each firm also decides how much to invest in cost-saving strategies and the investment decision of a firm i is in $[0, \bar{I}_i(s_k)]$, so that the maximum level of investment possible in any period depends on the current state. The single-period payoff of a firm is given by

$$u_i(s_k, I_i) = \pi_i(s_k) - c_i(I_i).$$

where $\pi_i(s_k)$ can be thought to be the profit of the firm when the state of the market is given by s_k. The profit $\pi_i(s_k)$ is the profit of the firm from the Cournot equilibrium when the state is s_k. The state s_k itself is determined by the state of the market demand and the realized marginal costs of the firms. We will denote the state for firm i to be s_{0i} when the firm draws the marginal cost c_0. When a firm draws marginal cost c_0, then

$$\pi_i(s_{0i}) \leq 0.$$

We now assume that

$$q(s_j | s_k, I_i, \bar{I}_{-i}(s_k)) \geq q(s_j | s_{0i}, I_i, 0) \text{ for all } s_k, s_j \neq s_{0i}. \tag{5}$$

Condition (i) implies that when probability of reaching a state $s_j \not{s}_{0i}$ when the current state is $s_k \neq s_{0i}$ is at least as high as when the current state is s_{0i}, even if the other firms have not invested any amount in cost saving strategies. Again it is worth noting that condition (i) is only a condition on the probability of reaching a particular state and the probability of reaching state $s_j \neq s_{0i}$ can be positive and relatively high even if the current state is s_{0i}, especially if the firm invests in cost-saving strategies in the current period. If transition probabilities are now increasing in cost-saving investments and decreasing in the cost-saving investments of the other firms, we have the following result

Proposition 2 *If* $q(s_j|s_k, I_i, I_{-i})$ *is increasing in* I_i *with*

$$\frac{\partial q(s_j|s_k, I_i, I_{-i})}{\partial I_i} \geq \frac{\partial q(s_j|s_{0i}, I_i, I_{-i})}{\partial I_i}$$

for all $s_j, s_k \neq s_{0i}$*, and decreasing in* I_{-i}*, then both (V) and (VI) hold.*

Proof: From the assumptions about the single-period profit functions we have

$$u_i(s_{0i}, I_i, a_{-i}) = \pi_i(s_{0i}) - c_i(I_i) \leq \pi_i(s_k) - c_i(I_i) = u_i(s_k, I_i, a'_{-i}) \tag{6}$$

for all I_i and a_{-i}, a'_{-i}. This gives (V).
Condition (VI) follows quite quickly from (5) and the above conditions. ∎

Example 3 *Resource extraction Games*

Consider the game in which players are engaged in extracting a common property resource. There are a finite number of possible states that indicate the amount of the resource available at the beginning of the period. The payoff of a player is given by

$$u_i(s_k, c_i) = r_i(c_i) - \alpha_i(s_k, c_i)$$

where $r_i(c_i)$ is the benefit derived by player i from consuming $c_i > 0$ amount of the resource. $\alpha_i(s_k, c_i)$ is the cost of extracting c_i amount of the resource by player i when the state is s_k. The amount of the resource that player i can extract lies in $[0, \bar{c}_i(s_k)]$ when the aggregate stock of the resource is s_k[4]. Therefore, when s_k is the aggregate state, $c_i \in [0, \bar{c}_i(s_k)]$. Let s_0 be the state in which the stock of the resource is below the level at which the agents can extract the resource profitably so that $\alpha_i(s_0, c_i) > \alpha_i(s_k, c_i)$ for any $c_i > 0$. Hence, for any pair (c_{-i}, c'_{-i}) and any $c_i > 0$ we have

$$u_i(s_0, c_i, c_{-i}) = r_i(c_i) - \alpha_i(s_0, c_i) \leq r_i(c_i) - \alpha_i(s_k, c_i) = u_i(s_k, c_i, c'_{-i}). \tag{7}$$

We assume that

$$q(s_j|s_k, 0, \bar{c}_{-i}(s_j)) \geq q(s_j|s_0, 0, 0) \text{ for all } s_j, s_k \neq s_0. \tag{8}$$

[4]This is therefore a situation where each player can independently access some of the aggregate resource, and the maximum amount that player i can access when the state is s_k is $\bar{c}_i(s_k)$. Thus, if one considers the amount of fish available to a country, then the amount $\bar{c}_i(s_k)$ is the amount of fish in the fishing grounds to which the country has exclusive rights. The choice of all the players then affects the aggregate stock in the next period, although the action of player i might have a greater impact on $\bar{c}_i$.

This thus implies that if the stock of the resource is given by the state s_0, then the probability of reaching a state in which the stock of the resource is higher and is given by the state s_j, is lower than if the current stock of the resource is higher and the state is s_k. This is true even if the other players extract the maximum possible amount available of the resource $\bar{c}_{-i}(s_k)$ available to them in state s_k.[5]

We now assume that for any c_i and c_{-i}, we have

$$\frac{\partial q(s_j|s_0, c_i, c_{-i})}{\partial c_i} \leq \frac{\partial q(s_j|s_k, c_i, c_{-i})}{\partial c_i} \leq 0, \tag{9}$$

and that

$$\frac{\partial q(s_j|s_0, c_i, c_{-i})}{\partial c_{-i}} \leq \frac{\partial q(s_j|s_k, c_i, c_{-i})}{\partial c_{-i}} \leq 0. \tag{10}$$

The conditions in (9) and (10) imply that the probability of reaching a normal state declines more slowly if the current state is a normal state than if the current state is s_0. This leads to the following observation

Proposition 3 *For any pair* (c_{-i}, c'_{-i}) *and any* c_i

$$q(s_j|s_k, c_i, c'_{-i}) \geq q(s_j|s_0, c_i, c_{-i}) \text{ for all } s_j, s_k \neq s_0.$$

Proof: From (8) we have

$$q(s_j|s_k, 0, \bar{c}_{-i}(s_j)) \geq q(s_j|s_0, 0, 0)$$

and applying (9) to this we get for any c_i

$$q(s_j|s_k, c_i, \bar{c}_{-i}(s_j)) \geq q(s_j|s_0, c_i, 0).$$

From (10) we now have that

$$q(s_j|s_k, c_i, c_{-i}) \geq q(s_j|s_0, c_i, 0) \geq q(s_j|s_0, c_i, c'_{-i}).$$

This concludes the proof. ■

[5] If we thus use the example of countries with exclusive fishing grounds, this condition implies that once the stock of the resource is given by the state s_0, the probability of the stock being replenished and reaching normal levels is lower than if the state is a normal state. This holds even if there is intensive fishing by the other countries in their own exclusive fishing zones.

From (7) we note that condition (V) is satisfied for the resource extraction games, and proposition 3 shows that condition (VI) also holds. Hence, both conditions (V) and (VI) hold for the resource extraction game. ∎

The examples show that in large classes of stochastic games that have applications in economics, conditions (V) and (VI) hold under quite reasonably and in addition a low payoff state seems to occur quite naturally. In the case of the dynamic market games as in Pakes and McGuire [1994], [2001], Ericson and Pakes [1995] and Doraszelski and Satherthwaite [6], the low-payoff state $s_{)i}$ is the one in which a firm is inactive.

Let $f_i^\star(f_{-i})$ denote an optimal stationary Markov Strategy of player i given the stationary strategy of the players other than i and $g_i^\star(f_{-i}) : S \to \mathbb{R}$ denote the value function of player i given the stationary Markov strategies of the players other than i. We now show that the value function of player i for state s_0 never exceeds the amount of the value function for any other state.

Lemma 2 *Given any stationary strategy f_{-i} of the players other than i, when conditions (V) and (VI) hold the value function of player i satisfies $g_i^\star(f_{-i})(s_0) \leq g_i^\star(f_{-i})(s_k)$ for any $s_k \neq s_0$.*

Proof: We fix f_{-i} and denote the optimal value function of player i by $g_i^\star(f_{-i})$. We first note that if $M(S)$ denotes the set of bounded functions from S to $\mathbb{R}$, then $g_i^\star(f_{-i})$ is the fixed point of the operator $U : M(S) \to M(S)$, where the operator U is defined as

$$U(g_i(f_{-i}))(s_k) = \max_{a_i}\{u_i(s_k, a_i) + \delta[\sum_{\ell=1}^{K} v_i'(s_\ell)q'(s_\ell|s_k, a_i)]\}.$$

From results in Dynamic Programming we know that this operator is a contraction mapping and thus has a fixed point. The fixed point of the operator U is the value function $g_i^\star(f_{-i})$.

We now claim that the operator U maps value functions v_i for which $v_i(s_0) \leq v_i(s_k)$ for $s_k \neq s_0$, to value functions that also satisfy that condition. Given a value function $v_i : S \to \mathbb{R}$ such that $v_i(s_0) \leq v_i(s_k)$ for $s_k \neq s_0$, we have

$$\begin{aligned} U(v_i)(s_0) &= \max_{a \in A_i}[u_i(s_0, a, f_{-i}(s_0)) + \delta \sum_s v_i(s)q(s|s_0, a, f_{-i}(s_0))] \\ &= u_i(s_0, a_0^\star, f_{-i}(s_0)) + \delta \sum_s v_i(s)q(s|s_0, a_o^\star, f_{-i}(s_0)) \\ &\leq u_i(s_k, a_0^\star, f_{-i}(s_k)) + \delta \sum_s v_i(s)q(s|s_k, a_0^\star, f_{-i}(s_k)) \\ &\leq U(v_i)(s_k), \end{aligned} \tag{11}$$

where the third inequality in (11) follows because of conditions (V) and (VI). Clearly,

$$U^2(v_i)(s_k) \geq U^2(v_i)(s_0)$$

for all $s_k \neq s_0$ and in general

$$U^n(v_i)(s_k) \geq U^n(v_i)(s_0)$$

for any $s_k \neq s_0$.

We thus have for any $s_k \neq s_0$

$$g_i^\star(f_{-i})(s_k) = \lim_{n\to\infty} U^n(v_i)(s_k) \geq \lim_{n\to\infty} U^n(v_i)(s_0) = g_i^\star(f_{-i})(s_0). \tag{12}$$

This concludes the proof. ∎

We now show that the set of optimal Markov Stationary strategies of a player i is convex. We do this by showing that player $i's$ payoff function is concave both in the single-period payoff as well as in the optimal expected future payoff.

Lemma 3 *The set of optimal stationary Markov strategies of player i, given the stationary Markov strategy f_{-i} of players other than i, is convex.*

Proof: We first note that in any period t, player i will choose $a_{it} \in A_i$ such that

$$\max_{a_{it}\in A_i} [u_i(s_t, a_{it}, f_{-i}(s_t)) + \delta \int_S g_i^\star(f_{-i})(s')q(ds'|s_j, a_{it}, f_{-i}(s_t))] \tag{13}$$

Thus to prove the claim it is sufficient to show that the payoff function given above is concave in a_{it}. By condition (iii)) of the basic model we have that $u_i(s, a_{it}, a_{-i})$ is concave in a_i. Thus we need to show that

$$\sum_s g_i^\star(f_{-i})(s)q(s|s_t, a_{it}, f_{-i}(s_t))$$

is concave in a_{it}.

We show this by first observing that for all $s_k \neq s_0$, by condition (vi),

$q(s_k|s, a, f_{-i}(s))$ is concave in a and $q(s_0|s, a, f_{-i}(s)$ is convex in a.

Therefore,

$$\sum_{s_k\neq s_0} q(s_k|s, a, f_{-i}(s)) = -q(s_0|s, a, f_{-i}(s)) \text{ is concave in } a. \tag{14}$$

Using (8) we get

$$\begin{aligned}
&\sum_{s} g_i^\star(f_{-i})(s)q(s|s,a_i,f_{-i}(s)) \\
= &\sum_{s_k\neq s_0} g_i^\star(f_{-i})(s_k)q(s_k|s,a,f_{-i}(s) + g_i^\star(f_{-i})(s_0)[1-\sum_{s_k\neq s_0} q(s_k|s,a,f_{-i}(s)] \\
= &\sum_{s_k\neq s_0} [g_i^\star(f_{-i})(s_k) - g_i^\star(f_{-i})(s_0)]q(s_k|s,a,f_{-i}(s)) + g_i^\star(f_{-i})(s_0). \qquad (15)
\end{aligned}$$

From lemma 2 we have $g_i^\star(f_{-i})(s_k) - g_i^\star(f_{-i})(s_0) \geq 0$ for all $s_k \neq s_0$. Therefore, as $q(s_k|s,a,f_{-i}(s))$ is concave on a for all $s_k \neq s_0$,

$$\sum_{s_k\neq s_0} [g_i^\star(f_{-i})(s) - g_i^\star(f_{-i})(s_0)]q(s_k|s,a,f_{-i}(s))$$

is concave in a. This concludes the proof. ■

Note that in showing that the payoff function of a player is concave in each period, the result from lemma 2 played an important role. The condition that there is a low payoff state is used to deal with non-convexity issues that would otherwise arise in these cases.

In the next section we explore a different class of games that need not have a *low-payoff* state but have optimal value functions that are monotonic in the state variable.

4 Stochastic Games with Nondecreasing State Dependent Payoffs

The class of Stochastic Games that we analyze here has a slightly different structure than the class of Stochastic Games that we analyzed in the previous section. Here again we show that when the payoff of a player depends only on the state and his own actions in a single-period, and the payoff is non decreasing in the payoff relevant states, then the set of optimal stationary strategies that are best responses to the stationary strategies of the other players is a convex set. In this class of Stochastic Games the actions of the other players affect the future expected payoff of a player through their impact on the state variable in the future. The transition probabilities are increasing in the actions of a player in the sense of first-order stochastic dominance. It is easy to verify that these conditions hold for the games illustrated in examples 4 and 5. To analyze these classes of games we replace conditions (V) and (VI) of the class of games analyzed in section 3 with conditions (VII) and $(VIII)$ here.

(VII) The single-period payoff function of a player is a function only of the state variable $s \in S$ and the actions $a_i \in A_i$ of player i, and is concave in a_i and increasing in the state variable. That is, $u_i(s_j, a_i) \leq u_i(s_{j+1}, a_i)$ for $j = 0, \cdots, K$ for each $a_i \in A_i$.

(VIII) The transition probabilities satisfy a first-order stochastic dominance condition, namely for all $\ell \geq 1$ and for all $j = 1, \cdots, K$

$$\sum_{\nu=\ell}^{K} q(s_\nu | s_{j+1}, a_i, a_{-i}) \geq \sum_{\nu=\ell}^{K} q(s_\nu | s_j, a_i, a'_{-i}), \text{ for all } a_{-i}, a'_{-i} \in A_{-i}.$$

Also, for any $s \in S$, there is an $s_j \in S$ such that $q(s_k | s, a_i, a_{-i})$ is convex in a_i for $k \leq j$ and concave in a_i for $k > j$. Thus the probability density on s_k is convex in a_i for $k \leq j$ and concave in a_i for $k > j$[6].

Condition (VII) indicates that the single-period payoff of a player is increasing in the state variable. Condition $(VIII)$ indicates that if the current state is a higher payoff state, then the likelihood of the higher payoff states occurring in the next period remains higher even if the other players took actions a'_{-i} rather than a_{-i}. Thus, the current state has a relatively strong effect on the next period's probability densities than the actions of the other players. The next two examples illustrate how these conditions can arise quite naturally in economic models.

Example 4 *R & D competition between two firms*

In these games the firms choose investment levels that lead to innovations that would enhance the market for the good in the next period. There are two possible states of the market, a high state (H) and a low state (L). The state is specific to the firm so that if the state of the market was high for both firms the state of the market is (H, H) so that there four possible states in this game. The single-period payoff of a firm is the profit that would be generated by the firm given the state of the market for the firm and the level of investment undertaken. Thus, the single-period payoff is give by

$$\pi_i(s_i) - c_i(I_i)$$

[6] Thus, in examples where the actions of the players are effort levels, one may view such a condition as indicating that as the effort level of a player increases, the probability of the good state increases but in a concave manner and the probability of a bad state decreases but in a convex manner.

where $\pi_i(s_i)$ is the profit of the firm given the state $s_i \in \{H, L\}$ and $c_i(I_i)$ is the cost of investing in product development and $I_i \in [\underline{I}, \bar{I}]$. The transition probability $q_i(.|s_i, I_i, I_j)$ of firm i gives the probability of the state of the market for firm i in the next period and depends on the current state of the market for the firm and the level of investment in the current period by both the firms.

We make the following assumptions about these transition probabilities

$$q_i(H|H, \underline{I}_i, \bar{I}_j) \geq q_i(H|L, \underline{I}_i, \underline{I}_j) \tag{16}$$

so that the probability of the state H for firm i is at least as large as the probability of state H when the last period's state is H, and firm i has invested the smallest possible amount $\underline{I}_i$, and firm j has invested the largest possible amount $\bar{I}_j$. We now observe that the following assumptions are fairly natural.

$$\frac{\partial q_i(H|s, I_i, I_j)}{\partial I_i} > 0, \text{ and } \frac{\partial^2 q_i(H|s, I_i, I_j)}{\partial^2 I_i} < 0$$

for $s = H, L$. This immediately implies that

$$\frac{\partial q_i(L|s, I_i, I_j)}{\partial I_i} < 0, \text{ and } \frac{\partial^2 q_i(L|s, I_i, I_j)}{\partial^2 I_i} > 0.$$

Also

$$\frac{\partial q_i(H|s, I_i, I_j)}{\partial I_j} \leq 0.$$

The next proposition shows that these conditions imply that the transition probabilities will satisfy the stochastic dominance condition.

Proposition 4 *If the transition probabilities satisfy the given assumptions then $q_i(.|H, I_i, I_j)$ stochastically dominates $q_i(.|L, I_i, I'_j)$ for any pair I_i and any pair (I_j, I'_j).*

Proof: In order to show this we need to show that

$$q_i(H|H, I_i, I_j) \geq q_i(H|L, I_i, I'_j)$$

for I_i and any pair (I_j, I'_j). As

$$q_i(H|H, \underline{I}_i, \bar{I}_j) \geq q_i(H|L, \underline{I}_i, \underline{I}_j)$$

and

$$\frac{\partial q_i(H|s, I_i, I_j)}{\partial I_i} > 0$$

we have

$$q_i(H|H, I_i, \bar{I}_j) \geq q_i(H|L, I_i, \underline{I}_j) \tag{17}$$

for any $I_i \in [\underline{I}_i, \bar{I}_i]$. From the condition

$$\frac{\partial q_i(H|s, I_i, I_j)}{\partial I_j} \leq 0$$

and (17) it now follows that

$$q_i(H|H, I_i, I_j) \geq q_i(H|L, I_i, I'_j) \tag{18}$$

for any pair (I_j, I'_j). ∎

Therefore in these models condition (VII) holds because the single-period payoffs depend only on the state and the current investment by the firm. The profit level of a firm is increasing in the state of the market for the firm, and the cost of investment increases with the level of current investment by the firm. Proposition 4 shows that condition $(vIII)$ will if the condition in (16) is satisfied. Condition (16) indicates that while the level of investment of the other firms can affect the transition probability of a firm, it is the current state of the market for the firm that has the stronger effect. One again needs to observe that the condition is a condition only about the transition probabilities. ∎

The next example, while it belongs to the same class of stochastic games as the previous example, is motivated by a different economic model. Also in this example the number of states for each player is three, so that the example better illustrates the conditions required when there are more than two states.

Example 5 Sales competition between two rivals

Two salespersons are engaged in competing for sales. The state of the market for a player i, $i = 1, 2$ can vary from a lowest payoff state s_1 to the highest payoff state s_K. For instance the state can be high (H). medium (M) and low (L) for a players i. Thus, between the two players there are six possible states with various combinations of high, medium and low for the two players. The single-period payoff of player is affected only by whether the state is high, medium or low for the player, so that it is possible that the state is high for both players, or that the state is low for one player but high for the other player. The single-period profit of a player is independent of the state for the other player and is only affected by the state of the market for that player.

The actions of the salespersons are the effort levels they choose and is assumed to belong to $[\underline{e}, \bar{e}]$. The effort levels impose a cost in the current period but increases the probability of higher sales in the next period, with higher effort levels in the current period associated with higher probability of a better state in the next period. The single-period payoff of a player in this game is thus

$$\pi_i(s_i) - c_i(e_i)$$

where $\pi_i(s_i)$ is the profit in the current period, and depends on the current state of the market for player i given by s_i, $c_i(e_i)$ is cost of the effort level e_i for player i in the current period. We assume that $c_i(.)$ is a convex function.

The transition probability which gives the probabilities of the various states of a player in the next period depends on the effort levels of both the players in the current period. Thus it is possible that the higher effort level of the rival player can lower the probabilities of the high payoff states of a player.

We will assume that the transition probability of a player i satisfies the following conditions.

$$\begin{aligned} q_i(H|H, \underline{e}_i, \bar{e}_j) &\geq q_i(H|M, \underline{e}_i, \underline{e}_j) \\ q_i(L|H, \underline{e}_i, \bar{e}_j) &\leq q_i(L|M, \underline{e}_i, \bar{e}_j). \end{aligned} \tag{19}$$

And for states M and L we will assume that

$$\begin{aligned} q_i(H|M, \underline{e}_i, \bar{e}_j) &\geq q_i(H|L, \underline{e}_i, \underline{e}_j) \\ q_i(L|M, \underline{e}_i, \bar{e}_j) &\leq q_i(L|L, \underline{e}_i, \underline{e}_j). \end{aligned} \tag{20}$$

The conditions in (19) and (20) are conditions on the transition probabilities. Thus if the current state is a higher payoff state, then the probability of a higher payoff state being realized in the future remains at least as high than if the current state is a lower payoff state, even when the rival player is exerting maximum effort. It is worth noting that this is a statement only about the probability with which a state may be realized and not about the *actual realizations*.

The following set of conditions about the direction of the change of the probabilities with respect to the effort levels of the player is fairly natural. We assume that

$$\frac{\partial q_i(s_k|s_k, e_i, e_j)}{\partial e_i} > 0, \text{ for } s_k = H, M \text{ and } \frac{\partial q_i(L|s_k, e_i, e_j)}{\partial e_i} < 0 \text{ for any } s_k. \tag{21}$$

Also

$$\frac{\partial q_i(H|H, e_i, e_j)}{\partial e_i} \geq \frac{\partial q_i(H|M, e_i, e_j)}{\partial e_i}, \text{ and } \frac{\partial q_i(L|H, e_i, e_j)}{\partial e_i} \leq \frac{\partial q_i(L|M, e_i, e_j)}{\partial e_i}. \tag{22}$$

It is also reasonable to expect that the following conditions will hold with respect to the effort levels of the rival player.

$$\frac{\partial q_i(H|H, e_i, e_j)}{\partial e_j} \leq \frac{\partial q_i(H|M, e_i, e_j)}{\partial e_j} < 0,$$

and

$$\frac{\partial q_i(L|H, e_i, e_j)}{\partial e_j} \geq \frac{\partial q_i(L|M, e_i, e_j)}{\partial e_j} > 0.$$

As the result below shows these conditions together lead to the first-order stochastic dominance condition, condition (VIII), on the transition probability.

Proposition 5 *For any pair of effort levels e_j and e_j' of player j, for any effort level e_i of player i, the transition probability $q_i(.|H, e_i, e_j)$ stochastically dominates the transition probability $q_i(.|M, e_i, e_j')$, and the transition probability $q_i(.|M, e_i, e_j)$ stochastically dominates the transition probability $q_i(.|L, e_i, e_j')$.*

Proof: We show this for the transition probabilities $q_i(.|H, e_i, e_j)$ and $q_i(.|M, e_i, e_j')$ as the proof for the other case is identical. We first note that as

$$q_i(H|H, \underline{e}_i, \bar{e}_j) \geq q_i(H|M, \underline{e}_i, \bar{e}_j)$$

and

$$\frac{\partial q_i(H|H, e_i, e_j)}{\partial e_i} \geq \frac{\partial q_i(H|M, e_i, e_j)}{\partial e_i} > 0$$

therefore

$$q_i(H|H, e_i, \bar{e}_j) \geq q_i(H|M, e_i, \underline{e}_j) \tag{23}$$

for any e_i. From (23) and the condition that

$$\frac{\partial q_i(H|H, e_i, e_j)}{\partial e_j} \leq \frac{\partial q_i(H|M, e_i, e_j)}{\partial e_j} < 0$$

we have for any e'_j that

$$q_i(H|H, e_i, \bar{e}_j) \geq q_i(H|M, e_i, e'_j)$$

and hence,

$$q_i(H|H, e_i, e_j) \geq q_i(H|M, e_i, e'_j) \tag{24}$$

for any e_j.

Similarly, from the conditions

$$q_i(L|H, \underline{e}_i, \bar{e}_j) \leq q_i(L|M, \underline{e}_i, \underline{e}_j)$$

and

$$\frac{\partial q_i(L|H, e_i, e_j)}{\partial e_j} \geq \frac{\partial q_i(L|M, e_i, e_j)}{\partial e_j} > 0, \ \frac{\partial q_i(L|H, e_i, e_j)}{\partial e_i} \leq \frac{\partial q_i(L|M, e_i, e_j)}{\partial e_i} < 0,$$

we have for any e_i and any pair $e_j,\ e'_j$

$$q_i(L|H, e_i, e_j) \leq q_i(L|M, e_i, e'_j)$$

but this then implies that

$$q_i(H|H, e_i, e_j) + q_i(M|H, e_i, e_j) \geq q_i(H|M, e_i, e'_j) + q_i(M|M, e_i, e'_j) \tag{25}$$

for any e_i and any pair $e_j,\ e'_j$. The fact that the transition probability $q_i(.|H, e_i, e_j)$ stochastically dominates the transition probability $q_i(.|M, e_i, e'_j)$ for any e_i and any pair $e_j,\ e'_j$ then follows from (24) and (25). ∎

We first show that given conditions (VII) and $(VIII)$, the value function of player i, given by $g_i^\star(f_{-i}) : S \to \mathbb{R}$ when the stationary Markov strategies of the players other than i is f_{-i}, is increasing in the state variable. We then show that the sum of the current and the expected payoff of a player i is concave in his actions a_i.

Lemma 4 *Under conditions (VII) and $(VIII)$, the value functions $g_i^\star(f_{-i}) : S \to \mathbb{R}$ are increasing in the state variable. That is,*

$$g_i^\star(f_{-i})(s_{j+1}) > g_i(f_{-i})(s_j)$$

for all $j = 1, \cdots, K-1$.

Proof: We first note that given the stationary Markov strategies f_{-i} of the other players, by lemma 1 there is a stationary Markov strategy that is an optimal strategy of player i. We also note that this is obtained by solving the dynamic programming problem in which in time period t player i solves

$$\max_{a_i}[u_i(s_k, a_i) + \delta \sum_{\ell=1}^{K} g_i(f_{-i})(s_\ell) q(s_\ell | s_k, a_i, f_{-i}(s_k))] \tag{26}$$

where $g_i(f_{-i}) : S \to \mathbb{R}$ denotes the expected future payoff of player i given the stationary strategies of the players other than i.

We need to show that the optimal value function $g_i^\star(f_{-i}) : S \to \mathbb{R}$ is increasing in s_k, that is, $g_i^\star(f_{-i})(s_{j+1}) > g_i^\star(f_{-i})(s_j)$ for $j = 1, \cdots, K-1$. We first note that if $M(S)$ denotes the set of bounded functions from S to $\mathbb{R}$, then $g_i^\star(f_{-i})$ is the fixed point of the operator $U : M(S) \to M(S)$, where the operator U is defined as

$$U(g_i(f_{-i}))(s_k) = \max_{a_i}[u_i(s_k, a_i) + \delta \sum_{\ell=1}^{K} g_i(f_{-i})(s_\ell) q(s_\ell | s_k, a_i, f_{-i}(s_k))].$$

From the literature in Dynamic Programming, again one may refer to [16] and [3], we know that this operator is a contraction mapping and thus has a fixed point. The fixed point of the operator U is the value function $g_i^\star(f_{-i})$.

We now claim that the operator U maps nondecreasing value functions to value functions that are increasing in the state variables. Given a value function $v_i : S \to \mathbb{R}$ such that $v_i(s_{j+1}) \geq v_i(s_j)$ for all $j = 1, \cdots, K-1$, if $a_1^\star$ is such that

$$U(v_i)(s_1) = u_i(s_1, a_1^\star) + \delta[\sum_{k=1}^{K} v_i(s_k) q(s_k | s_k, a_1^\star, f_{-i}(s_k))]$$

then from condition $(viii)$ of first-order stochastic dominance of the transition probabilities, and from the fact that the $v_i(.)$ is nondecreasing in the states, it follows that

$$\delta[\sum_{k=1}^{K} v_i(s_k) q(s_k | s_2, a_1^\star, f_{-i}(s_2))] \geq \delta[\sum_{k=1}^{K} v_i(s_k) q(s_k | s_1, a_1^\star, f_{-i}(s_1))] \tag{27}$$

therefore, as condition (VII) implies that $u_i(s_2, a_1^\star) > u_i(s_1, a_1^\star)$, from (27) we have

$$\begin{aligned} U(v_i)(s_2) &\geq u_i(s_2, a_1^\star) + \delta[\sum_{k=1}^{K} v_i(s_k) q(s_k | s_2, a_1^\star, f_{-i}(s_2))] \\ &> u_i(s_1, a_1^\star) + \delta[\sum_{k=1}^{K} v_i(s_k) q(s_k | s_1, a_1^\star, f_{-i}(s_1))] \\ &= U(v_i)(s_1), \end{aligned}$$

where the second inequality above follows from conditions (VII) and (VIII). Clearly,

$$U^2(v_i)(s_2) \geq U^2(v_i)(s_1)$$

and in general

$$U^n(v_i)(s_2) \geq U^n(v_i)(s_1).$$

We thus have for any s_2, s_1 that

$$g_i^\star(f_{-i})(s_2) = \lim_{n\to\infty} U^n(v_i)(s_2) \geq \lim_{n\to\infty} U^n(v_i)(s_1) = g_i^\star(f_{-i})(s_1). \tag{28}$$

Following the above steps for $U(v_i)(s_3)$ and $U(v_i)(s_2)$ we can show that

$$g_i^\star(f_{-i})(s_3) \geq g_i^\star(f_{-i})(s_2)$$

and, in general, for s_{k+1} and s_k that

$$g_i^\star(f_{-i})(s_{k+1}) \geq g_i^\star(f_{-i})(s_k). \tag{29}$$

Thus the optimal value function of player i is nondecreasing in the state variable s. This completes the proof. ∎

We now go on to show that the sum of the single-period payoff and the expected payoff from the optimal value function of player i is concave in the actions of player i.

Lemma 5 *The expected payoff of player i in any period t when player i plans to play optimally in the future given by*

$$u_i(s_k, a_i) + \delta[\sum_{\ell=1}^{K} g_i^\star(f_{-i})(s_\ell) q(s_\ell | s_k, a_i, f_{-i}(s_k))]$$

is concave in a_i. Therefore, the optimal stationary strategies belong to a convex subset of functions from S to A_i.

Proof: We need to only show that $\sum_{\ell=1}^{K} g_i^\star(f_{-i})(s_\ell) q(s_\ell | s_k, a_i, f_{-i}(s_k))$ is concave in a_i. From condition $(viii)$ it follows that for each s there is a state s_j such that for $k > j$, $q(s_k|s, a_i, f_{-i})(s))$ is concave in a_i and for $k \leq j$, $q(s_k|s, a_i, f_{-i})(s))$ is convex in a_i. We first note that, for any $a_i, a_i' \in A_i$, we have

$$\begin{aligned} &\sum_{\ell=1}^{K} q(s_\ell | s_k, \lambda a_i + (1-\lambda) a_i'), f_{-i}(s_k)) = 1 \\ &= \lambda \sum_{\ell=1}^{K} q(s_\ell | s_k, a_i, f_{-i}(s_k)) + (1-\lambda) \sum_{\ell=1}^{K} q(s_\ell | s_k, a_i', f_{-i}(s_k)) \end{aligned}$$

so that

$$\begin{aligned}
&[\lambda \sum_{\ell=1}^{j} q(s_\ell|s_k, a_i, f_{-i}(s_k)) + (1-\lambda) \sum_{\ell=1}^{j} q(s_\ell|s_k, a_i', f_{-i}(s_k))] \\
&- \sum_{\ell=1}^{j} q(s_\ell|s_k, \lambda a_i + (1-\lambda)a_i'), f_{-i}(s_k)) \\
= &\sum_{\ell=j+1}^{K} q(s_\ell|s_k, \lambda a_i + (1-\lambda)a_i'), f_{-i}(s_k)) \\
&-[\lambda \sum_{\ell=j+1}^{K} q(s_\ell|s_k, a_i, f_{-i}(s_k)) + (1-\lambda) \sum_{\ell=j+1}^{K} q(s_\ell|s_k, a_i', f_{-i}(s_k))].
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
&[\sum_{\ell=1}^{j}[\lambda q(s_\ell|s_k, a_i, f_{-i}(s_k)) + (1-\lambda) q(s\ell|s_k, a_i', f_{-i}(s_k)) \\
&- q(s_\ell|s_k, \lambda a_i + (1-\lambda)a_i'), f_{-i}(s_k))] \\
= &\sum_{\ell=j+1}^{K} [q(s_\ell|s_k, \lambda a_i + (1-\lambda)a_i'), f_{-i}(s_k)) \\
&- (\lambda q(s_\ell|s_k, a_i, f_{-i}(s_k)) + (1-\lambda) q(s_\ell|s_k, a_i', f_{-i}(s_k)))]. \qquad (30)
\end{aligned}$$

Because of condition (VIII), which states that for $\ell \leq j$, $q(s|s_k, a_i, f_{-i})$ is convex in a_i and for $\ell > j$, $q(s|s_k, a_i, f_{-i})$ is concave in a_i, each term in (30) is nonnegative. From lemma 4 which shows that the optimal value functions are increasing in the state variable, and (30), we then have

$$\begin{aligned}
&[\lambda \sum_{\ell=1}^{j} g_i^\star(s_\ell) q(s_\ell|s_k, a_i, f_{-i}(s_k)) + (1-\lambda) \sum_{\ell=1}^{j} g_i^\star(s_\ell) q(s_\ell|s_k, a_i', f_{-i}(s_k))] \\
&- \sum_{\ell=1}^{j} g_i^\star(s_\ell) q(s_\ell|s_k, \lambda a_i + (1-\lambda)a_i'), f_{-i}(s_k)) \\
\leq &\sum_{\ell=j+1}^{K} g_i^\star(s_\ell) q(s_\ell|s_k, \lambda a_i + (1-\lambda)a_i'), f_{-i}(s_k)) \\
&-[\lambda \sum_{\ell=j+1}^{K} g_i^\star(s_\ell) q(s_\ell|s_k, a_i, f_{-i}(s_k)) + (1-\lambda) \sum_{\ell=j+1}^{K} g_i^\star(s_\ell) q(s_\ell|s_k, a_i', f_{-i}(s_k))]. \qquad (31)
\end{aligned}$$

But from (31) it now follows that

$$[\lambda \sum_{\ell=1}^{K} g_i^\star(s_\ell) q(s_\ell|s_k, a_i, f_{-i}(s_k)) + (1-\lambda) \sum_{\ell=1}^{K} g_i^\star(s_\ell) q(s_\ell|s_k, a_i', f_{-i}(s_k))]$$

$$\leq \sum_{\ell=1}^{K} g_i^\star(s_\ell) q(s_\ell | s_k, \lambda a_i + (1-\lambda) a_i'), f_{-i}(s_k)). \quad (32)$$

This shows that $\sum_{\ell=1}^{K} g_i^\star(s_\ell) q(s_\ell | s_k, a_i, f_{-i}(s_k))$ is concave in a_i for any s_k. This concludes the proof. ∎

5 Existence of Equilibrium

In this section we show that the two classes of Stochastic Games described in sections 3 and 4 have a stationary Markov perfect equilibrium. From this stage on the analysis of the two classes of games are identical and hence the results are common to both these classes of games. We first show that the set of optimal stationary strategies of a player i has the right continuity properties with respect to the stationary strategy choices of the other players. We first note that the payoff function

$$u_i(s_t, f_i(s_t), f_{-it}(s_t)) + \delta \int_S g_i(s') q(ds' | s_t, f_i(s_t) f_{-it}(s_t))$$

where f_i and f_{-i} denotes the stationary strategies of player i and of the players other then i respectively.

Let $F_i = \{f_i : S \to A_i\}$ and $F_{-i} = \{f_{-i} : S \to A_{-i}\}$. That is, F_i is the set of all possible stationary strategies of player i and F_{-i} is the set of all possible stationary strategies of the players other than i. As A_i is a convex set for each i, it follows that both F_i and F_{-i} are convex sets. As S is finite and A_i is a compact set of a Euclidean space, it also follows that both F_i and F_{-i} are also compact subsets of a Euclidean space.

Now define $B_i : F_{-i} \longrightarrow F_i$ as

$$B_i(f_{-i}) = \{f_i^\star : S \to A_i |\ f_i^\star \text{ is an optimal stationary strategy of player } i\}$$

That is the correspondence B_i is the set of optimal stationary strategies of player i when the players other than i use the stationary strategy f_{-i}. The next result shows that the correspondence B_i has a useful regularity property.

Lemma 6 *The correspondence B_i has a closed graph.*

Proof: Let $\{f_{-i}^\nu\}$ be a sequence of stationary strategies of the players other than i in F_{-i} that converges to $\hat{f}_{-i} \in F_{-i}$. Let $\{f_i^\nu\}$ be a sequence of stationary strategies of player i in F_i such that $f_i^\nu \in B_i(f_{-i}^\nu)$ for each $\nu = 1, 2, \cdots$.

Define $g_i^\nu : S \to \mathbb{R}$ to be the optimal stationary payoff function of player i when the other players use the optimal strategy f_{-i}^ν. Then because of compactness it follows that f_i converges to some $\hat{f}_i$ and the payoff functions g_i^ν converges to some $g_i : S \to \mathbb{R}$. We now note that for each $\nu \geq 1$ at any time t and for any s, player i solves the following problem

$$\max_{a \in A_i} [u_i(s, a, f_{-i}^\nu(s)) + \delta \int_S g_i^\nu(s') q_i(ds'|s, a, f_{-i}^\nu(s))].$$

From the definition of f_{-i}^ν, f_i^ν and g_i^ν it follows that

$$\begin{aligned} g_i^\nu(s) &= \max_{a \in A_i} [u_i(s, a, f_{-i}^\nu(s)) + \delta \int_S g_i^\nu(s') q_i(ds'|s, a, f_{-i}^\nu(s))] \\ &= u_i(s, f_i^\nu(s), f_{-i}^\nu(s)) + \delta \int_S g_i^\nu(s') q_i(ds'|s, f_i^\nu(s), f_{-i}^\nu(s)) \end{aligned}$$

for every $s \in S$. From continuity it now follows that

$$g_i(s) = u_i(s, \hat{f}_i(s), \hat{f}_{-i}(s)) + \delta \int_S g_i(s') q_i(ds'|s, \hat{f}_i(s), \hat{f}_{-i}(s)) \tag{33}$$

for each $s \in S$. But this shows that $\hat{f}_i \in B_i(\hat{f}_{-i})$ and completes the proof. ■

We can now give the main result.

Theorem 1 *There exists a stationary Markov Perfect Equilibrium for both classes of Stochastic Games.*

Proof: Let $F : S \to A$ denote the set of all Markov stationary strategies in the stochastic game. Define the correspondence $B : F \to F$ as

$$B(f) = B_i(f_{-1}) \times B_2(f_{-2}) \times \cdots \times B_n(f_{-n}).$$

Then B is a correspondence from a compact and convex set that is convex-valued (see lemma 2), compact-valued, and has a closed-valued (see lemma 3). Therefore, by Kakutani's fixed theorem it has a fixed point $f^\star$. We claim that $f^\star$ is a Markov perfect equilibrium of the stochastic game. This follows from noting that as $f_i^\star \in B_i(f_{-i}^\star)$, for every player i, and at any time t and any $s \in S$, we have

$$\begin{aligned} &\max_{a \in A_i} [u_i(s, a, f_{-i}^\star(s)) + \delta \int_S g_i^\star(s') q_i(ds'|s, a, f_{-i}^\star(s))] \\ = \; & u_i(s, f_i^\star(s), f_{-i}^\star(s)) + \delta \int_S g_i^\star(s') q_i(ds'|s, f_i^\star(s), f_{-i}^\star(s)) \end{aligned} \tag{34}$$

where $g_i^\star : S \to \mathbb{R}$ is the expected payoff of player i when the Markov Stationary strategy combination $f^\star$ is played is played in the future. This completes the proof. ■

6 Conclusion

The result depends quite crucially on the assumption that the transition probability splits into a concave and convex part and that in one case there is a low payoff state that absorbs the convex part of the transition probability and in the other case the convex part of the transition probability density describes the densities on the lower payoff states with the densities being concave in the higher payoff states. This makes the expected equilibrium future payoffs of a player to be concave in the current actions. This together with the assumption that the single-period utility functions are concave in the actions of a player allows for the crucial concavity property of the sum of the single-period payoff function and the continuation payoff of a player in any period. This as should be evident, plays a central in the derivation of the result here. Without these two conditions, even if the single-period utility function is concave, the result of lemma 2 may not hold as the *correct* expected future payoff or the *right value function* may not satisfy the concavity property and the best reply correspondence may not be convex-valued.

It is important to observe here that even with the convexity of the best reply correspondence, the fact that the best reply correspondence has a closed graph depends crucially on the fact that the state space is finite. If the state space is not finite then even if the best reply correspondence is convex-valued the graph of the best reply correspondence may not have a closed graph in the weak topology and thus the fixed point result used here may not work.

It also needs to be pointed out that the results here provide only sufficient conditions so it may be possible to derive other parallel results on the existence of equilibrium in pure strategies. The appeal of the results here are that these provide clear and plausible sufficient conditions under which one can get a Markov perfect stationary equilibrium in pure strategies. As the literature has shown this is not always easy to obtain. While the results themselves are useful it would seem that the method used here is especially of interest as it indicates a line of analysis that may be useful in finding other results. The method uses much of the classical results in stochastic dynamic programming to construct the optimal stationary strategies and then applies the fixed point arguments in a judicious way. The close ties between stochastic games and stochastic dynamic programming should tell us much about the nature of optimal strategies in stochastic games; the method used here show a way in which this relationship can be used profitably to generate solutions for stochastic games.

It is instructive to note here that the results perhaps cannot be demonstrated as easily by using a backward induction argument as is common in some analysis of dynamic games. The backward induction argument does not usually allow one to extract the properties of the value function or the continuation payoffs as transparently as is possible here, when the problem for a player is reduced to a stochastic dynamic programming problem. Clearly, the results have been obtained by finding the right mix of relationships between the actions of the players and their impact on the state of the game, and to draw on the results in stochastic dynamic programming to extract properties of the optimal value function of a player.

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